My A Level Maths Notes

Core Maths C1–C4

Kathy

Updated 07-Apr-2013

'My A Level Core Maths Notes'

These are my private notes, and it is up to you, dear reader, to ensure the facts are correct. Although I have done my best to proof read the notes, please use sensibly - check your facts.

The notes can be downloaded from www.brookhaven.plus.com/maths
If you see any problems please send corrections to: mymathsnotes@gmail.com

My thanks to Fritz K for his comments and corrections.

These notes have been produced entirely on an RISC OS Iyonix computer, using Martin Würthner's TechWriter for the typesetting and equations. Illustrations have been created in Martin Würthner's Artworks vector drawing package.
See www.mw-software.com for further information.

Please check the website for the latest version.

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Preface

Introduction

These are my class notes for C1 to C4 which my Dad has transcribed on to the computer for me, although he has gone a bit OTT with them! My cousin has been studying the AQA syllabus and so some of the chapters have been marked to show the differences.

Although a lot of my hand written mistakes have been corrected - there may be a few deliberate errors still in the script. If you find any, then please let us know so that we can correct them.

I have tried to put a * next to formulæ that are on the Formulæ sheet and a ** if I need to learn something.

Finally, there is no better way of learning than doing lots and lots of practise papers. Not least to get the hang of how the questions are worded and how you are often expected to use information from the previous part of a question. Sometimes this is not very obvious.

Thanks to Fritz K for his comments and corrections.

Kathy

Aug 2012

Required Knowledge

Algebra

A good grounding in handling algebraic expressions and equations, including the expansion of brackets, collection of like terms and simplifying is required. Revise how to deal with basic fractions - yes really. Can you do $\frac{7}{16} - \frac{1}{64}$ without using the calculator? How is your mental maths?

Studying for A Level

According to the papers, everyone seems to have achieved a raft of A*'s at GCSE, and you will be forgiven for thinking that A level can't be that much harder. Sorry, but you are in for a rude shock.

In maths alone you will have 6 modules to complete, and the first AS exams will probably be in the January after your first term of 6th form. Take note of these pointers:

- Compared to GCSE, the difficulty of work increases with many new concepts introduced.
- The amount of work increases, and the time to do the work is limited.
- The AS exams account for 50% of the marks and these exams are easier than the A2 exams. It is imperative to get the highest mark possible in AS, and avoid having to resit them.
- There is no substitute for doing lots and lots of practise papers.

By the time many students wake up to the reality of the work required, it may be too late to catch up without the added pressure of the inevitable resits.

Meaning of symbols

In addition to the usual mathematical symbols, ensure you have these committed to memory:

$\equiv$ is identical to

$\approx$ is approximately equal to

$\Rightarrow$ implies

$\Leftarrow$ is implied by

$\Leftrightarrow$ implies and is implied by

$\in$ is a member of

: is such that

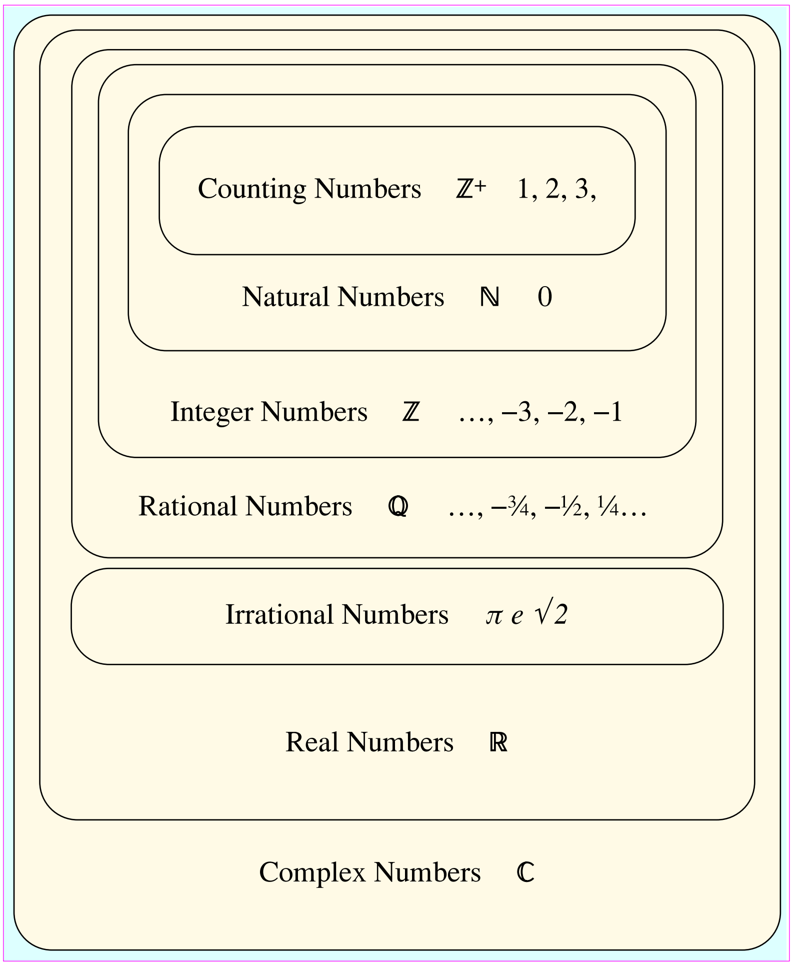
Sets of Numbers

The 'open face' letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are often used to define certain infinite sets of numbers. Unfortunately, there is no universal standard definition for the natural and counting numbers. Different authors have slight differences between them. The following should suffice for A level studies.

$\mathbb{Z}^+$	the counting numbers — whole numbers (from 1 upwards)
$\mathbb{N}$	the natural numbers — 0, 1, 2, 3… (0, plus all the counting numbers)
$\mathbb{Z}$	the integers — all whole numbers, includes negatives numbers, and all the natural numbers above (from the German *Zahlen*, meaning numbers)
$\mathbb{R}$	the real numbers — all the measurable numbers which includes integers above and the rational & irrational numbers (i.e. all fractions & decimals)
$\mathbb{Q}$	the rational numbers — from the word ratio, includes any number that can be expressed as a fraction with integers top and bottom, (this includes recurring decimals). $\mathbb{Q}$ stands for quotient
	the irrational numbers — any number that can't be expressed as a fraction, e.g. π, $\sqrt{2}$
$\mathbb{C}$	the complex numbers — e.g. $a + bi$ where $i = \sqrt{-1}$ (imaginary number)

Irrational numbers, when expressed as a decimal, are never ending, non repeating decimal fractions. Any irrational number that can be expressed exactly as a root term, such as $\sqrt{2}$, is called a **surd**.

A venn diagram may be helpful to sort them out.



Calculators in Exams

Check with exam board!

You cannot have a calculator that does symbolic algebra, nor can you have one that you have preprogrammed with your own stuff.

For A-Level the Casio FX-991 ES calculator is a excellent choice, and one that has a solar cell too.

If you want a graphical one, then the Texas TI 83+ seems to be highly regarded, although I used an older Casio one.

Get a newer version with the latest natural data entry method.

I prefer a Casio one so that data entry is similar between the two calculators.

Exam Tips

- Read the examiners reports into the previous exams. Very illuminating words of wisdom buried in the text.
- Write down formulae before substituting values.
- You should use a greater degree of accuracy for intermediate values than that asked for in the question. Using intermediate values to two decimal places will not result in a correct final answer if asked to use three decimal places.
- For geometrical transformations the word translation should be used rather than "trans" or "shift" etc.
- When finding areas under a curve a negative result may be obtained. However, the area of a region is a positive quantity and an integral may need to be interpreted accordingly.
- When asked to use the Factor Theorem, candidates are expected to make a statement such as "therefore $(x - 2)$ is a factor of $p(x)$" after showing that $p(2) = 0$.
- When asked to use the Remainder Theorem no marks will be given for using long division.

Module C1

Core 1 Basic Info

Indices and surds; Polynomials; Coordinate geometry and graphs; Differentiation.

The C1 exam is 1 hour 30 minutes long and normally consists of 10 question. The paper is worth 72 marks (75 AQA).

No calculator allowed for C1

Section A (36 marks) consists of 5—7 shorter questions worth at most 8 marks each.
Section B (36 marks) consists of 3 to 4 longer questions worth between 11—14 marks each.

OCR Grade Boundaries.

These vary from exam to exam, but in general, for C1, the approximate raw mark boundaries are:

Grade	100%	*A*	*B*	*C*
Raw marks	72	57 ± 3	50 ± 3	44 ± 3
UMS %	100%	80%	70%	60%

The raw marks are converted to a unified marking scheme and the UMS boundary figures are the same for all exams.

C1 Contents

Disclaimer

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Kathy, Feb 2013

C1 Assumed Basic Knowledge

You should know the following formulae, (many of which are NOT included in the Formulae Book).

1 Basic Algebra

Difference of squares is always the sum times the difference:

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 - b = \left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right)$$

2 Quadratic Equations

$$ax^2 + bx + c = 0 \quad \text{has roots} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Discriminant is $b^2 - 4ac$

3 Geometry

$$y = mx + c$$

$$y - y_1 = m(x - x_1)$$

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 m_2 = -1$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Hence: $$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Length of line between 2 points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Co-ordinate of the Mid point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

4 Circle

A circle, centre (a, b) and radius r, has equation

$$(x - a)^2 + (y - b)^2 = r^2$$

5 Differentiation and Integration

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
ax^n	anx^{n-1}
$f(x) + g(x)$	$f'(x) + g'(x)$

Function $f(x)$	Integral $\int f(x)\,dx$	
ax^n	$\frac{a}{n+1}x^{n+1} + c$	$n \neq -1$
$f'(x) + g'(x)$	$f(x) + g(x) + c$	

Area under curve $$A_x = \int_a^b y\,dx \qquad (y \geqslant 0)$$

C1 Brief Syllabus

1 Indices & Surds

- understand rational indices (positive, negative & zero), use laws of indices with algebraic problems
- recognise the equivalence of surd and index notation (e.g.$\sqrt{a} = a^{\frac{1}{2}}$)
- use the properties of surds, including rationalising denominators of the form $a + \sqrt{b}$

2 Polynomials

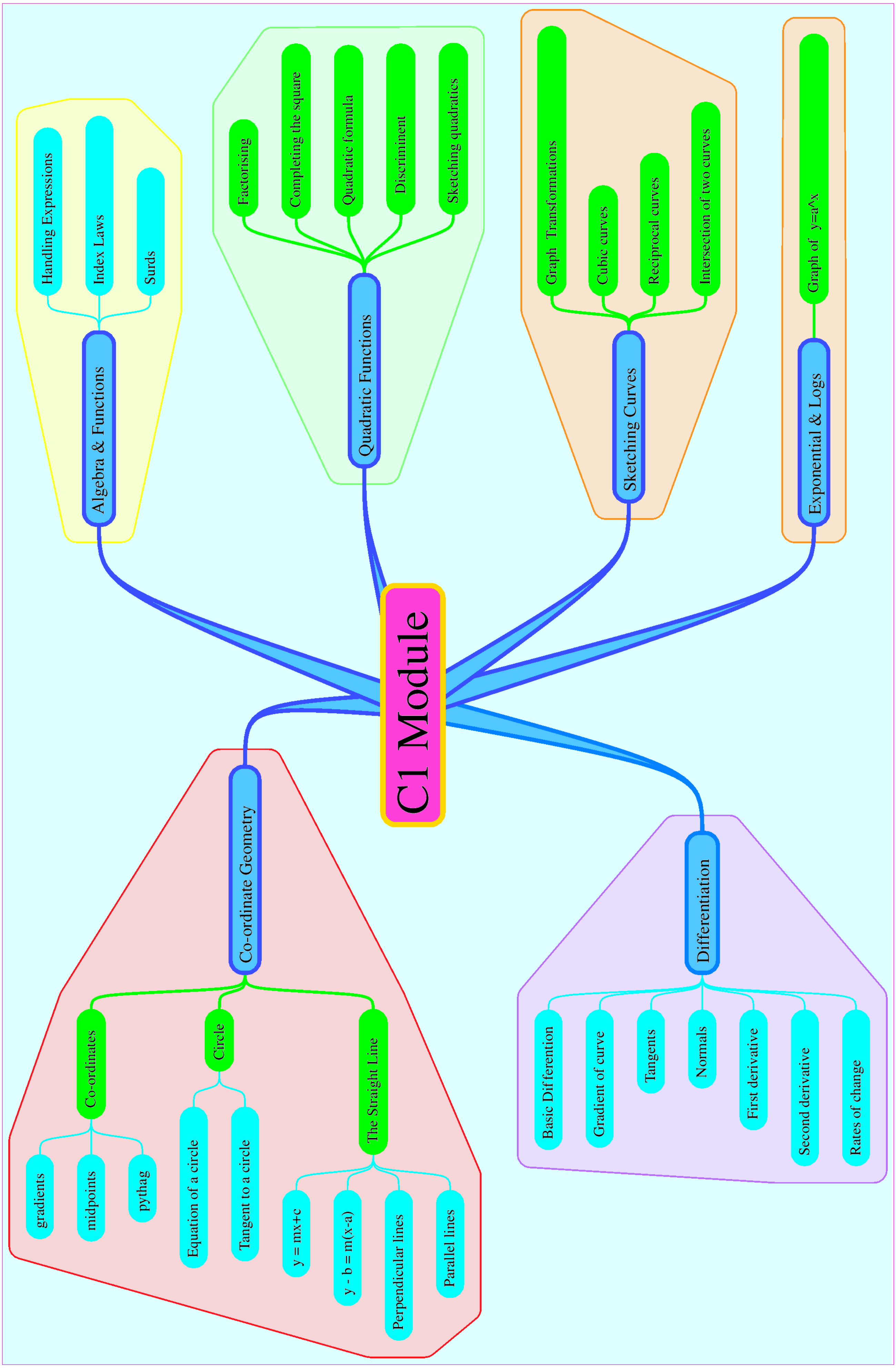
- carry out addition, subtraction, multiplication, expansion of brackets, collection of like terms and simplifying
- completing the square for a quadratic polynomial
- find and use the discriminant of a quadratic polynomial
- solve quadratic equations, and linear & quadratic inequalities, (one unknown)
- solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic
- recognise and solve equations in x which are quadratic in some function of x, e.g. $8x^{\frac{2}{3}} - x^{\frac{1}{3}} + 4 = 0$

3 Coordinate Geometry and Graphs

- find the length, gradient and mid-point of a line-segment, given the coordinates of the endpoints
- find the equation of a straight line
- understand the relationship between the gradients of parallel and perpendicular lines
- be able to use linear equations, of the forms $y = mx + c$, $y - y_1 = m(x - x_1)$, $ax + by + c = 0$
- understand that the equation $(x - a)^2 + (y - b)^2 = r^2$ represents the circle with centre (a, b) and radius r
- use algebraic methods to solve problems involving lines and circles, including the use of the equation of a circle in expanded form $x^2 + y^2 + 2px + 2qy + r = 0$. Know the angle in a semicircle is a right angle; the perpendicular from the centre to a chord bisects the chord; the perpendicularity of radius and tangent
- understand the relationship between graphs and associated algebraic equations, use points of intersection of graphs to solve equations, interpret geometrically the algebraic solution of equations (to include, in simple cases, understanding of the correspondence between a line being tangent to a curve and a repeated root of an equation)
- sketch curves with equations of the form:
    - $y = kx^n$, where n is a positive or negative integer and k is a constant
    - $y = k\sqrt{x}$, where k is a constant
    - $y = ax^2 + bx + c$, where a, b, c are constants
    - $y = f(x)$ where $f(x)$ is the product of at most 3 linear factors, not necessarily all distinct
- understand and use the relationships between the graphs of $y = f(x)$, $y = kf(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(kx)$, where a and k are constants, and express the transformations involved in terms of translations, reflections and stretches.

4 Differentiation

- understand the gradient of a curve at a point as the limit of the gradients of a suitable sequence of chords (an informal understanding only is required, differentiation from first principles is not included)
- understand the ideas of a derived function and second order derivative, and use the standard notations $f'(x)$, $\frac{dy}{dx}$, $f''(x)$, $\frac{d^2y}{dx^2}$
- use the derivative of x^n (for any rational n), together with constant multiples, sums and differences
- apply differentiation to gradients, tangents and normals, rates of change, increasing and decreasing functions, and the location of stationary points (the ability to distinguish between maximum points and minimum points is required, but identification of points of inflexion is not included)


C1 Module
Algebra & Functions
Handling Expressions
Index Laws
Surds
Quadratic Functions
Factorising
Completing the square
Quadratic formula
Discriminent
Sketching quadratics
Sketching Curves
Graph Transformations
Cubic curves
Reciprocal curves
Intersection of two curves
Exponential & Logs
Graph of y=a^x
Co-ordinate Geometry
Co-ordinates
gradients
midpoints
pythag
Circle
Equation of a circle
Tangent to a circle
The Straight Line
y = mx+c
y - b = m(x-a)
Perpendicular lines
Parallel lines
Differentiation
Basic Differentiation
Gradient of curve
Tangents
Normals
First derivative
Second derivative
Rates of change

1 • C1 • Indices & Power Rules

1.1 The Power Rules - OK

Recall that:

2^{10} is read as "2 raised to the power of 10" or just "2 to the power of 10"
where 2 is the base and 10 is the index, power or exponent.

The Law of Indices should all be familiar from GCSE or equivalent. Recall:

$$a^m \times a^n = a^{m+n} \qquad \text{Law ①}$$

$$\frac{a^m}{a^n} = a^{m-n} \qquad \text{Law ②}$$

$$\left(a^m\right)^n = a^{mn} \qquad \text{Law ③}$$

$$a^0 = 1 \qquad \text{Law ④}$$

$$a^{-n} = \frac{1}{a^n} \qquad \text{Law ⑤}$$

$$\sqrt[n]{a} = a^{\frac{1}{n}} \qquad \text{Law ⑥}$$

$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} \qquad (n \neq 0)$$

$$a^{\frac{1}{mn}} = \sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}} \qquad (m \neq 0,\ n \neq 0)$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

From the above rules, these common examples should be remembered:

$$\sqrt{a} = \sqrt[2]{a} = a^{\frac{1}{2}}$$

$$\sqrt[3]{a} = a^{\frac{1}{3}}$$

$$\frac{1}{a} = a^{-1}$$

$$a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}$$

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a$$

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$$

$$a^{\frac{3}{2}} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a\sqrt{a}$$

$$\left(\sqrt{a}\right)^2 = a \qquad \left(\sqrt[n]{a}\right)^n = a$$

$$a^0 = 1 \qquad a^1 = a$$

1.2 Examples

1 Solve for x: $\dfrac{6^x \times 6^5}{36} = 6^9$

$$\frac{6^x \times 6^5}{6^2} = 6^9 \quad \Rightarrow \quad 6^{x+5-2} = 6^9$$

Compare indices $\quad x + 3 = 9 \quad \Rightarrow \quad x = 6$

2 Solve for x and y with the following simultaneous equations:

$$5^x \times 25^{2y} = 1 \quad \text{and} \quad 3^{5x} \times 9^y = \frac{1}{9}$$

$$5^x \times \left(5^2\right)^{2y} = 5^0 \quad \Rightarrow \quad 5^x \times 5^{4y} = 5^0$$

$$\therefore \quad x + 4y = 0$$

$$3^{5x} \times 9^y = \frac{1}{9} \quad \Rightarrow \quad 3^{5x} \times 3^{2y} = 3^{-2}$$

$$\therefore \quad 5x + 2y = -2$$

Hence: $\quad x = -\dfrac{1}{6} \quad$ *and* $\quad y = \dfrac{1}{24}$

3 Simplify: $4a^2b \times \left(3ab^{-1}\right)^{-2}$

$$4a^2b \times 3^{-2}a^{-2}b^2 \quad \Rightarrow \quad \frac{4}{9}a^0b^3 \quad \Rightarrow \quad \frac{4}{9}b^3$$

4 Simplify: $\left(\dfrac{MLT^{-2}}{L^2}\right) \div \left(\dfrac{LT^{-1}}{L}\right) \quad \Rightarrow \quad \left(\dfrac{MT^{-2}}{L}\right) \div T^{-1}$

$$\left(\frac{MLT^{-2}}{L^2}\right) \div \frac{1}{T} \quad \Rightarrow \quad \left(\frac{MT^{-2}}{L}\right) \times T \quad \Rightarrow \quad \frac{M}{LT}$$

5 Solve for x: $2^{x+1} \div 4^{x+2} = 8^{x+3}$

Express as powers of 2 $\quad 2^{x+1} \div \left(2^2\right)^{x+2} = \left(2^2\right)^{x+3}$

$$2^{x+1} \div 2^{2x+4} = 2^{3x+9}$$

$$2^{x+1-(2x+4)} = 2^{3x+9}$$

$$2^{-x-3} = 2^{3x+9}$$

Compare indices $\quad -x - 3 = 3x + 9$

$$\therefore \quad x = -3$$

6 Simplify

Ex 1 $\quad 2x\sqrt{x} = 2x \times x^{\frac{1}{2}} = 2x^{1\frac{1}{2}} = 2x^{\frac{3}{2}}$ (usually left in top heavy form)

Ex 1 $\quad \dfrac{6}{\sqrt[3]{x}} = \dfrac{6}{x^{\frac{1}{3}}} = 6x^{-\frac{1}{3}}$

Ex 1 $\quad \dfrac{1}{x^2\sqrt{x}} = \dfrac{1}{x^{\frac{5}{2}}} = x^{-\frac{5}{2}}$

7 Evaluate

Ex 1	$\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$	(Cube root)
Ex 2	$(64)^{-\frac{1}{3}} \Rightarrow \left(\frac{1}{64}\right)^{\frac{1}{3}} \Rightarrow \frac{1}{4}$	(Cube root)
Ex 3	$\left(\frac{1}{4}\right)^{-\frac{1}{2}} \Rightarrow 4^{\frac{1}{2}} \Rightarrow \pm 2$	(Square root)
Ex 4	$16^{-\frac{3}{4}} \Rightarrow \left(\frac{1}{16}\right)^{\frac{3}{4}} \Rightarrow \left(\frac{1}{2}\right)^{3} \Rightarrow \frac{1}{8}$	(4-th root, cubed)
Ex 5	$\left(2\frac{1}{4}\right)^{-\frac{1}{2}} \Rightarrow \left(\frac{9}{4}\right)^{-\frac{1}{2}} \Rightarrow \left(\frac{4}{9}\right)^{\frac{1}{2}} \Rightarrow \frac{2}{3}$	

8 Solve

$$x^{\frac{3}{4}} = 27$$
$$x = 27^{\frac{4}{3}}$$
$$x = 3^4 = 81$$

9 Solve: $5x^{\frac{1}{3}} = x^{\frac{2}{3}} + 4$

$$x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 4 = 0$$

This is a quadratic in $x^{\frac{1}{3}}$ so let $y = x^{\frac{1}{3}}$

$$y^2 - 5y + 4 = 0 \quad \Rightarrow \quad (y - 1)(y - 4) = 0$$
$$y = 1 \text{ or } 4$$
$$\therefore \quad x^{\frac{1}{3}} = 1 \text{ or } 4$$
$$\therefore \quad x = 1^3 \text{ or } 4^3 \quad \Rightarrow 1,\ 64$$

10 Solve: $2^{2x} - 5\left(2^{x+1}\right) + 16 = 0$

Solution:

This should be a quadratic in $2x$ but middle term needs simplifying:

$$2^{x+1} = 2^x \times 2$$
$$\therefore \quad 5\left(2^{x+1}\right) = 5 \times 2^x \times 2 = 10\left(2^x\right)$$

Hence: $\quad \left(2^x\right)^2 - 10\left(2^x\right) + 16 = 0$

Let $y = 2^x$ $\quad y^2 - 10y + 16 = 0$

$$(y - 2)(y - 8) = 0$$
$$y = 2 \text{ or } 8$$
$$2^x = 2 \text{ or } 8$$
$$2^x = 2^1 \text{ or } 2^3$$
$$\therefore \quad x = 1 \text{ or } 3$$

11 Solve:

$$10^p = 0.1$$
$$= \frac{1}{10} = 10^{-1}$$
$$\therefore p = -1$$

12 Solve:

$$135^x \times 5^{5x} = 75$$

Solution:

Convert all numbers to prime factors:

$$135 = 3^3 \times 5$$
$$75 = 3 \times 5^2$$
$$\therefore \quad (3^3 \times 5)^x \times 5^{5x} = 3 \times 5^2$$
$$3^{3x} \times 5^x \times 5^{5x} = 3 \times 5^2$$
$$3^{3x} \times 5^{6x} = 3^1 \times 5^2$$

Compare indices for each base $\quad \therefore \quad 3x = 1 \quad \& \quad 6x = 2$

$$x = \frac{1}{3}$$

13 Solve:

$$27^{x+2} = 9^{2x-1}$$

Solution:

$$(3^3)^{x+2} = (3^2)^{2x-1}$$
$$3^{3x+6} = 3^{4x-2}$$
$$\therefore \quad 3x + 6 = 4x - 2$$
$$6 + 2 = 4x - 3x$$
$$x = 8$$

14 Evaluate: $8^{\frac{2}{3}}$

Three ways to achieve this:

(1) $\quad 8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} \Rightarrow 64^{\frac{1}{3}} = 4$

(2) $\quad 8^{\frac{2}{3}} = 8^{\frac{2}{3}} \times 8^{\frac{2}{3}} \Rightarrow 2 \times 2 = 4$

(3) $\quad 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 \Rightarrow 2^2 = 4$

15 Simplify: $\left(\frac{3x^2y^3z^6}{-6y^5}\right)^0$

$$\left(\frac{3x^2y^3z^6}{-6y^5}\right)^0 = 1$$

16 Simplify: $(-6y^5z^3)^0$

$$(-6y^5z^3)^0 = 1$$

17 Evaluate:

$$\left(27^{\frac{1}{3}} + 25^{\frac{1}{2}}\right)^{\frac{1}{3}}$$

Solution:

$$\left(27^{\frac{1}{3}} + 25^{\frac{1}{2}}\right)^{\frac{1}{3}} \Rightarrow (3 + 5)^{\frac{1}{3}}$$
$$= (8)^{\frac{1}{3}}$$
$$= 2$$

18 Evaluate:

$$16^{4.5} = 16^{\frac{9}{2}}$$
$$= \left(16^{\frac{1}{2}}\right)^9$$
$$= (4)^9$$
$$= 16 \times 16 \times 16 \times 16 \times 4$$
$$= 65536$$

19 Show that the function:

$$f(x) = \left(\sqrt{x} + 4\right)^2 + \left(1 - 4\sqrt{x}\right)$$

can be written as:

$$f(x) = ax + b$$

Solution:

$$f(x) = \left(\sqrt{x} + 4\right)^2 + \left(1 - 4\sqrt{x}\right)$$
$$= \left(x + 8\sqrt{x} + 16\right) + \left(1 - 8\sqrt{x} + 16x\right)$$
$$= 17x + 17$$

20 Evaluate:

$$\left(3\frac{3}{16} + 4\frac{3}{8}\right)^{-\frac{1}{2}}$$

Solution:

$$\left(3\frac{3}{16} + 4\frac{3}{8}\right)^{-\frac{1}{2}} = \left(7\frac{9}{16}\right)^{-\frac{1}{2}}$$

Recall that: $7\frac{9}{16} \equiv 7 + \frac{9}{16}$

$$= \left(\frac{112}{16} + \frac{9}{16}\right)^{-\frac{1}{2}}$$
$$= \left(\frac{121}{16}\right)^{-\frac{1}{2}}$$
$$= \left(\frac{16}{121}\right)^{\frac{1}{2}}$$
$$= \frac{\sqrt{16}}{\sqrt{121}}$$
$$= \frac{4}{11}$$

21 Solve:

$$\left(49k^4\right)^{\frac{1}{2}} = 63$$

Solution:

$$7k^2 = 63$$

$$k^2 = \frac{63}{7} = 9$$

$$k = 3$$

22 Solve:

$$3\,(x)^{-\frac{1}{2}} - 4 = 0$$

Solution:

$$\frac{3}{\sqrt{x}} = 4$$

$$\frac{3}{4} = \sqrt{x}$$

$$x = \left(\frac{3}{4}\right)^2$$

$$= \frac{9}{16}$$

2 • C1 • Surds

2.1 Intro to Surds

A surd is any expression which contains a square or cube root, and which cannot be simplified to a rational number, i.e. it is irrational.

Recall the set of real numbers includes rational & irrational numbers:

ℝ the real numbers — all the measurable numbers which includes integers and the rational & irrational numbers (i.e. all fractions & decimals)

ℚ the rational numbers — from the word ratio, includes any number that can be expressed as a ratio or fraction with integers top and bottom, (this includes all terminating & recurring decimals).
(ℚ stands for quotient)

NS the irrational numbers — any number that **cannot** be expressed as a fraction, e.g. π, $\sqrt{2}$
(includes the square root of any non square number, & the cube root of any non cube number)
(*NS* – there is No Symbol for irrational numbers)

Irrational numbers, when expressed as a decimal, are never ending, non repeating decimal fractions with no pattern. Any irrational number that can be expressed exactly as a root, such as $\sqrt{2}$, is called a **surd**.

It is often convenient to leave an answer in surd form because:

- surds can be manipulated like algebraic expressions
- **surds are exact** – use when a question asks for an exact answer!
- the decimal expansion is never wholly accurate and can only be an approximation
- a surd will often reveal a pattern that the decimal would hide

The word 'surd' was often used as an alternative name for 'irrational', but it is now used for any root that is irrational.

Some examples:

Number	Simplified	Decimal	Type	Root is :
$\sqrt{2}$	$\sqrt{2}$	1·414213562…	**Irrational**	**Surd**
$\sqrt{3}$	$\sqrt{3}$	1·732050808…	**Irrational**	**Surd**
$\sqrt{9}$	3	3·0	Integer	
$\sqrt{\frac{4}{9}}$	$\frac{2}{3}$	0·666’	Rational	
$\sqrt[3]{13}$	$\sqrt[3]{13}$	2·351334688…	**Irrational**	**Surd**
$\sqrt[3]{64}$	4	4·0	Integer	
$\sqrt[4]{625}$	5	5·0	Integer	
$\sqrt{Prime\ No}$			**Irrational**	**Surd**
π	π	3·141592654…	Irrational	
e	e	2·718281828…	Irrational	

In trying to solve questions involving surds it is essential to be familiar with square numbers thus:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144…

and with cube numbers thus:

1, 8, 27, 64, 125, 216…

2.2 Handling Surds — Basic Rules

These rules are useful when simplifying surds:

$$\sqrt{x} \times \sqrt{x} = \left(\sqrt{x}\right)^2 = x$$

Rearranging gives some useful results:

$$\sqrt{x} = \frac{x}{\sqrt{x}}$$

$$\frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x}$$

From the law of indices

Law 1 $$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$$

Law 2 $$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$$

Also

$$x = \sqrt{x^2}$$

$$a\sqrt{c} + b\sqrt{c} = (a + b)\sqrt{c}$$

- If it is a **root** and **irrational**, it is a **surd**, e.g. $\sqrt{3}$, $\sqrt[3]{6}$
- Not all roots are surds, e.g. $\sqrt{9}$, $\sqrt[3]{64}$
- Square roots of integers that are square numbers are rational
- The square root of all prime numbers are surds and irrational

2.3 Factorising Surds

In factorising a surd, look for square numbers that can be used as factors of the required number. Recall the square numbers of 4, 9, 16, 25, 36, 49, 64…

2.3.1 Example:

Simplify:

Ex 1 $$\sqrt{54} = \sqrt{9 \times 6} = \sqrt{9} \times \sqrt{6} = 3\sqrt{6}$$

Ex 2 $$\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$$

2.4 Simplifying Surds

Since surds can be handled like algebraic expressions, you can easily multiply terms out or add & subtract 'like' terms.

2.4.1 Example:

Simplify the following:

Ex 1 $$\sqrt{12}\sqrt{3} = \sqrt{36} = 6$$

Ex 2 $$\frac{\sqrt{27}}{\sqrt{3}} = \frac{\sqrt{9 \times 3}}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}} = 3$$

Ex 3 $$\sqrt{28} + \sqrt{63} = 2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$$

Ex 4 $$\sqrt[3]{16} = \sqrt[3]{2 \times 8} = 2\sqrt[3]{2}$$

2.5 Multiplying Surd Expressions

Handle these in the same way as expanding brackets in algebraic expressions.

2.5.1 Example:

Simplify $(1 - \sqrt{3})(2 + 4\sqrt{3})$

Solution:

$$\begin{aligned}(1 - \sqrt{3})(2 + 4\sqrt{3}) &= 2 + 4\sqrt{3} - 2\sqrt{3} - 4\sqrt{3}\sqrt{3} \\ &= 2 + 2\sqrt{3} - 4 \times 3 \\ &= -10 + 2\sqrt{3}\end{aligned}$$

2.6 Surds in Exponent Form

If you are a bit confused by the surd form, try thinking in terms of indices:

E.g.

Ex 1

$$\begin{aligned}\frac{x}{\sqrt{x}} &= \frac{x}{x^{\frac{1}{2}}} \\ &= x \times x^{-\frac{1}{2}} \\ &= x^{\frac{1}{2}} \\ &= \sqrt{x}\end{aligned}$$

Ex 2

$$\begin{aligned}\frac{\sqrt{x}}{x} &= \frac{x^{\frac{1}{2}}}{x} \\ &= x^{\frac{1}{2}} \times x^{-1} \\ &= x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

2.7 Rationalising Denominators (Division of Surds)

By convention, it is normal to clear any surds in the denominator. This is called **rationalising the denominator**, and is easier than attempting to divide by a surd.

In general, simplify any answer to give the smallest surd.

There are three cases to explore:

- A denominator of the form $\sqrt{a}$ $\qquad \dfrac{k}{\sqrt{a}}$
- A denominator of the form $a \pm \sqrt{b}$ $\qquad \dfrac{k}{a + \sqrt{b}}$
- A denominator of the form $\sqrt{a} \pm \sqrt{b}$ $\qquad \dfrac{k}{\sqrt{a} - \sqrt{b}}$

The first case is the simplest and just requires multiplying top and bottom by the surd on the bottom:

2.7.1 Example:

Ex 1 $$\frac{7}{\sqrt{3}} = \frac{7}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

Ex 2 $$\frac{3\sqrt{5}}{\sqrt{3}} = \frac{3\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{15}}{3} = \sqrt{15}$$

The second case has a denominator of the form $a \pm \sqrt{b}$, which requires you to multiplying top and bottom by $a \mp \sqrt{b}$. So if the denominator has the form $a + \sqrt{b}$, then multiply top and bottom by $a - \sqrt{b}$, which gives us a denominator of the form $a^2 - b$. The section on the differences of squares, above, will show why you do this. Obviously, if the denominator is $b - \sqrt{c}$ then multiply top and bottom by $b + \sqrt{c}$.

2.7.2 Example:

Ex 1 $$\frac{1}{3 - \sqrt{2}} = \frac{1}{3 - \sqrt{2}} \times \frac{3 + \sqrt{2}}{3 + \sqrt{2}} = \frac{3 + \sqrt{2}}{9 - 2} = \frac{3 + \sqrt{2}}{7}$$

Ex 2 $$\frac{2\sqrt{2}}{\sqrt{3} - \sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{3} - \sqrt{5}} \times \frac{\sqrt{3} + \sqrt{5}}{\sqrt{3} + \sqrt{5}} = \frac{2\sqrt{6} + 2\sqrt{10}}{3 - 5} = -\left(\sqrt{6} + \sqrt{10}\right)$$

The third case has a denominator of the form $\sqrt{a} \pm \sqrt{b}$, which requires you to multiplying top and bottom by $\sqrt{a} \mp \sqrt{b}$, which gives us a denominator of the form $a - b$.

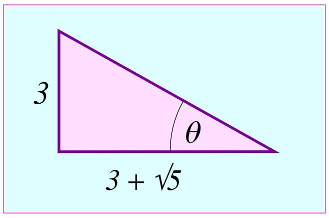
2.7.3 Example:

$$\frac{1}{\sqrt{3} - \sqrt{2}} = \frac{1}{\sqrt{3} - \sqrt{2}} \times \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} = \frac{\sqrt{3} + \sqrt{2}}{3 - 2} = \sqrt{3} + \sqrt{2}$$

2.8 Geometrical Applications

2.8.1 Example:

1 Find $\tan\theta$:



Solution:

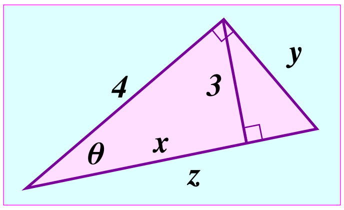
$$\tan\theta = \frac{3}{3+\sqrt{5}}$$

$$\tan\theta = \frac{3}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}}$$

$$\tan\theta = \frac{9-3\sqrt{5}}{9-5} = \frac{9-3\sqrt{5}}{4}$$

2 Find:

$x,\ \cos\theta,\ z,\ y$



Solution:

Find x $\quad 4^2 = x^2 + 3^2 \Rightarrow 16 = x^2 + 9$

$\therefore \quad x = \sqrt{7}$

Find Cos θ $\quad \cos\theta = \dfrac{\sqrt{7}}{4}$

Find z $\quad z^2 = 4^2 + y^2$

$$\cos\theta = \frac{4}{z} \qquad \therefore \frac{4}{z} = \frac{\sqrt{7}}{4}$$

$$z = \frac{16}{\sqrt{7}} = \frac{16\sqrt{7}}{7}$$

Find y $\quad y = \sqrt{\left(\dfrac{16\sqrt{7}}{7}\right)^2 - 16} = \sqrt{\dfrac{256}{7} - 16}$

$$y = \sqrt{\frac{256}{7} - \frac{112}{7}} = \sqrt{\frac{144}{7}} = \frac{12}{\sqrt{7}}$$

$$y = \frac{12\sqrt{7}}{7}$$

3 Express $(3-\sqrt{5})^2$ in the form of $a + b\sqrt{5}$

Solution:

$$(3-\sqrt{5})^2 = 9 - 3\sqrt{5} - 3\sqrt{5} + 5$$

$$= 14 - 6\sqrt{5}$$

$\therefore \quad a = 14, \qquad b = -6$

2.9 Topical Tip

Whenever an exam question asks for an **exact** answer, leave the answer as a surd. Don't evaluate with a calculator (which you can't have in C1:-)

2.10 The Difference of Two Squares

This is a favourite of examiners.

Note the LH & RH relationships — the difference of squares (LHS) always equals the sum times the difference (RHS):

$$a^2 - b^2 = (a + b)(a - b)$$

This will always result in an rational number.

A common trick exam question is to ask you to factorise something like: $(a^2 - 1)$.

2.10.1 Example:

1 Simplify $\left(\sqrt{5} + 2\right)\left(\sqrt{5} - 2\right)$

Solution:

$$\left(\sqrt{5} + 2\right)\left(\sqrt{5} - 2\right) = \left(\sqrt{5}\right)^2 - 2^2$$
$$= 5 - 4 = 1$$

2 A common trick question is to ask you to factorise $(a^2 - 1)$.

Solution:

$$\left(a^2 - 1\right) = \left(a^2 - 1^2\right) = (a + 1)(a - 1)$$

3 The difference of squares can be used to calculate numerical expressions such as:

Solution:

$$\left(25^2 - 15^2\right) = (25 + 15)(25 - 15)$$
$$= 40 \times 10 = 400$$

2.11 Heinous Howlers

Do not confuse yourself.

$\sqrt{7} \times \sqrt{7} \neq 49$ ✘ $\sqrt{7} \times \sqrt{7} = 7$ ✔

$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$ ✘

$(a + b)^2 \neq a^2 + b^2$ ✘

3 • C1 • Algebraic Fractions

3.1 Handling Algebra Questions

Two golden rules:

- If a polynomial is given e.g. a quadratic, FACTORISE IT
- If bracketed expressions are given e.g. $(x - 4)^2$ EXPAND THE BRACKETS

3.2 Simplifying Algebraic Fractions

The basic rules are:

- If more than one term in the numerator (top line): put it in brackets
- Repeat for the denominator (bottom line)
- Factorise the top line
- Factorise the bottom line
- Cancel any common factors outside the brackets and any common brackets

Remember:

- B — Brackets
- F — Factorise
- C — Cancel

3.2.1 Example:

1

$$\frac{x-3}{2x-6}$$

$$\frac{x-3}{2x-6} \overset{(B)}{\Rightarrow} \frac{(x-3)}{(2x-6)} \overset{(F)}{\Rightarrow} \frac{(x-3)}{2(x-3)} \overset{(C)}{\Rightarrow} \frac{\cancel{(x-3)}}{2\cancel{(x-3)}} = \frac{1}{2}$$

2

$$\frac{2x-3}{6x^2-x-12}$$

$$\frac{2x-3}{6x^2-x-12} \overset{(B)}{\Rightarrow} \frac{(2x-3)}{(6x^2-x-12)} \overset{(F)}{\Rightarrow} \frac{(2x-3)}{(2x-3)(3x+4)} \overset{(C)}{\Rightarrow} \frac{\cancel{(2x-3)}}{\cancel{(2x-3)}(3x+4)}$$

$$= \frac{1}{(3x+4)}$$

3

$$\frac{3x^2-8x+4}{6x^2-7x+2}$$

$$\frac{3x^2-8x+4}{6x^2-7x+2} \Rightarrow \frac{(3x^2-8x+4)}{(6x^2-7x+2)} \Rightarrow \frac{(x-2)(3x-2)}{(2x-1)(3x-2)} = \frac{(x-2)}{(2x-1)}$$

4

$$\frac{x-2}{2-x}$$

Watch out for the change of sign:

$$\frac{x-2}{2-x} \Rightarrow \frac{(x-2)}{(2-x)} \Rightarrow \frac{-(2-x)}{(2-x)} = -1$$

3.3 Adding & Subtracting Algebraic Fractions

The basic rules are the same as normal number fractions (remember 11+ exams???):

- Put terms in brackets for both top and bottom lines
- Factorise top & bottom lines, if necessary
- Find common denominator
- Put all fractions over the common denominator
- Add/subtract numerators
- Simplify

3.3.1 Example:

1

$$\frac{1}{x} - \frac{2}{3}$$

$$\frac{1}{x} - \frac{2}{3} \Rightarrow \frac{3}{3x} - \frac{2x}{3x} = \frac{3 - 2x}{3x}$$

2

$$\frac{3}{x + 2} - \frac{6}{2x - 1}$$

Solution:

$$\begin{aligned}\frac{3}{(x + 2)} - \frac{6}{(2x - 1)} &= \frac{3(2x - 1)}{(x + 2)(2x - 1)} - \frac{6(x + 2)}{(x + 2)(2x - 1)} \\ &= \frac{3(2x - 1) - 6(x + 2)}{(x + 2)(2x - 1)} \\ &= \frac{6x - 3 - 6x + 12}{(x + 2)(2x - 1)} \\ &= \frac{-15}{(x + 2)(2x - 1)}\end{aligned}$$

3

$$\frac{31x - 8}{2x^2 + 3x - 2} - \frac{14}{x + 2}$$

Solution:

$$\begin{aligned}\frac{(31x - 8)}{(2x^2 + 3x - 2)} - \frac{14}{(x + 2)} &= \frac{(31x - 8)}{(x + 2)(2x - 1)} - \frac{14}{(x + 2)} \\ &= \frac{(31x - 8)}{(x + 2)(2x - 1)} - \frac{14(2x - 1)}{(x + 2)(2x - 1)} \\ &= \frac{(31x - 8) - 14(2x - 1)}{(x + 2)(2x - 1)} \\ &= \frac{31x - 8 - 28x + 14}{(x + 2)(2x - 1)} \\ &= \frac{(3x + 6)}{(x + 2)(2x - 1)} \\ &= \frac{3(x + 2)}{(x + 2)(2x - 1)} \\ &= \frac{3}{2x - 1}\end{aligned}$$

3.4 Multiplying & Dividing Algebraic Fractions

Basic rules are:

- Multiplication:
  - Simplify if possible
  - Multiply out: $\dfrac{top \times top}{bottom \times bottom}$
  - Simplify
- Division
  - Turn second fraction upside down: $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c}$
  - Follow multiplication rules above

3.4.1 Example:

1

$$\frac{2}{x} \times \frac{x^2 - 2x}{x - 2}$$

Solution:

$$\frac{2}{x} \times \frac{x^2 - 2x}{x - 2} = \frac{2}{\cancel{x}} \times \frac{\cancel{x}\cancel{(x-2)}}{\cancel{(x-2)}} = 2$$

2

$$\frac{x - 2}{x^2 - 4x + 3} \div \frac{x}{2x^2 - 7x + 3}$$

Solution:

$$\frac{x - 2}{x^2 - 4x + 3} \div \frac{x}{2x^2 - 7x + 3} = \frac{(x - 2)}{(x^2 - 4x + 3)} \times \frac{(2x^2 - 7x + 3)}{x}$$

$$= \frac{(x - 2)}{(x - 1)\cancel{(x-3)}} \times \frac{\cancel{(x-3)}(2x - 1)}{x}$$

$$= \frac{(x - 2)(2x - 1)}{x(x - 1)}$$

3

Express $\dfrac{x^8 - 1}{x^3}$ in the form of $x^p - x^q$

Solution:

$$\frac{x^8 - 1}{x^3} = x^5 - x^{-3}$$

4

Show that $5\left(\dfrac{n}{2}(n - 1) + 3n\right)$ is the same as $\dfrac{5n(n + 5)}{2}$

Solution:

$$5\left(\frac{n}{2}(n - 1) + 3n\right) = \frac{5n}{2}(n - 1) + 15n$$

$$= \frac{5n(n - 1) + 30n}{2}$$

$$= \frac{5n^2 - 5n + 30n}{2} = \frac{5n^2 + 25n}{2}$$

$$= \frac{5n(n + 5)}{2}$$

3.5 Further Examples

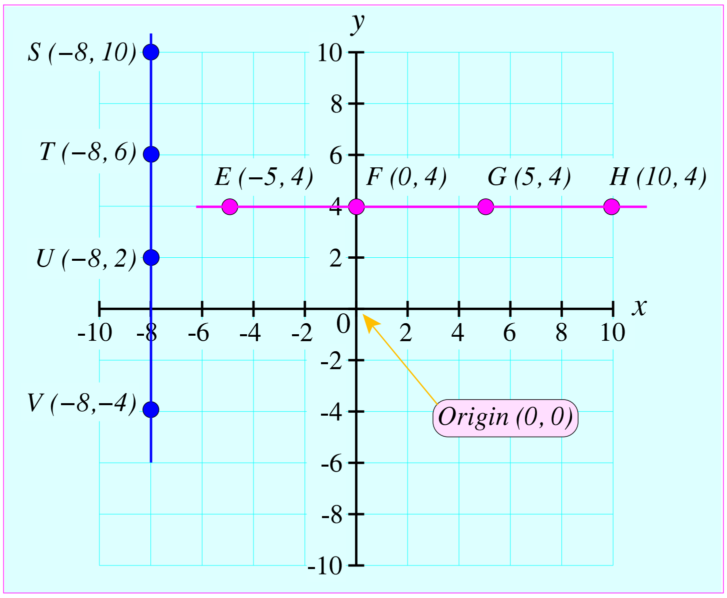
4 • C1 • Straight Line Graphs

Co-ordinate geometry is the link between algebra and geometry. The co-ordinate system allows algebraic expressions to be plotted on a graph and shown in pictorial form. Algebraic expressions which plot as straight lines are called **linear equations**.

A line is the joining of two co-ordinates, thus creating a series of additional co-ordinates between the original two points.

4.1 Plotting Horizontal & Vertical Lines

The simplest lines to plot are horizontal & vertical lines.



Notice that the horizontal line, with points E to H, all have the same y coordinate of 4.

The equation of the line is said to be:

$$y = 4$$

or, in general: $y = a$ (where a = a number)

Similarly the vertical line, with points S to V, all have the same x coordinate of −8.

The equation of the line is said to be:

$$x = -8$$

or, in general: $x = b$ (where b = a number)

4.2 Plotting Diagonal Lines

Take the equations:

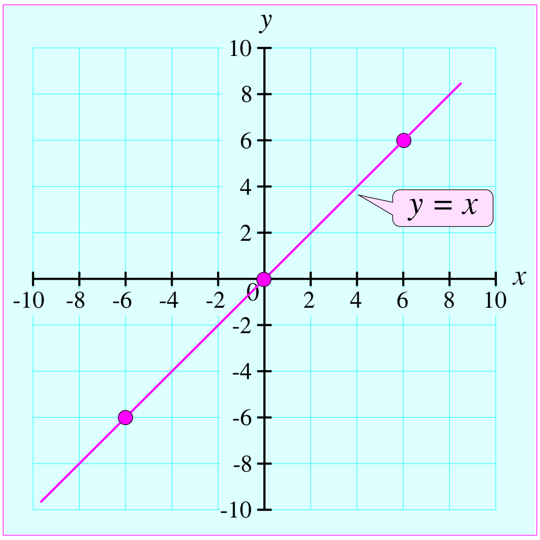
$$y = x$$

$$y = -x$$

In the first case, y is always equal to the value of x.
In the second case, y is always equal to the value of $-x$.

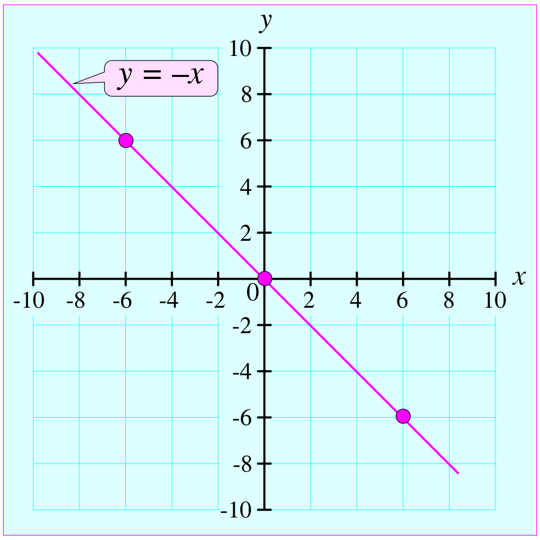
For each equation, a simple table of values will show this. The results can be plotted as shown:

$y = x$			
x	− 6	0	6
y	**− 6**	**0**	**6**
Co-ords	**(−6, −6)**	**(0, 0)**	**(6, 6)**



In this case y has the same value as x, and produces a diagonal line which slopes upwards.

$y = -x$			
x	− 6	0	6
y	**6**	**0**	**− 6**
Co-ords	**(−6, 6)**	**(0, 0)**	**(6, −6)**



In this case, y has the same value as $-x$, and produces another diagonal line, but sloping downwards.

Notice also that both lines pass through the origin.

4.3 The Equation of a Straight Line

4.3.1 The Equation

So far we have seen 4 special cases of the straight line.

$$x = a \quad \text{where } a \text{ is a number,}$$

$$y = b \quad \text{where } b \text{ is a number,}$$

$$y = x$$

$$y = -x$$

In fact, these are special cases of the more general equation of a straight line, which, by convention, is expressed as:

$$y = mx + c \quad \text{where } m \text{ \& } c \text{ are constants.}$$

4.3.2 Solving the equation

Whereas an equation such as $2y = 10$ has only one solution (i.e. $y = 5$), an equation with two variables (x and y), must have a pair of values for a solution. These pairs can be used as co-ordinates and plotted. A line has an infinite number of pairs as solutions.

4.3.3 Rearranging the equation

Any equation with two variables (x and y), will produce a straight line, but it may not be conveniently written in the ideal form of $y = mx + c$.

4.3.3.1 Example:

Rearrange the equation $4y - 12x - 8 = 0$ to the standard form for a straight line.

Solution:

$$4y - 12x - 8 = 0 \quad \text{A non standard straight line equation}$$

$$4y = 12x + 8 \quad \text{Transpose the terms } 12x \text{ and } 8$$

$$y = 3x + 2 \quad \text{Divide by 4, giving the standard equation.}$$

4.3.4 Interpreting the Straight Line Equation

When thinking about plotting equations, think of y as being the output of a function machine (the y co-ordinate), whilst x is the input (the x coordinate).

For example, the straight line $y = 3x + 2$. The y co-ordinate is just the x coordinate multiplied by 3 with 2 added on. Plotting all the values of x and y will give our straight line.

4.4 Plotting Any Straight Line on a Graph

Take the simple equation:

$$y = 2x + 1$$

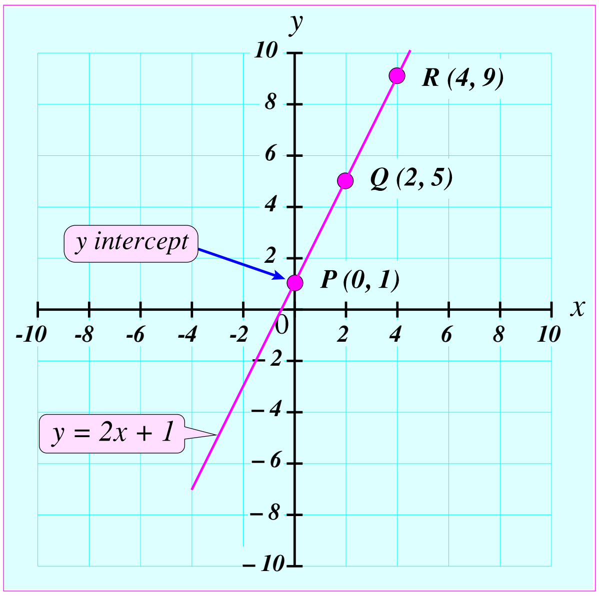
In order to plot this equation, y has to be calculated for various values of x, which can then be used as co-ordinates on the graph. Of course, only two points are required to plot a straight line but a minimum of three points and preferably 4 should be used, in order to spot any errors. If one point is not in line with the others then you know there is a mistake.

Draw a table of values, choose some easy values of x (like 0, 2, 4), then calculate y:

$y = 2x + 1$			
x	0	2	4
y	**1**	**5**	**9**
Co-ords	**(0, 1)**	**(2, 5)**	**(4, 9)**

Notice how the values of x and y both increase in a linear sequence. As x increases by 2, y increases by 4. The two variables are connected by the rule: 'The y coordinate is found by multiplying the x coordinate by 2 and adding 1'.

Plot the co-ordinates as shown:



Notice that the line cuts the y-axis at $y = 1$.

4.5 Properties of a Straight Line

From the previous diagram, note that the straight line:

- is sloping—we call this a gradient,
- and crosses the y axis at a certain point, we call the y intercept.

4.5.1 Gradient or Slope

Gradient is a measure of how steep the slope is rising or falling. It is the ratio of the vertical rise over the horizontal distance, measured between two points on the straight line.

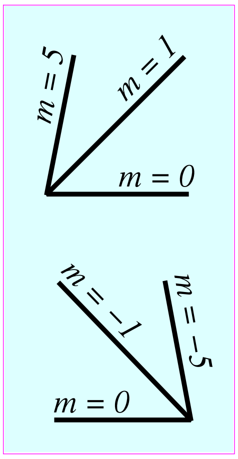
Remember 'rise over run'.

By convention, the gradient is usually assigned the letter m (after the French word 'monter', meaning 'to climb'). The gradient can be either positive or negative.

$$\text{Slope or Gradient},\ m = \frac{\text{Vertical rise}}{\text{Horizontal run}}$$

$$m = \frac{\text{Change in } y \text{ values}}{\text{Change in } x \text{ values}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) are the co-ordinates of the first point and (x_2, y_2) are the co-ordinates of the second point.



The larger the number m, the steeper the line.
Imagine walking left to right, the slope is uphill and is said to be positive.

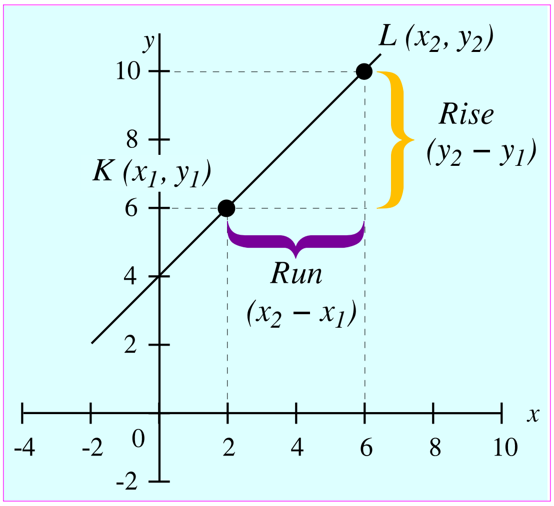
A horizontal line has a slope of zero, $m = 0$.

Walking (or falling) downhill, left to right, the slope is said to be negative.

The slope of a vertical line is not determined as the sum would involve division by zero, or it could be regarded as infinite.

4.5.2 Positive Gradients

A line in which both the x and y values increase at the same time is said to be positive, and has a positive gradient. In other words, as we move from left to right along the x-axis, y increases. We say this is a positive slope or gradient.



Positive Gradient

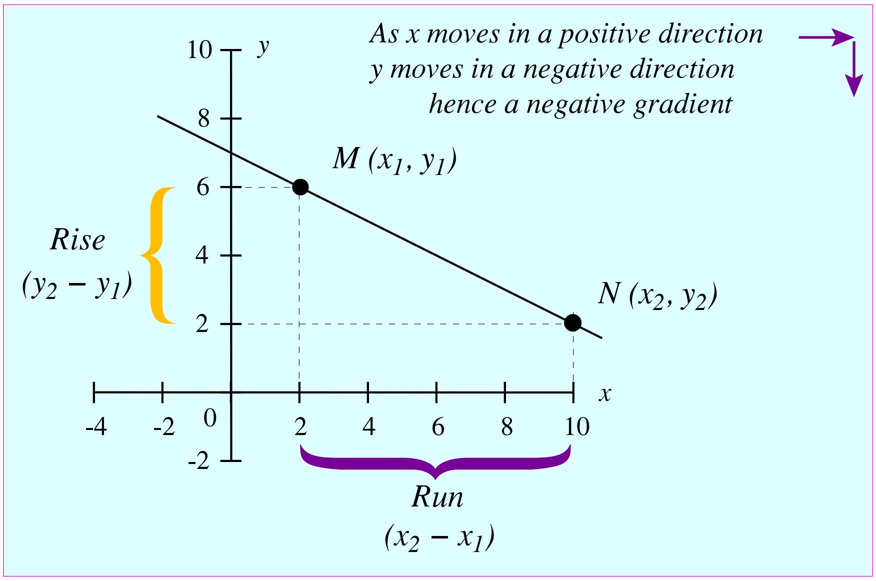
4.5.2.1 Example: Positive Slope

In the above diagram, point K has co-ordinates (2, 6) and point L (6, 10).

$$\begin{aligned} \text{Gradient},\ m &= \frac{rise}{run} = \frac{\text{Change in } y \text{ values}}{\text{Change in } x \text{ values}} \\ &= \frac{y \text{ coord of } L - y \text{ coord } K}{x \text{ coord of } L - x \text{ coord } K} = \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{10 - 6}{6 - 2} = \frac{4}{4} = 1 \\ m &= 1 \end{aligned}$$

4.5.3 Negative Gradients

As we move from left to right along the x-axis, y decreases. We say this is a negative slope or gradient.



Negative Gradient

4.5.3.1 Example: Negative Slope

In the above diagram, point M has co-ordinates (2, 6), labelled $(x_1,\ y_1)$, and point N (10, 2), labelled $(x_2,\ y_2)$. Notice that in this case subtracting the y co-ordinates produces a negative number.

$$\textit{Gradient},\ m = \frac{\textit{rise}}{\textit{run}} = \frac{\textit{Change in y values}}{\textit{Change in x values}}$$

$$= \frac{y \text{ coord of } N - y \text{ coord } M}{x \text{ coord of } N - x \text{ coord } M} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2 - 6}{10 - 2} = \frac{-4}{8} = -0.5$$

$$m = -0.5$$

If the order of the co-ordinates are swapped round, so that point N (10, 2) is the first point $(x_1,\ y_1)$, and M (2, 6) the second $(x_2,\ y_2)$, then the gradient is calculated in a similar manner:

$$\textit{Gradient},\ m = \frac{y \text{ coord of } M - y \text{ coord } N}{x \text{ coord of } M - x \text{ coord } N} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{6 - 2}{2 - 10} = \frac{4}{-8} = -0.5$$

$$m = -0.5$$

It's a relief to find the answers are the same!!!!!

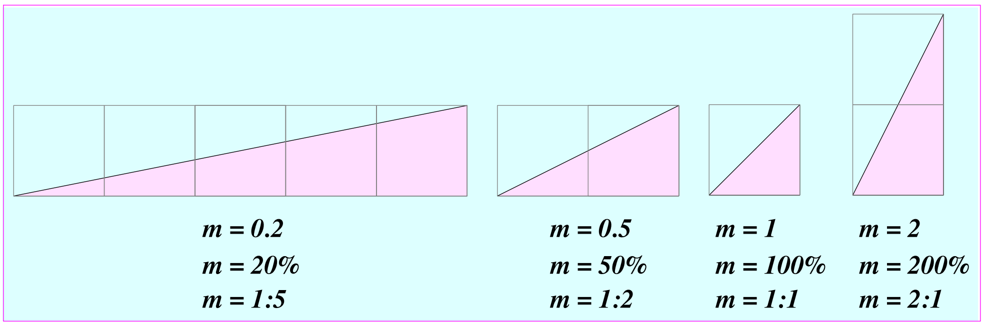
4.5.4 Expressing Gradients

So far, a gradient has been expressed as a number, and the steeper the gradient the bigger the number. Gradients can also expressed as a ratio or a percentage.

A gradient of 0.2 is often quoted as "1 in 5", meaning it rises (or falls) 1 metre in every 5 metres distance.

This can also be expressed as a percentage value, thus: $0.2 \times 100 = 20\%$

This is summarised below:



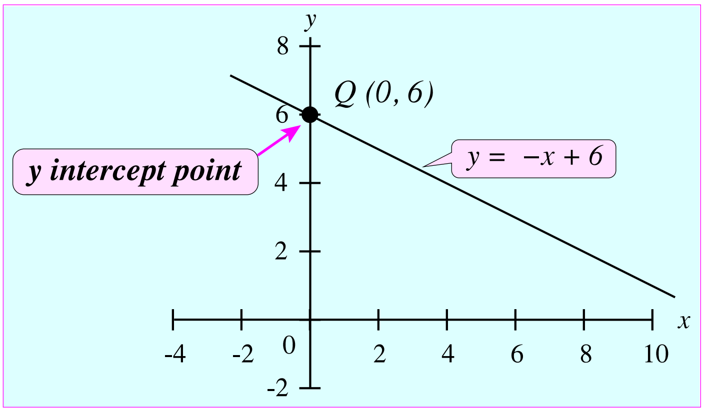
4.5.5 Intercept point of the y axis

In the diagram below, note how the straight line crosses the y axis at some point. The y intercept point always has the x coordinate of zero. (Point Q has a coordinate of (0, 6)).

$$y = mx + c$$

The y intercept point can be found if $x = 0$, then:

$$y = c$$

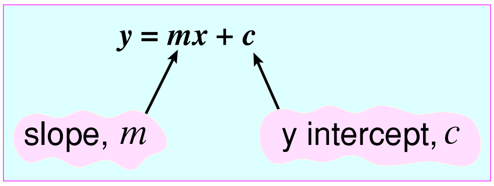


Intercept point of the y axis

4.6 Decoding the Straight Line Equation

We can now see that the equation of a line can be rewritten as:

$y = (slope)x + (y\,intercept)$

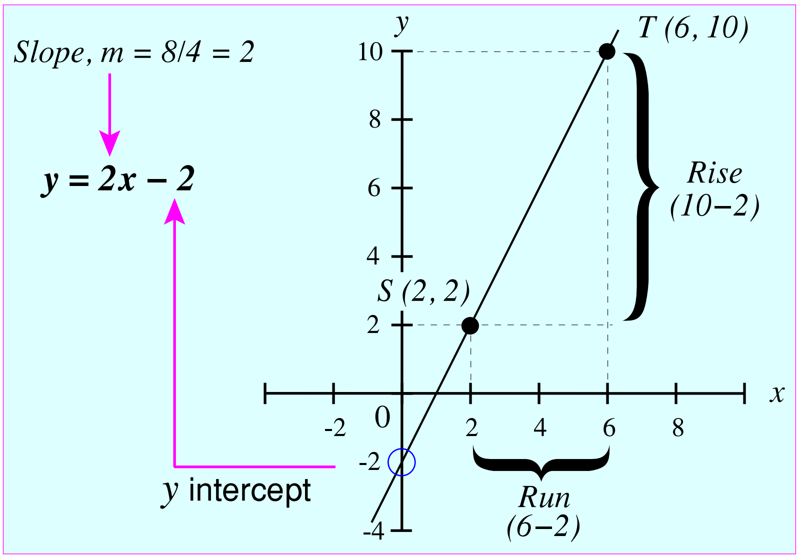


Notice that:

- If $m = 0$; then $y = c$. — A horizontal line with y intercept c.
- If $m = 1$; then $y = x + c$. — A 45° diagonal line with y intercept c.
- If the line is vertical then the horizontal run is zero. This means that the gradient cannot be determined as division by zero is not allowed, or indeterminate. Try it on a calculator!
  If you consider the 'run' as being very small (say 0·00001) then it is easy to see that m would be very large and so m could be regarded as being infinite.

$$m = \frac{rise}{run} = \frac{rise}{0} = \infty$$

The relationship between gradient and the constant c can be seen below. The points S and T are convenient points chosen to measure the rise and run of the graph.



Decoding the Straight Line Equation

4.7 Plotting a Straight Line Directly from the Standard Form

Once you understand the standard form of $y = mx + c$ then it is easy to plot the straight line directly on the graph.

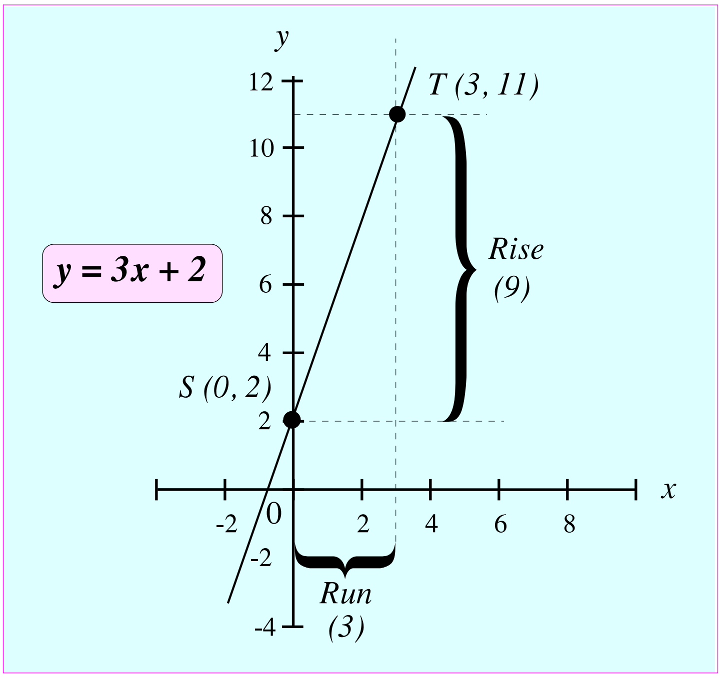
4.7.1 Example:

Plot the equation $y = 3x + 2$.

Solution:

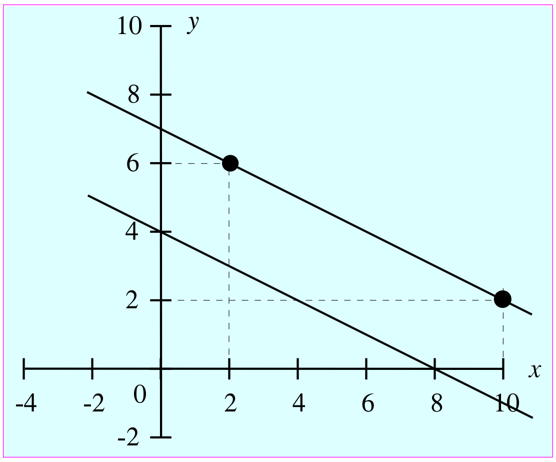
From the equation the gradient is 3 and the y intercept is 2.

The gradient means that for every unit of x, y increases by 3. To improve the accuracy when drawing the line, we can draw the gradient over (say) 3 units of x. In which case y increases by 9 etc.



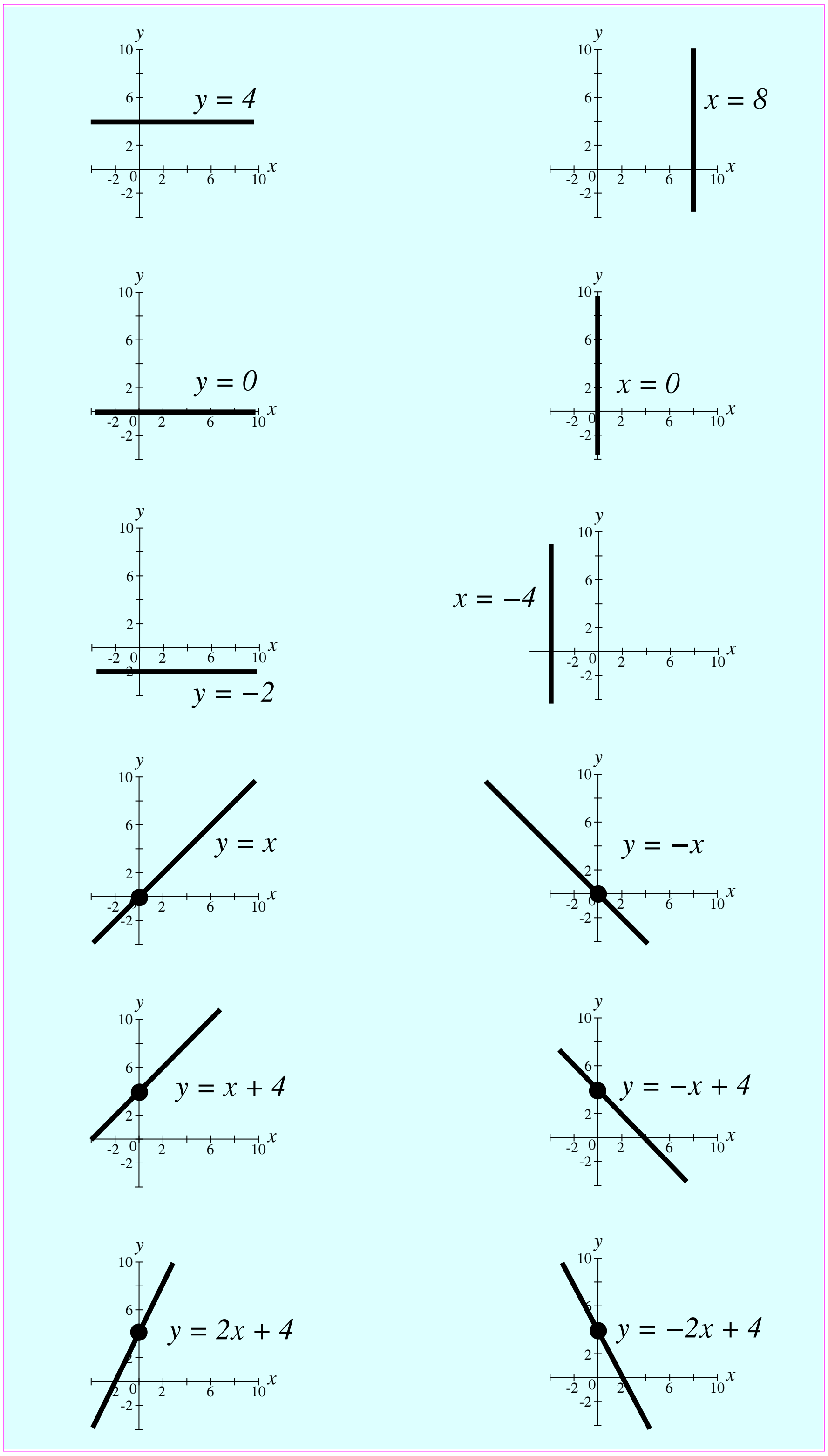
4.8 Parallel Lines

It is worth pointing out the parallel lines have the same gradient - always.



Parallel lines have the same gradient - always

4.9 Straight Line Summary



4.10 Topical Tips

- The quick way to plot a straight line is to calculate the point where the line crosses the x and y axis (i.e. find x if $y = 0$ and find y if $x = 0$), and then join the two points. However, when plotting graphs it is always best to use a minimum of 3 points, preferably 4. Errors will then stand out, as all lines should be dead straight.
- The slope or gradient of a line will only look correct if the x & y scales are the same.
- Always use the x and y axis values to calculate the slope. Do not rely on the graph paper grid alone to find the slope, as this is only correct if the x & y scales are the same.
- If the given equation is $y = 6 - 3x$, take care to write the gradient down as -3 and not 6. It is the coefficient of x that gives the gradient.
- The equation of the x-axis is $y = 0$

  The equation of the y-axis is $x = 0$

  Don't get confused.

5 • C1 • Geometry of a Straight Line

5.1 General Equations of a Straight Line

There are three general equations that may be used. Sometimes an exam question may ask for the answer to be written in a certain way, e.g. $ax + by = k$.

5.1.1 Version 1

$$y = mx + c$$

where m = gradient, and the graph cuts the y-axis at c.

5.1.2 Version 2

$$y - y_1 = m(x - x_1)$$

where m = gradient, and (x_1, y_1) are the co-ordinates of a given point on the line.

Example Find the equation of a line with gradient 2 which passes through the point (1, 7)

$$y - 7 = 2(x - 1) \Rightarrow y - 7 = 2x - 2$$

$$y = 2x + 5$$

5.1.3 Version 3

$$ax + by = k$$

Note that you cannot read the gradient and the y-intercept from this equation directly, but they can be calculated using:

$$y = -\frac{a}{b}x + \frac{k}{b}$$

5.1.3.1 Example:

1 Find the gradient of $3x - 4y - 2 = 0$

$$3x - 2 = 4y$$

$$y = \frac{3}{4}x - \frac{2}{4}$$

$$\text{Gradient} = \frac{3}{4}$$

2 One side of a parallelogram is on the line $2x + 3y + 5 = 0$ and point P (3, 2) is one vertex of the parallelogram. Find the equation of the other side in the form $ax + by + k = 0$.

Gradient of given line: $3y = -2x - 5$

$$y = -\frac{2}{3}x - 5$$

$$\text{Gradient} = -\frac{2}{3}$$

Equation of line through P $y - 2 = -\frac{2}{3}(x - 3)$

$$y = -\frac{2}{3} + 4 \quad \Rightarrow \quad 2x + 3y - 12 = 0$$

5.2 Distance Between Two Points on a Line

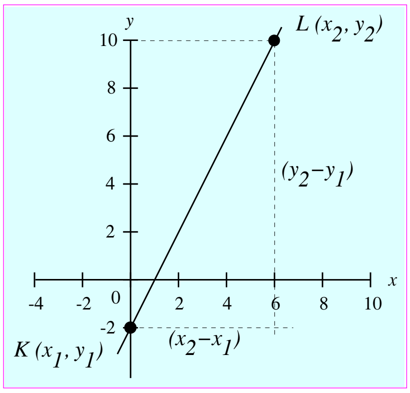
Finding the distance between two points on a straight line uses Pythagoras.

$$\textit{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It should be noted that any distance found will be the +ve square root.

5.2.1 Example:

Find the length of the line segment KL.



$$\textit{Distance} = \sqrt{(6 - 0)^2 + (10 - (-2))^2}$$

$$= \sqrt{6^2 + 12^2} = \sqrt{36 + 144} = \sqrt{180}$$

$$= 6\sqrt{5}$$

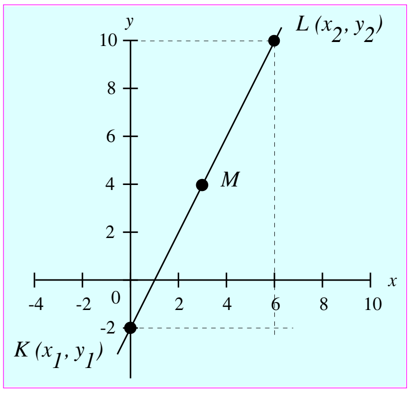
5.3 Mid Point of a Line Segment

The mid point is just the average of the given co-ordinates.

$$\text{Mid point co-ordinates} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

5.3.1 Example:

Find the mid point co-ordinate M.



$$M = \left(\frac{0 + 6}{2}, \frac{-2 + 10}{2}\right) = (3, 4)$$

5.4 Gradient of a Straight Line

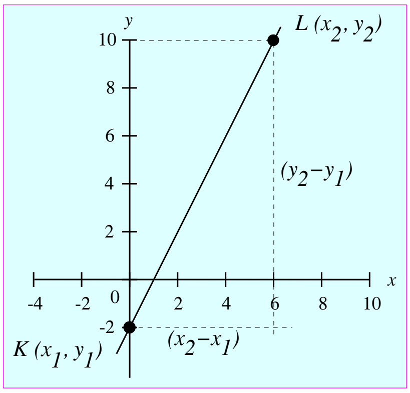
Gradient is the rise over the run. Note that a vertical line can be said to have a gradient of ∞.

$$Gradient,\ m = \frac{Rise}{Run} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is equivalent to the amount of vertical rise for every 1 unit of horizontal run.

5.4.1 Example:

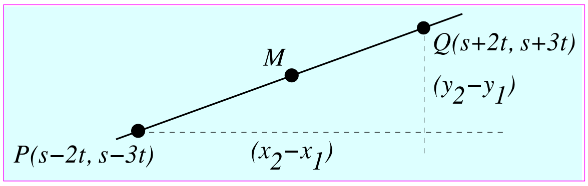
1 Find the gradient of line segment KL.



$$Gradient = \frac{Rise}{Run} = \frac{10 - (-2)}{6 - 0} = \frac{12}{6} = 2$$

2 The ends of a line segment are $P\,(s - 2t,\ s - 3t)$ and $Q\,(s + 2t,\ s + 3t)$.
Find the length and gradient of the line segment, and the co-ordinates of the mid point.

Solution:



$$x_2 - x_1 = s + 2t - (s - 2t) = 4t$$

$$y_2 - y_1 = s + 3t - (s - 3t) = 6t$$

$$Distance\ PQ = \sqrt{(4t)^2 + (6t)^2} = \sqrt{16t^2 + 36t^2}$$

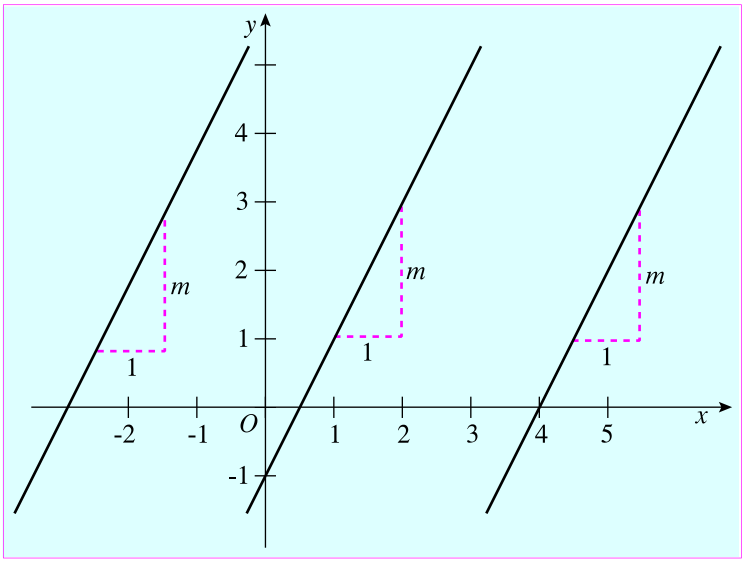
$$= t\sqrt{52}$$

$$Gradient = \frac{Rise}{Run} = \frac{6t}{4t} = 1{\cdot}5$$

$$\text{Mid point } M = \left(\frac{4t}{2}, \frac{6t}{2}\right) = (2t,\ 3t)$$

5.5 Parallel Lines

The important point about parallel lines is that they all have the same gradient.



As seen earlier, one way of expressing a straight line is:

$$ax + by = k$$

The gradient only depends on the ratio of a and b.

$$y = -\frac{a}{b}x - \frac{k}{b}$$

Hence, for any given values of a and b, say a_1 and b_1, then all the lines…

$$a_1x + b_1y = k_1$$

$$a_1x + b_1y = k_2$$

$$a_1x + b_1y = k_3 \qquad etc$$

…are parallel.

5.5.1 Example:

1 Find the equation of a straight line, parallel to $2x + 3y = 6$, and which passes through the point (2, 8).

Solution:

Since an equation of the form $2x + 3y = k$ is parallel to $2x + 3y = 6$, the problem reduces to one of finding the value of k, when x and y take on the values of the given point (2, 8).

$$2 \times 2 + 3 \times 8 = k$$

$$4 + 24 = k$$

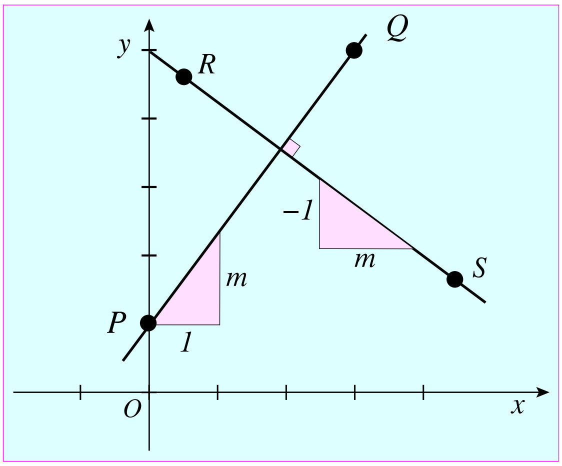
$$\therefore \quad k = 28$$

Equation of the required line is: $\quad 2x + 3y = 28$

5.6 Perpendicular Lines

Lines perpendicular to each other have their gradients linked by the equation:

$$m_1 m_2 = -1$$



From the diagram:

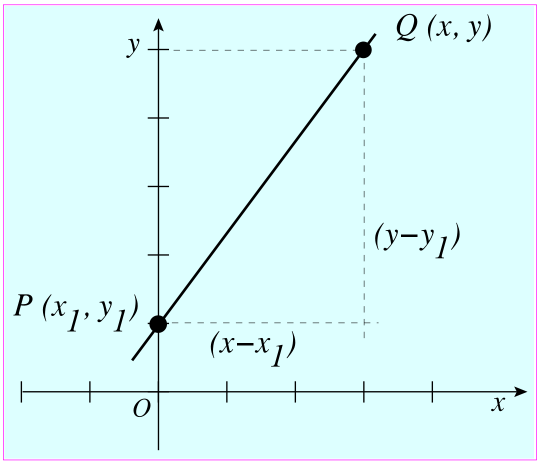
Gradient of PQ $\quad m_1 = m$

Gradient of RS $\quad m_2 = \frac{-1}{m}$

$\therefore \quad m_1 m_2 = m \times \frac{-1}{m} = -1$

5.7 Finding the Equation of a Line

A very common question is to find the equation of a straight line, be it a tangent or a normal to a curve.



From the definition of the gradient we can derive the equation of a line that passes through a point $P(x_1, y_1)$:

$$m = \frac{Rise}{Run} = \frac{y - y_1}{x - x_1}$$

$$\therefore \quad y - y_1 = m(x - x_1)$$

This is the best equation to use for this type of question as it is more direct than using $y = mx + c$.

5.7.1 Example:

1 Find the equation of the line which is perpendicular to $3x - 4y + 8 = 0$ and which passes though the point P (7, 10).

Solution:

$$3x - 4y + 8 = 0$$

$$y = \frac{3x + 8}{4} = \frac{3x}{4} + 2$$

$$\therefore \textit{Gradient} = \frac{3}{4}$$

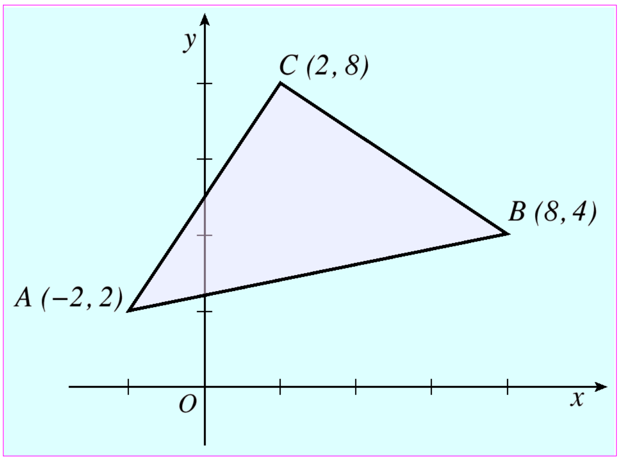
$$\text{Gradient of perpendicular line} = -\frac{4}{3}$$

$$\text{Equation of line thro' } P \Rightarrow \quad y - 10 = -\frac{4}{3}(x - 7)$$

$$3y - 30 = -4x - 28$$

$$4x + 3y - 58 = 0$$

2 Prove that the triangle ABC is a right angled triangle. The co-ordinates of the triangle are given in the diagram.



Solution:

To prove a right angle we need to examine the gradients of each side to see if they fit the formula for perpendicular lines.

$$\text{Gradient of } AB = \frac{4 - 2}{8 - (-2)} = \frac{2}{10} = \frac{1}{5}$$

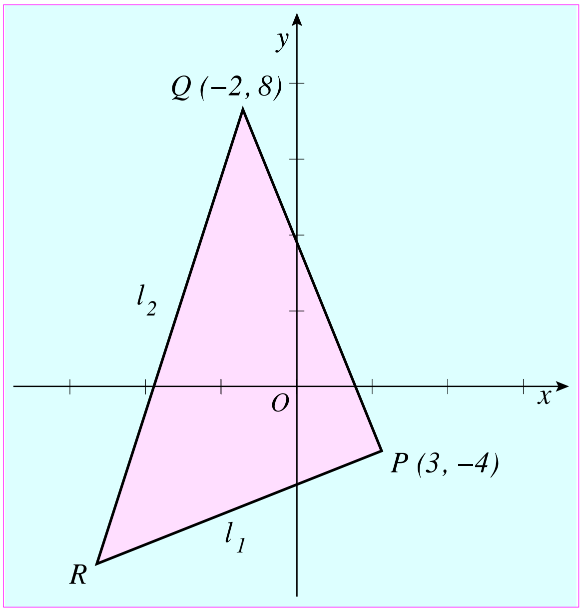
$$\text{Gradient of } BC = \frac{8 - 4}{2 - 8} = \frac{4}{-6} = -\frac{2}{3}$$

$$\text{Gradient of } AC = \frac{8 - 2}{2 - (-2)} = \frac{6}{4} = \frac{3}{2}$$

$$\text{Test for perpendicularity:} \quad m_{BC} \times m_{AC} = -\frac{2}{3} \times \frac{3}{2} = -1$$

Sides AC & BC are perpendicular, therefore it is a right angled triangle.

3 A line l_1 $12y - 5x + 63 = 0$ passes though the point P (3, −4),
and the line l_2 $7y - 12x - 90 = 0$ passes though the point Q (−2, 8).
Find the co-ordinates of the intersection of the two lines, point R, and hence or otherwise show that the triangle PQR is a right angled isosceles triangle.



Solution:

To find the co-ordinates of the intersection, set up a simultaneous equation:

$$12y - 5x = -63 \qquad (1)$$
$$7y - 12x = 90 \qquad (2)$$

$$\times 7 \qquad 84y - 5x = -441 \qquad (3)$$
$$\times 5 \qquad 84y - 204x = 1080 \qquad (4)$$

$$169x = -1521$$
$$x = -9$$

Substitute into (1) $\qquad 12y + 45 = -63 \quad \Rightarrow \quad y = -9$

$\therefore$ Co-ordinates of R (−9, −9)

To test if two lines are perpendicular to each other, find the gradients of each line.

$$\text{Gradient of } \quad l_1 = \frac{y_Q - y_R}{x_Q - x_R} = \frac{8 - (-9)}{-2 - (-9)} = \frac{17}{7}$$

$$\text{Gradient of } \quad l_2 = \frac{y_P - y_R}{x_P - x_R} = \frac{-4 - (-9)}{3 - (-9)} = \frac{5}{12}$$

$$\text{Gradient of } \quad PQ = \frac{y_P - y_Q}{x_P - x_Q} = \frac{-4 - 8}{3 - (-2)} = -\frac{12}{5}$$

From this, note that the gradients of $l_2 \times PQ = -1$

Therefore, the triangle is a right angled triangle.

To test for an isosceles triangle, find the lengths of l_1 and PQ.

Length of $l_1 = \sqrt{(x_R - x_P)^2 + (y_R - y_P)^2}$

Length of $l_1 = \sqrt{(-9 - 3)^2 + (-9 - (-4))^2} = \sqrt{12^2 + 5^2} = 13$

Length of $PQ = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$

Length of $PQ = \sqrt{(-2 - 3)^2 + (8 - (-4))^2} = \sqrt{5^2 + 12^2} = 13$

Therefore, the triangle is a right angled isosceles triangle.

5.8 Heinous Howlers

Always use the x and y axis values to calculate the slope. Do not rely on the graph paper grid alone to find the slope, as this is only correct if the x & y scales are the same.

6 • C1 • The Quadratic Function

6.1 Intro to Polynomials

A Polynomial expression has the form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where a_0, a_1, $a_2 \ldots$ a_n are the terms coefficient, and n is a positive integer. Negative powers are not allowed in a polynomial. The variable shown here is x, but it can be any other convenient letter.

The degree, or order, of the polynomial is given by the highest power of the variable.

In general, multiplying two linear expressions will give a second degree polynomial (a quadratic), and multiplying a linear expression with a quadratic will give a third degree polynomial (a cubic).

A polynomial can be ‘solved’ by setting the expression to zero. This is the same as asking ‘what are the values of x when the curve crosses the x-axis’. The number of possible solutions or roots, matches the order of the polynomial. A cubic function will have up to 3 roots, whilst a quadratic has up to 2 roots.

Think of this as solving a simultaneous equation of (say): $y = ax^2 + bx + c$ & $y = 0$

6.2 The Quadratic Function

A quadratic function is a second order polynomial with the general form:

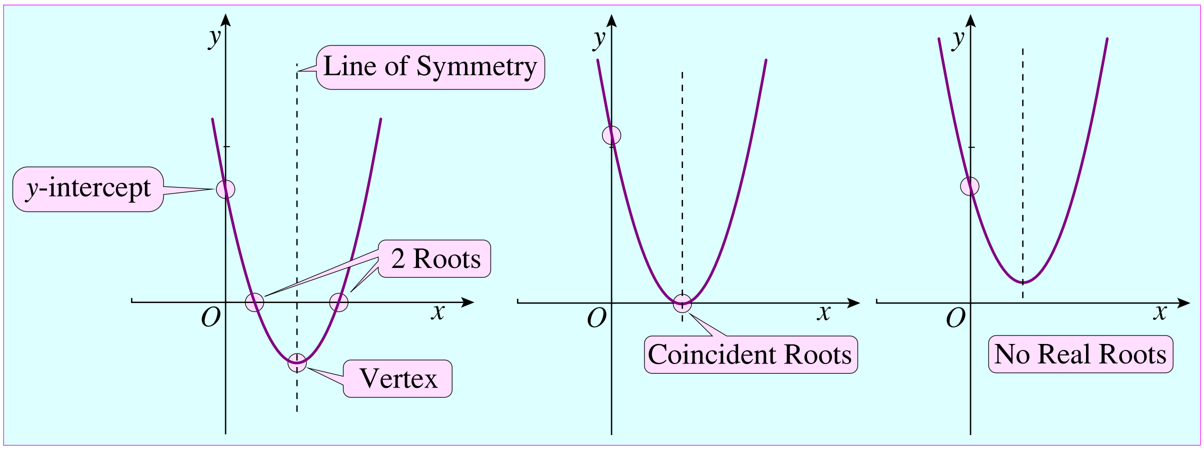
$$ax^2 + bx + c = 0 \qquad a \neq 0$$

When plotted, a parabolic curve is produced that is useful in engineering and physics. e.g. footballs in motion follow a parabolic curve very closely, and designs for headlamp reflectors are also parabolic in shape.

A quadratic curve is symmetrical about a line of symmetry which passes through the vertex of the curve (the minimum or maximum point of the curve).

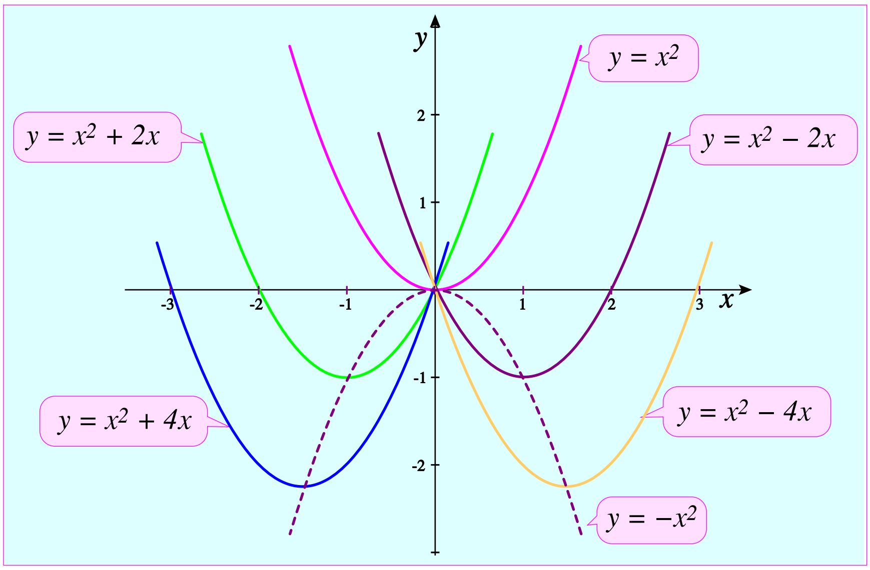
A quadratic function has up to two solutions or roots which may be:

- Two distinct real roots
- Two equal roots, (coincident roots)
- No real roots, (actually there are roots, but they involve imaginary or complex numbers which is not part of C1)



Quadratic Features

It is interesting to note the shape and way that a family of curves relates to each other.



Family of Quadratic Curves

6.3 Quadratic Types

There are three general form of quadratic function:

- Standard form: $ax^2 + bx + c = 0$
- Factored form: $(x + s)(x + t) = 0$
- Square or Vertex form: $a(x + p)^2 + q = 0$

From the standard form, and assuming that $a = 1$, there are three cases to deal with:

- $b = 0$ Hence: $x^2 - c = 0$
  $x^2 = c \quad \Rightarrow \quad x = \pm\sqrt{c}$
- $c = 0$ Hence: $x^2 + bx = 0$
  $x(x + b) = 0 \quad \Rightarrow \quad x = 0$ or $x = -b$
- $b \neq 0, \ c \neq 0$ Hence: $x^2 + bx + c = 0$
  $(x + s)(x + t) = 0 \quad \Rightarrow \quad x = -s$ or $x = -t$

  Note that: $(s + t) = b; \quad st = c$
  c is +ve when s & t have the same sign.
  c is −ve when s & t have opposite signs.

6.4 Quadratic Syllabus Requirements

You need to be able to:

- Factorise them
- Solve them by:
  - Factorising
  - Completing the square
  - Using the quadratic formula
- Sketch them – either by completing the square, finding the factors, or knowing the relationship between the equation and its various features.
- Understand the significance of the discriminant
- Recognise that some complex looking equations can be solved by reduction to a standard quadratic.

7 • C1 • Factorising Quadratics

7.1 Methods for Factorising

Factorising is the opposite of expanding the brackets of an expression.

The key is to recognise the different sorts of expressions that might be presented. Most are listed below:

- Expressions with a common factor: *e.g.* $2x^2 + 6x + 8 = 2(x^2 + 3x + 4)$
- Expressions of the form: $(u + v)^2 = k$
- Difference of two squares: $u^2 - v^2$
- Perfect square: (see completing the square below)
- Quadratic factorisation, type: $x^2 + bx + c$ $\quad a = 1$
- Quadratic factorisation, type: $ax^2 + bx + c$ $\quad a > 1$
- Completing the square, (see separate section)
- Quadratic formula, (see separate section)

Some other key pointers are:

- Factorisation is made easier when the coefficients a & c are prime numbers
- If $f(1) = 0$ then $(x - 1)$ is a factor, i.e. $x = 1$ if all the coefficients add up to 0
- A quadratic will only factorise if $b^2 - 4ac$ is a perfect square (see section on discriminants).

7.2 Zero Factor Property

Recall that solving any quadratic is based on the Zero Factor Property which says that if the product of two (or more) variables is zero, then each variable can take the value of zero, thus:

$$\text{If } uv = 0 \quad \text{then} \quad u = 0 \ \textbf{ OR } \ v = 0$$

which is why we go to so much trouble to factorise polynomials.

7.3 Expressions with a Common Factor

Expressions with the form:

$$ax^2 + bx = x(ax + b)$$

Always remove any common factors before factorising a polynomial.

7.3.1 Example:

1

$$2x^2 + 16x + 24 \Rightarrow 2(x^2 + 8x + 12)$$
$$= 2(x + 2)(x + 6)$$

2 Solve $6x^2 - 2x = 0$

Solution

$$6x^2 - 2x = 0$$
$$2x(3x - 1) = 0$$
$$2x = 0 \quad \Rightarrow \quad x = 0$$
$$3x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{3}$$

7.4 Expressions of the form $(u + v)^2 = k$

Expressions with the form $(u \pm v)^2 = k$ can be solved without factorisation simply by taking roots each side and solving for x.

$$(x + v)^2 = k$$
$$(x + v) = \pm\sqrt{k}$$
$$x = -v \pm \sqrt{k}$$

7.4.1 Example:

1 Solve:

$$(x + 3)^2 = 16$$

Solution

$$(x + 3) = \pm 4$$
$$x = -3 \pm 4$$
$$x = 1, \quad or \quad -7$$

2 Solve:

$$(3x - 2)^2 = 12$$

Solution

$$(3x - 2) = \pm\sqrt{12}$$
$$x = \frac{2 \pm \sqrt{12}}{3}$$
$$x = \frac{2 + 2\sqrt{3}}{3} \quad or \quad \frac{2 - 2\sqrt{3}}{3}$$

7.5 Difference of Two Squares

Expressions with the form $u^2 - v^2$, called the difference of squares (LHS), is always the sum times the difference (RHS):

$$u^2 - v^2 = (u + v)(u - v)$$

7.5.1 Example:

1 Factorise: $x^2 - 1$
(A favourite expression in exams, as it disguises the fact that it is the difference of squares).

Solution

$$x^2 - 1 = (x + 1)(x - 1)$$

2 Factorise: $x^4 - 36y^2$
(Another favourite expression in which you need to recognise that each term can be expressed as a squared term).

Solution

$$x^4 - 36y^2 \Rightarrow (x^2)^2 - (6y)^2$$
$$\Rightarrow (x^2 + 6y)(x^2 - 6y)$$

7.6 Perfect Squares

There is more on this in the next section dealing with Completing the Square, but for now you need to recognise expressions with the form $(u \pm v)^2$, which expand to this:

$$(u + v)^2 = u^2 + 2uv + v^2$$

$$(u - v)^2 = u^2 - 2uv + v^2$$

Note that: $u^2 + v^2$ has no factors

7.6.1 Example:

1 Solve $4x^2 + 20x + 25 = 0$

Solution

$$ax^2 + bx + c = 0$$

$$4x^2 + 20x + 25 = 0$$

$$(2x + 5)(2x + 5) = 0$$

Recognise that a and c are square numbers and that the middle term in x, is $2(2x \times 5) = 20x$

7.7 Finding Possible Factors

The heart of factorising a quadratic is finding any possible factors without having to guess wildly.

Using our standard quadratic equation $ax^2 + bx + c$, if the roots are rational, possible solutions are given by:

$$\pm\frac{\textit{factors of coefficient } c}{\textit{factors of coefficient } a}$$

7.7.1 Example:

Find the possible factors for $3x^2 - 14x - 5$

Since $c = 5$ factors for c are 1 & 5 and for $a = 3$, factors for a are 1 & 3.

Possible solutions are: $\pm\frac{1}{1}, \pm\frac{5}{1}, \pm\frac{1}{3}, \pm\frac{5}{3} \Rightarrow \pm 1, \pm 5, \pm\frac{1}{3}, \pm\frac{5}{3}$

Actual factors are: $(3x + 1)$ *and* $(x - 5)$

Note, this only gives you a 'starter for 10' not the solution, and it only works for rational roots. However, it does work for all polynomials.

An example with irrational roots is: $x^3 - 3 = 0$ which has potential roots of ± 1 and ± 3, but the real roots are: $\sqrt[3]{3} = 1{\cdot}4422$

Large values of a and c, can lead to a large number of potential solutions, so this method has its limits.

We find that for the standard quadratic: $x^2 + bx + c$

$$x^2 + bx + c = (x + \square)(x + \square)$$

Factors of c

and for the standard quadratic: $ax^2 + bx + c$

Factors of a

$$ax^2 + bx + c = (\blacksquare x + \square)(\blacksquare x + \square)$$

Factors of c

7.8 Quadratic Factorisation, type $x^2 + bx + c$

Consider how factors s and t combine to form a quadratic with the form $x^2 + bx + c$:

$$(x + s)(x + t) = x^2 + (s + t)x + st$$
$$(x + s)(x - t) = x^2 + (s - t)x - st$$
$$(x - s)(x + t) = x^2 + (-s + t)x - st$$
$$(x - s)(x - t) = x^2 - (s + t)x + st$$

Notice how the product of the factors s and t combine to form the constant part of the quadratic, c, and the sum or difference combine to form the x coordinate b.

The signs of the coefficients need to be handled with care:

$$(x + s)(x + t) \Rightarrow x^2 + bx + c$$
$$(x \pm s)(x \mp t) \Rightarrow x^2 \pm bx - c$$
$$(x - s)(x - t) \Rightarrow x^2 - bx + c$$

Set up a small table to find the factors of c and to explore the sum and difference to make the coefficient of x:

7.8.1 Example:

1 Factorise: $x^2 + 8x + 12$

Since the coefficient of $x^2 = 1$, and signs of both the following terms are positive, then the form of factors must be $(x + \ldots)(x + \ldots)$.

c		b
1	12	
2	6	$2 + 6 = 8$
3	4	

$$\therefore \quad x^2 + 8x + 12 = (x + 2)(x + 6)$$

2 Factorise: $x^2 - x - 12$

Since the coefficient of $x^2 = 1$, and signs of both the following terms are negative, then the form of factors must be $(x + \ldots)(x - \ldots)$.

c		b
1	12	
2	6	
3	4	$3 - 4 = -1$

$$\therefore \quad x^2 - x - 12 = (x + 3)(x - 4)$$

3 Factorise: $x^2 - 8x + 16$

Since the coefficient of $x^2 = 1$, and sign of the x term is negative, and the constant term is positive, then the form of factors must be $(x - \ldots)(x - \ldots)$.

c		b
1	16	
2	8	
3	/	
4	4	$-4 - 4 = -8$

$$\therefore \quad x^2 - 8x + 16 = (x - 4)(x - 4)$$

7.9 Factorising Quadratic of Type: $ax^2 + bx + c$

So far we have only dealt with quadratics where $a = 1$. Now for some problems with $a > 1$.

7.9.1 Traditional Method

Consider how factors s and t combine to form a quadratic with the form $ax^2 + bx + c$, assuming that a is factored as $a \times 1$:

$$(ax + s)(x + t) = ax^2 + (s + at)x + st$$
$$(ax + s)(x - t) = ax^2 + (s - at)x - st$$
$$(ax - s)(x + t) = ax^2 + (-s + at)x - st$$
$$(ax - s)(x - t) = ax^2 - (s + at)x + st$$

Notice how the product of the factors s and t combine to form the constant part of the quadratic, c, and the sum or difference combine with the coefficient a to form the x coefficient.

Set up a small table to find the factors of c and to explore the sum and difference to make up the coefficient of x. One of the factors has to be multiplied by a as shown:

7.9.1.1 Example:

1 Factorise: $3x^2 + 11x + 10$

Since the coefficient of $x^2 = 3$, and signs of both the following terms are positive, then the form of factors must be $(3x + \dots)(x + \dots)$.

c		b
s	t	$s + at$
1	10	
2	5	$2 + (3 \times 5) \neq 11$
5	2	$5 + (3 \times 2) = 11$
10	1	

$\therefore \quad 3x^2 + 11x + 10 = (3x + 5)(x + 6)$

2 Factorise: $5x^2 - 21x + 18$

Since the coefficient of $x^2 = 5$, and sign of the x term is negative, and the constant term is positive, then the form of factors must be $(5x - \dots)(x - \dots)$.

c		b
s	t	$-s - at$
1	18	
2	9	
3	6	
6	3	$-6 - (3 \times 5) = -21$

$\therefore \quad 5x^2 - 21x + 18 = (5x - 3)(x - 6)$

3 Factorise: $-5x^2 + 7x - 2$

Step one is to rewrite the expression in such a way as to give a +ve x^2 term: $-(5x^2 - 7x + 2)$

c		b
s	t	$-s - at$
1	2	
2	1	$-2 - 5 = -7$

$\therefore \quad -(5x^2 - 7x + 2) = -[(5x - 2)(x - 1)]$

$\therefore \quad -5x^2 + 7x - 2 = (-5x + 2)(x - 1)$

7.9.2 Factoring by Grouping

This method works for any polynomial, not just quadratics. This method relies on manipulating the terms to find a common factor between the terms, which may involve splitting the terms to achieve the required grouping. However, it it is not always obvious how to arrange the grouping of the terms, hence lots of practise is required.

7.9.2.1 Example:

1 Factorise by grouping:

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 \\ &= (2x^2 + 6x) - (x + 3) \\ &= 2x(x + 3) - 1(x + 3) \\ &= (2x - 1)(x + 3) \end{aligned}$$

2 Factorise by grouping:

$$\begin{aligned} 5x^2 - 12x + 4 &= 5x^2 - 10x - 2x + 4 \\ &= (5x^2 - 10x) - (2x - 4) \\ &= 5x(x - 2) - 2(x - 2) \\ &= (5x - 2)(x - 2) \end{aligned}$$

7.9.3 Vieta's Theorem

All the quick methods below are based on Vieta's theorem which says that if a quadratic has roots, p & q, then:

$$\begin{aligned} x^2 + bx + c &= (x - p)(x - q) \\ &= x^2 - (p + q)x + pq \end{aligned}$$

Multiply by a :

$$\begin{aligned} a(x^2 + bx + c) &= a(x^2 - (p + q)x + pq) \\ ax^2 + abx + ac &= ax^2 - a(p + q)x + apq \end{aligned}$$

Comparing coefficients:

$$ab = -a(p + q)$$

$$\rightarrow \quad p + q = -b$$

$$ac = apq$$

$$\rightarrow \quad pq = c$$

7.9.4 The 'ac' Method v1

This is my personal choice of method.

Starting wth the standard form: $ax^2 + bx + c$, (and having taken out any common factors), we convert this to the form: $x^2 + bx + ac$ which is now easier to factorise.

Start by factorising the value of ac:

ac	
f_1	f_2
f_3	f_4
f_5	f_6

then follow the method below:

- Find all the factor pairs of ac: $f_1 \times f_2$; $f_3 \times f_4$ *etc*.
- Find the factor pair that adds up to b. Say $f_3 \pm f_4$
- The solution to $x^2 + bx + ac$ is then $(x + f_3)(x + f_4)$
- The factors f_3 & f_4 are then divided by a and the solutions to $ax^2 + bx + c$ become:

  $$\left(x + \frac{f_3}{a}\right) = 0 \quad \text{and} \quad \left(x + \frac{f_4}{a}\right) = 0$$
- Simplify the fractions $\frac{f3}{a}$ and $\frac{f4}{a}$ into their lowest forms
- Remove the fractional elements by multiplying each solution by a.

This method can be shown to work by considering :

$$(mx + p)(nx + q) = mnx^2 + (mq + np)x + pq$$

7.9.4.1 Example:

1 Solve $2x^2 - 5x - 3 = 0$

Solution:

Multiply 2×-3 and reform the equation as $x^2 - 5x - 6 = 0$

Find the factor pairs for ac:

Two factors add up to −5: $(1 - 6) = -5$

ac		b
−6		
1	−6	$1 - 6 = -5$
−2	−3	This gives wrong soln!

Now: $x^2 - 5x - 6 = (x + 1)(x - 6) = 0$

Hence: $2x^2 - 5x - 3 = \left(x + \frac{1}{2}\right)\left(x - \frac{6}{2}\right) = 0$

Simplify: $\left(x + \frac{1}{2}\right)(x - 3) = 0$

Solutions are: $x = -\frac{1}{2}$ and $x = 3$

Rearranging the solutions, the factorised equation is: $(2x + 1)(x - 3) = 0$

After some practice it can be seen that you can multiply the factors with the fractional part by a to give the final factors, neatly presented.

e.g. $\left(x + \frac{1}{2}\right)(x - 3) = 0$

$$2\left(x + \frac{1}{2}\right) = 0 \quad \Rightarrow \quad (2x + 1) = 0$$

2 Solve $20x^2 - 7x - 6 = 0$

Solution:

Multiply 20×-6 and reform the equation as $x^2 - 7x - 120 = 0$

Find the factor pairs for ac:

Two factors add up to -7: $(8 - 15) = -7$

ac		b
-120		
1	120	
2	60	
3	40	
4	30	
5	24	
6	20	
8	**−15**	$8 - 15 = -7$

Now: $x^2 - 7x - 120 = (x + 8)(x - 15) = 0$

Hence: $20x^2 - 7x - 6 = \left(x + \frac{8}{20}\right)\left(x - \frac{15}{20}\right) = 0$

Simplify: $\left(x + \frac{2}{5}\right)\left(x - \frac{3}{4}\right) = 0$

Solutions are: $x = -\frac{2}{5}$ and $x = \frac{3}{4}$

Hence: $5x + 2 = 0$ and $4x - 3 = 0$

$\therefore$ $20x^2 - 7x - 6 = (5x + 2)(4x - 3)$

Alternatively, remove the fractional elements by multiplying by a:

$$\left(x + \frac{8}{20}\right)\left(x - \frac{15}{20}\right) = 0$$

$$20\left(x + \frac{8}{20}\right) = 0 \quad \Rightarrow \quad 20x + 8 = 0 \quad \Rightarrow \quad 5x + 2 = 0$$

$$20\left(x - \frac{15}{20}\right) = 0 \quad \Rightarrow \quad 20x - 15 = 0 \quad \Rightarrow \quad 4x - 3 = 0$$

After lots of practise, a short cut presents itself. Using the simplified factors to illustrate this:

$$\left(x + \frac{2}{5}\right)\left(x - \frac{3}{4}\right) = 0$$

Move the denominator of the fraction and make it the coefficient of the x term:

$$\left(\underset{\uparrow}{\square}x + \frac{2}{\leftarrow\ 5}\right)\left(\underset{\uparrow}{\square}x - \frac{3}{\leftarrow\ 4}\right) = 0$$

$$(5x + 2)(4x - 3) = 0$$

3 A special case arises when there is only one root, (sometimes called a double root).
Factorise $4x^2 + 12x + 9$

Solution:

Multiply 4×9 and find the factor pairs:

Two factors add up to 12: $(6 + 6) = 12$

36		b
1	36	
2	18	
3	12	
4	9	
6	**6**	$6 + 6 = 12$
9	4	

Hence: $\left(x + \frac{6}{4}\right)\left(x + \frac{6}{4}\right)$

Simplify: $\left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right)$

Solutions are: $x = -\frac{3}{2}$ and $x = -\frac{3}{2}$

Factors are: $(2x + 3)(2x + 3) = (2x + 3)^2$

4 Solve $3x^2 - 3x - 18 = 0$

Solution:

Multiply 3×-18 and find the factor pairs:

Two factors add up to −3: $(6 - 9) = -3$

−54		b
1	54	
2	27	
3	18	
6	**−9**	$6 - 9 = -3$

Hence: $\left(x + \frac{6}{3}\right)\left(x - \frac{9}{3}\right) = 0$

Simplify: $(x + 2)(x - 3) = 0$

Solutions are: $x = 2$ and $x = -3$

Factors are: $(x + 2)(x - 3)$

Note that the coefficients of x in the factorised expression are both 1. Look at the original equation and you can see that all the terms could have been divided by 3 to give:

$$x^2 - x - 6 = 0$$

5 Another double root example:
Solve $9x^2 - 6x + 1 = 0$

Solution:
Multiply 9×1 and find the factor pairs:

Two factors add up to −6: $(-3 - 3) = -6$

9		b	
1	9		
2	−		
−3	**−3**	**−3 − 3**	**−6**

Hence: $\left(x - \frac{3}{9}\right)\left(x - \frac{3}{9}\right) = 0$

$\left(x - \frac{1}{3}\right)\left(x - \frac{1}{3}\right) = 0$

Solution is: $x = \frac{1}{3}$

Factors are: $(3x - 1)(3x - 1) = 0$

6 For equations that have a −ve x^2 term, this method works best if the −ve part is changed to +ve by removing a common factor of −1:
Solve $-6x^2 + 5x - 1 = 0$

Solution:

$-6x^2 + 5x - 1 = 0 \quad \Rightarrow \quad (-1)(6x^2 - 5x + 1) = 0$

Work on solving: $6x^2 - 5x + 1 = 0$

Multiply 6×1 and find the factor pairs:

Two factors add up to −5: $(-2 - 3) = -5$

6		b	
1	6		
−2	**−3**	**−2 − 3**	**−5**

Hence: $\left(x - \frac{2}{6}\right)\left(x - \frac{3}{6}\right) = 0$

$\left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right) = 0$

Solution is: $x = \frac{1}{3}$ and $x = \frac{1}{2}$

Factors are: $(3x - 1)(2x - 1) = 0$

But include the -1 $\quad \Rightarrow \quad (-1)(3x - 1)(2x - 1) = 0$

$\Rightarrow \quad (-3x + 1)(2x - 1) = 0$

or $\quad \Rightarrow \quad (3x - 1)(-2x + 1) = 0$

7.9.5 The 'ac' Method v2

Another variation on a theme. Again we turn a hard quadratic into an easier one by multiplying a & c and replacing c with ac and give the x^2 term a coefficient of 1, giving the form $x^2 + bx + ac$.

7.9.5.1 Example:

Factorise $7x^2 - 11x + 6$

Solution:

Multiply 7×6 and change the quadratic thus:

$$7x^2 - 11x + 6 \quad (1)$$

$$\rightarrow \quad x^2 - 11x + 42 \quad (2)$$

Factorise Eq (2)

Two factors add up to -11: $(3 - 14) = -11$

42	
1	42
2	21
3	**–14**

Factors are: $(x + 3)(x - 14)$

However, we want a $7x^2$ term, so factorise the 14 to 7×2:

$$(x + 3)(x - \overset{7\times 2}{14})$$

Move the factor 7, to the x^2 term:

$$(7x + 3)(x - 2)$$

Check that the x term coefficient is correct:

$$(7x + 3)(x - 2)$$

$$3x$$

$$\rightarrow -14x$$

$$\overline{-11x}$$

7.9.5.2 Example:

A special case arises when there is only one root, (sometimes called a double root).
Factorise $4x^2 + 12x + 9$

Solution:

Multiply 4×9 and change the quadratic thus:

$$x^2 + 12x + 36$$

Factorise as normal:

36	
1	36
2	18
3	12
4	9
6	**6**
9	4

Two factors add up to 12: $(6 + 6) = 12$

Factors are: $(x + 6)(x + 6)$

The roots are the same, so in assigning value for a, the two required factors have to be the same:

$$(x + \overset{2\times 3}{6})(x + \overset{2\times 3}{6})$$

Move the factor 2, to both the x^2 terms:

$$(2x + 3)(2x + 3)$$

Check that the x term coefficient is correct:

$$(2x + 3)(2x + 3)$$

$\downarrow \quad 6x \quad \downarrow$

$\rightarrow \quad 6x \quad \hookleftarrow$

$\overline{12x}$

7.9.6 The Division Method

From the standard form:

$$ax^2 + bx + c$$

$$\frac{(ax + p)(ax + q)}{a}$$

The numbers p & q must add to b, and multiply to ac.

7.9.6.1 Example:

Factorise $10x^2 - 7x - 6$

Solution:

Multiply 10×6 and find the factors.

Two factors add up to −7: $(5 - 12) = -7$

60	
1	60
2	30
3	20
4	15
5	**−12**
6	10

Since $a = 10 \quad \Rightarrow \frac{(10x + p)(10x + q)}{10}$

$p = 5$ and $q = -12$

Substitute: $\Rightarrow \frac{(10x + 5)(10x - 12)}{10}$

Rearrange: $\Rightarrow \frac{(10x + 5)}{5} \times \frac{(10x - 12)}{2}$

Cancel: $\Rightarrow (2x + 1) \times (5x - 6)$

$x = -\frac{1}{2}$ and $x = \frac{6}{5}$

7.9.7 The Chinese Cross Product Method

I call this the Chinese Cross Product method, because I found it in a Chinese maths book! It tabulates the normal method of guessing factors.

Altering the standard form to:

$$(mx + p)(nx + q) = mnx^2 + (mq + np)x + pq$$

Notice how the cross product forms the x term coefficient:

$$\begin{array}{ccc} m & & p \\ & \times & \\ n & & q \\ \hline & np + mq & \end{array}$$

7.9.7.1 Example:

1 Factorise: $5x^2 - 12x + 4$

Solution:

The factors of 5 are 5×1, and factors of 4 are 1×4, 2×2, −2×−2

Use the cross product to form the x coefficient:

$$\begin{array}{ccccl} 5x & & -2 & & \Rightarrow (5x - 2) \\ & \times & & & \\ 1x & & -2 & & \Rightarrow (x - 2) \\ & & -10x - 2x & & = -12x \\ & & -12x & & \end{array}$$

$$\therefore \quad 5x^2 - 12x + 4 = (5x - 2)(x - 2)$$

2 Factorise: $8x^2 + 10x + 3$

Solution:

The factors of 8 are 8×1, 4×2 and factors of 3 are 1×3, 3×1, −3× −1, −1× −3

You need a separate table for each pair of a factors, and cross multiply with each pair of c factors and stop when you find the b coefficient.

$$\begin{array}{ccccc} 8x & & 3 & 1 & -3 \\ & \times & & & \\ 1x & & 1 & 3 & -1 \\ & & 8x + 3x & 24x + x & -ve \\ & & 11x & 25x & \therefore \textit{ not valid} \end{array}$$

$$\begin{array}{ccccl} 4x & & 3 & & \Rightarrow (4x + 3) \\ & \times & & & \\ 2x & & 1 & & \Rightarrow (2x + 1) \\ & & 4x + 6x & & = 10x \\ & & 10x & & \end{array}$$

$$\therefore \quad 8x^2 + 10x + 3 = (4x + 3)(2x + 1)$$

8 • C1 • Completing the Square

8.1 General Form of a Quadratic

The general form of a quadratic is:

$$ax^2 + bx + c$$

The object of **completing the square** is to put the quadratic into the square form:

$$a(x + p)^2 + q$$

This is sometimes called the vertex format, for reasons which will become obvious later.

The advantage of changing the standard quadratic into this square form is that we have just one term in x. The x^2 term has been eliminated.

In practice, when completing the square we need to set the leading coefficient, a, (of the x^2 term) to 1.

$$\text{e.g.} \qquad 2x^2 + 4x + 8 \Rightarrow 2(x^2 + 2x + 4)$$

8.2 A Perfect Square

The expressions $(x + k)^2$ and $(x - k)^2$ are both perfect squares. To complete the square of any quadratic, you need to get as close to the ideal perfect square as you can by adjusting the constant.

The general form of a perfect square is:

$$(x + k)^2 = x^2 + 2kx + k^2$$

$$(x - k)^2 = x^2 - 2kx + k^2$$

Notice the coefficient of the expanded x term is $2k$. i.e. in order to find k, we halve the coefficient of the x term.

Some practical examples make the point clearly:

$$(x + 1)^2 = x^2 + 2x + 1 = x^2 + 2(1)x + 1^2$$

$$(x + 2)^2 = x^2 + 4x + 4 = x^2 + 2(2)x + 2^2$$

$$(x + 3)^2 = x^2 + 6x + 9 = x^2 + 2(3)x + 3^2$$

$$(x + 4)^2 = x^2 + 8x + 16 = x^2 + 2(4)x + 4^2$$

Using this format it is easy to arrange an expression like $x^2 - 12x$ into a perfect square.

Thus:

$$\begin{aligned} x^2 - 12x &= x^2 - 2(6)x \\ &= x^2 - 2(6)x + 6^2 - 6^2 && \textit{Adding } 6^2 \textit{ makes a perfect square} \\ &= \{x^2 - 2(6)x + 6^2\} - 6^2 && \textit{Subtract } 6^2 \textit{ to balance the equation} \\ &= (x - 6)^2 - 6^2 \end{aligned}$$

Note that the following types are **not** perfect squares:

$$(x + s)(x + t) = x^2 + (s + t)x + st$$

$$\textit{e.g.} \qquad (x + 1)(x + 2) = x^2 + 3x + 2$$

8.3 Deriving the Square or Vertex Format

The square format of a quadratic $a(x + p)^2 + q$ can be derived as follows:

$$
\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + \frac{ac}{a} \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a} - c\right) \\
&= a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)
\end{aligned}
$$

Hence: $p = \dfrac{b}{2a}$ & $q = -\left(\dfrac{b^2 - 4ac}{4a}\right)$

8.4 Completing the Square

- Find the nearest perfect square by halving the coefficient of the x term, to give k
- Irrespective of whether the perfect square is $(x + k)^2$ **or** $(x - k)^2$, subtract k^2
- Add on the old '$+ c$' term.

This works by taking the $x^2 + bx$ part of the quadratic and turning this into a perfect square, and to balance the equation you have to subtract the value of k^2

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

But $$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

$$\therefore \quad x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Assuming $a = 1$, we can write:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

$$x^2 - bx + c = \left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

8.4.1 Example:

Ex: 1 $x^2 - 8x + 7 \Rightarrow (x-4)^2 - 16 + 7$

$$\Rightarrow (x-4)^2 - 9$$

Ex: 2 $x^2 + 5x - 12 \Rightarrow \left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 12$

$$\Rightarrow \left(x + \frac{5}{2}\right)^2 - 18\frac{1}{4}$$

Ex: 3 $x^2 - x - 12 \Rightarrow \left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 12$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 - 12\frac{1}{4}$$

Ex: 4 $2x^2 - 11x - 8 \Rightarrow 2\left(x^2 - \frac{11}{2}x - 4\right) = 2\left[\left(x - \frac{11}{4}\right)^2 - \left(\frac{11}{4}\right)^2 - 4\right]$

$$\Rightarrow 2\left[\left(x - \frac{11}{4}\right)^2 - \frac{185}{16}\right]$$

An alternative approach, which just factors out the coefficients of the terms in x:

Ex: 5 $-2x^2 + 12x + 5 \Rightarrow -2(x^2 - 6x) + 5 = -2[(x-3)^2 - 9] + 5$

$$\Rightarrow -2(x-3)^2 + 18 + 5$$

$$\Rightarrow -2(x-3)^2 + 23$$

8.5 Completing the Square in Use

There are several uses for this technique:

- Solve any quadratic
- Solving inequalities
- Graphing — finding the turning point (max / min value) or vertex, and the line of symmetry
- Simplify an equation ready for transformation questions
- Used in circle geometry to find the centre of a circle
- Derivation of the quadratic formula (see later section)
- Integration — used later to manipulate an inverse trig function ready for integration

One advantage of using the method is that x appears in the expression only once, unlike a standard quadratic where it appears twice. Completing the square can be used on any quadratic, but for solving quadratics, simple factorisation or the quadratic formula may be easier.

8.6 Solving Quadratics

We shall see later that the quadratic formula for solving quadratics is derived from completing the square, but completing the square can be used as a relatively simple way to solve quadratics.

8.6.1 Example: Solving Quadratics

Solve the quadratic $x^2 - 8x + 5 = 0$

Solution:

$$x^2 - 8x + 5 \Rightarrow (x - 4)^2 - 11$$
$$(x - 4)^2 = 11$$
$$x - 4 = \pm\sqrt{11}$$
$$x = 4 \pm \sqrt{11}$$

8.7 Solving Inequalities

It turns out that many inequalities can be rearranged as the sum of a square.

8.7.1 Example: Solving Inequalities

Show that $y = x^2 + 2x + 3$ is positive for all real values of x.

Solution:

$$x^2 + 2x + 3 \Rightarrow (x + 1)^2 - 1 + 3$$
$$\Rightarrow (x + 1)^2 + 2$$

Since $(x + 1)^2$ will always be positive (as it is squared) then the LHS must be equal or greater than 2, hence the expression is always positive.

8.8 Graphing – Finding the Turning Point (Max / Min Value)

Looking at the square form of the quadratic, you can see that the minimum or maximum value of the function is given when the squared term containing x equals zero.

Recall that any squared term is positive irrespective of the value of x, i.e. $(x + k)^2 > 0$.

If the coefficient of the squared term is positive we have a minimum value, if the coefficient is negative we have a maximum value.

The min or max value is sometimes referred to as a '**turning point**' or as the '**vertex**'. For a quadratic the vertex also defines a line of symmetry.

General form of a completed square: $y = a(x + p)^2 + q$

Min value of y is when: $x = -p \quad \therefore \quad y = q$

The coordinate of the turning point is: $(-p,\ q)$

For a quadratic of the form $x^2 + bx + c$ where $a = 1$

$$y = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Turning point is when: $x = -\frac{b}{2} \quad \therefore\ y = -\left(\frac{b}{2}\right)^2 + c$

For a quadratic of the form $ax^2 + bx + c$, completing the square gives:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$\therefore$ Turning point is when $x = -\frac{b}{2a}$

Substitute $x = -\frac{b}{2a}$ to find y

$$\therefore \qquad y = a\left[0 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] \Rightarrow -a\left(\frac{b}{2a}\right)^2 + \frac{ac}{a}$$

$$y = -a\frac{b^2}{4a^2} + c$$

$$y = -\frac{b^2}{4a} + c$$

For a quadratic of the form $ax^2 + bx + c$ where $a = 1$

Turning point is when $x = -\frac{b}{2}$

$$y = -\left(\frac{b}{2}\right)^2 + c$$

$$y = -\frac{b^2}{4} + c$$

8.8.1 Example: Graphing and Turning Points

1 Sketching the graph:

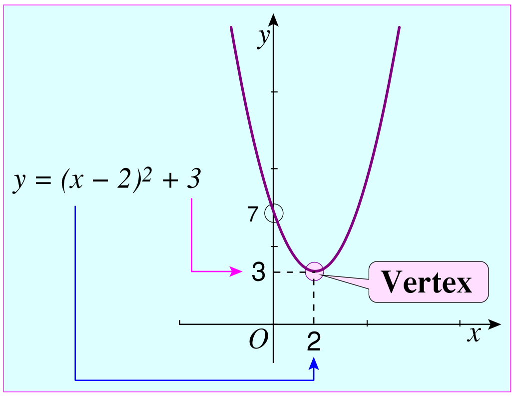
$x^2 - 4x + 7 \equiv (x - 2)^2 + 3$

Minimum point of graph is when $x = 2$

$\therefore \quad y = 3$

Vertex at (2, 3)

The quadratic is symmetrical about the line $x = 2$



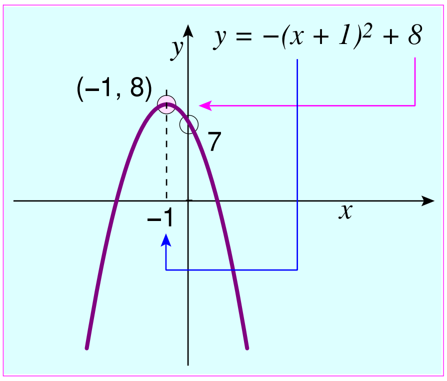
$-x^2 - 2x + 7 \equiv -1\,(x^2 + 2x - 7)$
$-x^2 - 2x + 7 \equiv -1\left[(x + 1)^2 - 1 - 7\right]$
$-x^2 - 2x + 7 \equiv -1\,(x + 1)^2 + 8$

Max point of graph is when $x = -1$

$\therefore \quad y = +8$

Vertex at (−1, 8)

The quadratic is symmetrical about the line $x = -1$



2 Find the equation of the tangent to the curve $y = x^2 - 4x + 2$ which is parallel to the x-axis.

Solution:

Since the tangent is parallel to the x-axis, we need to find the minimum by completing the square:

$$x^2 - 4x + 2 = (x - 2)^2 - 2^2 + 2$$
$$= (x - 2)^2 - 2$$

$\therefore$ min when $\quad x = 2$

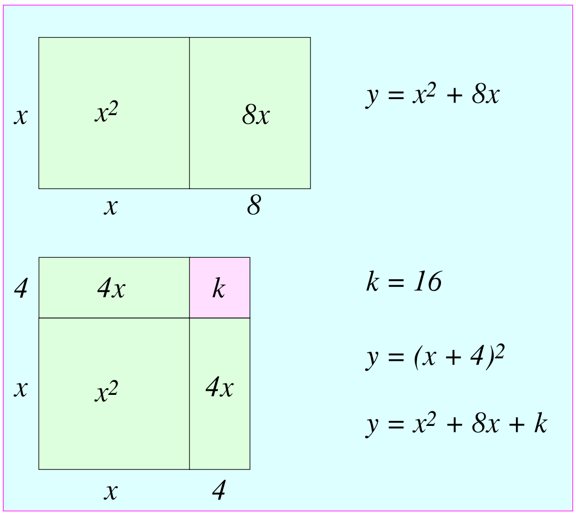
$\therefore \quad y = -2$

The equation of the tangent to the curve, and parallel to the x-axis is $y = -2$

8.9 A Geometric View of Completing the Square

Take a simple quadratic such as: $x^2 + 8x$.

This expression can be represented as a diagram, as shown in the first half of the sketch below:



The nearest perfect square for $x^2 + 8x$ is $(x + 4)^2$. From the diagram we can see that $(x + 4)^2$ is larger than $x^2 + 8x$ by an additional amount k. Thus:

$$\begin{aligned} x^2 + 8x &= (x + 4)^2 - k \\ &= (x + 4)^2 - 4^2 \\ &= (x + 4)^2 - 16 \end{aligned}$$

Note how any similar quadratic such as: $x^2 + 8x + 7$ can now be represented by:

$$\begin{aligned} x^2 + 8x + 7 &= (x + 4)^2 - 16 + 7 \\ &= (x + 4)^2 - 9 \end{aligned}$$

8.10 Topic Digest

Standard solution:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

$$x^2 - bx + c = \left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

For a quadratic of the form: $a(x + p)^2 + q$

$$y = a(x + p)^2 + q$$

Co-ordinates of vertex $(-p,\ q)$

Axis of symmetry $x = -p$

If $a > 0$, graph is ∪ shaped, vertex is a minimum point

If $a < 0$, graph is ∩ shaped, vertex is a maximum point

For a quadratic of the form: $ax^2 + bx + c$

Turning point is when $x = -\dfrac{b}{2a}$; $y = -\dfrac{b^2}{4a} + c$

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

9 • C1 • The Quadratic Formula

9.1 Deriving the Quadratic Formula by Completing the Square

The Quadratic Formula is just another method of completing the square to solve a quadratic. A sledge hammer to crack a nut. To derive the formula, complete the square for the general form of a quadratic:

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Divide by } a$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \qquad \text{Complete the square}$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Take square roots}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The roots of a quadratic are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It follows that with a ± symbol in the formula there will be **two** solutions.

Solution 1) $\quad x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

Solution 2) $\quad x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Note also that:

$$ax^2 + bx + c = a(x - root_1)(x - root_2)$$

$$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

9.2 Examples of the Quadratic Formulae

9.2.1 Example:

1 Find the roots of: $3x^2 + 17x + 10 = 0$

Solution:

$$3x^2 + 17x + 10 = 0$$

$$\therefore \quad a = 3,\ b = 17,\ c = 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-17 \pm \sqrt{17^2 - 4(3)(10)}}{6}$$

$$x = \frac{-17 \pm \sqrt{289 - 120}}{6}$$

$$x = \frac{-17 \pm \sqrt{169}}{6}$$

$$\therefore \quad x = \frac{-17 + \sqrt{169}}{6} = \frac{-17 + 13}{6} = -\frac{2}{3}$$

$$\text{and} \quad x = \frac{-17 - \sqrt{169}}{6} = \frac{-17 - 13}{6} = -5$$

$$x = -\frac{2}{3},\ \text{and}\ -5$$

Working backwards we see the quadratic factorises to:

$$3x^2 + 17x + 10 = (3x + 2)(x + 5)$$

2 Find the roots of: $2x^2 - 7x - 1 = 0$

Solution:

$$2x^2 - 7x - 1 = 0$$

$$\therefore \quad a = 2,\ b = -7,\ c = -1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(-7)^2 - (2)(-1)}}{4}$$

Watch the signs!

$$x = \frac{-7 \pm \sqrt{49 - 4(2)(-1)}}{4}$$

$$x = \frac{-7 \pm \sqrt{57}}{4}$$

$$x = \frac{-7 + \sqrt{57}}{4} \quad \text{or} \quad x = \frac{-7 - \sqrt{57}}{4}$$

$$x = 3{\cdot}64,\ \text{or}\ -0{\cdot}14$$

3 Solve $5 - 8x - x^2 = 0$

Solution:

First, rearrange to the correct format:

$$5 - 8x - x^2 = 0$$

$$-x^2 - 8x + 5 = 0$$

Let $a = -1,\ b = -8,\ c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4(-1)5}}{-2}$$

Watch the signs!

$$x = \frac{8 \pm \sqrt{64 - 4(-1)5}}{-2}$$

$$x = \frac{8 \pm \sqrt{64 + 20}}{-2}$$

$$x = \frac{8 \pm \sqrt{84}}{-2}$$

$$x = -4 \pm \sqrt{21}$$

Alternative solution (completing the square):

$$-x^2 - 8x + 5 = 0$$

$$x^2 + 8x - 5 = 0$$

$$(x + 4)^2 - 16 - 5 = 0$$

$$(x + 4)^2 - 21 = 0$$

$$(x + 4)^2 = 21$$

$$(x + 4) = \pm\sqrt{21}$$

$$x = -4 \pm \sqrt{21}$$

4 Solve $x + \frac{1}{x} = 6$

Solution:

First, rearrange to the correct format:

$$x + \frac{1}{x} = 6$$

$$x^2 + \frac{x}{x} = 6x$$

$$x^2 - 6x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{36 - 4(1)(1)}}{2}$$

$$x = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2}$$

$$x = 3 \pm 2\sqrt{2}$$

9.3 Finding the Vertex

See also section 8.6 Graphing – Finding the Turning Point.

Rearranging the standard quadratic formula we find:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Hence, we can see that the roots are either side of the vertex, where the x-coordinate of the vertex is given by:

$$x = \frac{-b}{2a}$$

9.3.1 Example:

1

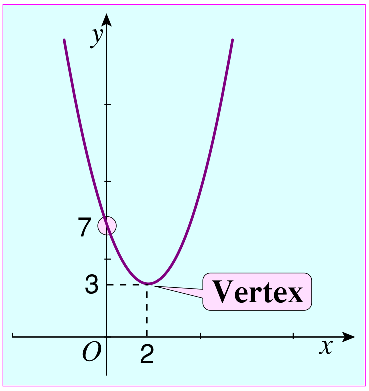
$y = x^2 - 4x + 7$

Vertex is when $x = \frac{-(-4)}{2} = 2$

$\therefore \quad y = 3$

Vertex at (2, 3)

The quadratic is symmetrical about the line $x = 2$



9.4 Heinous Howlers

In trying to solve something like $7 - 5x - 2x^2 = 0$ DO NOT set $a = 7$, $b = -5$ or $c = 2$!!!!!!

Watch the signs - a very common error is to square $-b$ and end up with a negative answer.

9.5 Topical Tips

In finding the roots of a quadratic, if all else fails, the quadratic formulae can always be used on any quadratic, providing that you pay attention to the signs.

10 • C1 • The Discriminant

10.1 Assessing the Roots of a Quadratic

The roots of a quadratic are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression "$b^2 - 4ac$" is part of the quadratic formula and is known as the **discriminant**. It determines how many solutions the equation has, or in other words, how many times does the graph cross the x-axis.

Very useful when sketching graphs, to test if the graph crosses the x-axis.

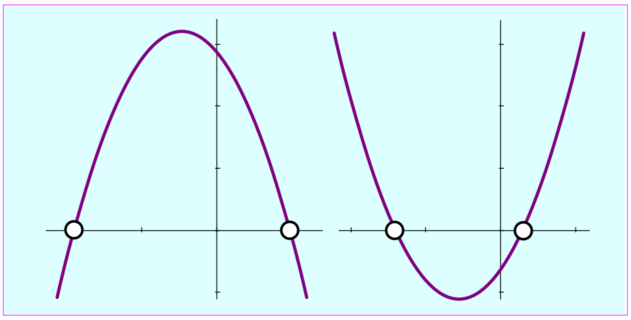
If the discriminant…	Then…	Roots or solutions	Notes
$b^2 - 4ac > 0$ or $b^2 > 4ac$	Graph intersects the x-axis twice	2 distinct real solutions	If the **discriminant** is a perfect square, the solution is rational and can be factorised.
$b^2 - 4ac = 0$ or $b^2 = 4ac$	Graph intersects the x-axis once	1 real solution $x = -\frac{b}{2a}$	Sometimes called repeated or coincident roots. The **quadratic** is a perfect square. The x-axis is a tangent to the curve.
$b^2 - 4ac < 0$ or $b^2 < 4ac$	Graph does not intersect the x-axis	No real solutions	Only complex roots, which involve imaginary numbers ($\sqrt{-1}$).

If the discriminant > 0
then $\sqrt{b^2 - 4ac}$ is positive and there are two real solutions:
one involves:

$$+\sqrt{b^2 - 4ac}$$

and the other involves:

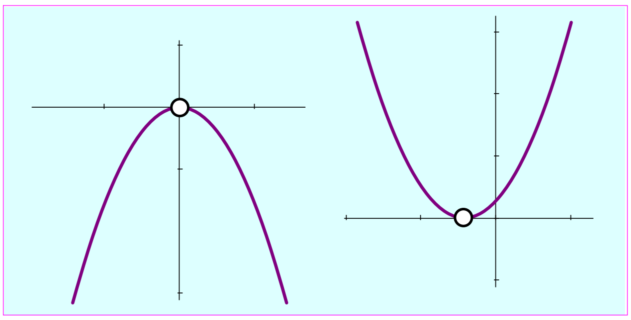
$$-\sqrt{b^2 - 4ac}$$



If the discriminant = 0
then only one solution since both

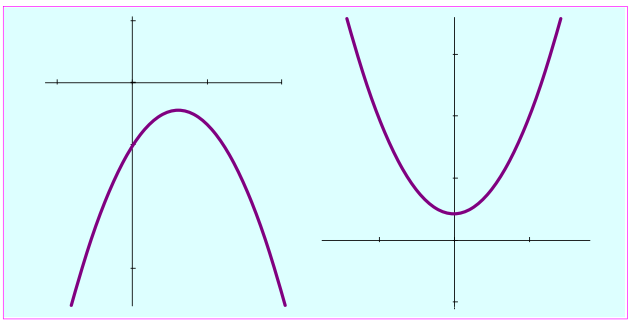
$$+\sqrt{0} \textit{ and } -\sqrt{0}$$

are both zero.



If the discriminant < 0

No **real** solutions are possible, as we can't evaluate the square root of a negative number, (at least in this module – there are in fact two 'complex' solutions - see later).



10.2 Discriminant = 0

When the discriminant = 0, i.e. when $b^2 = 4ac$, the **quadratic** is a perfect square of the form:

$$(px + q)^2 = p^2x^2 + 2pqx + q^2$$

Hence: $$b^2 - 4ac = (2pq)^2 - 4p^2q^2 = 0$$

$$\therefore \quad x = -\frac{b}{2a}$$

In this case, the x-axis is tangent to the quadratic curve at the vertex.

Note the distinction of the discriminant being a perfect square and the quadratic being a perfect square.

10.3 Topical Tips

In the exam, note how the question is phrased.

If asked to find 'two distinct roots', or find 'two distinct points of intersection', then use: $b^2 - 4a > 0$

For questions wanting the 'real roots', then use: $b^2 - 4a \geqslant 0$

For 'equal roots' use: $b^2 - 4a = 0$

Questions will often ask you to show that an inequality is true. They try to disguise the question by giving an inequality that is less than zero. Start with the basics above and you will find you will need to multiply by -1, which changes the inequality around. (See last example below).

Note that if a line and curve intersect with equal roots, then the line must be a tangent to the curve. Recall that setting a quadratic $ax^2 + bx + c = 0$ is really asking you to solve two simultaneous equations of $y = ax^2 + bx + c$ and $y = 0$. The same logic applies if you are asked to find the intersection of $y = ax^2 + bx + c$ and the line $y = mx + c$.

Remember that the discriminant is the bit inside the square root!

10.4 Examples

1 The equation $kx^2 - 2x - 7 = 0$ has two real roots. What can you deduce about the value of k.

Solution:

$$\therefore \quad b^2 - 4ac \geqslant 0$$
$$4 - (4k \times -7) \geqslant 0$$
$$4 + 28k \geqslant 0$$
$$28k \geqslant -4$$
$$k \geqslant -\frac{4}{28}$$
$$k \geqslant -\frac{1}{7}$$

2 The equation $x^2 - 7x + k = 0$ has repeated or equal roots. Find the value of k.

Solution:

$$\therefore \quad b^2 - 4ac = 0$$
$$49 - (4 \times 1 \times k) = 0$$
$$49 - 4k = 0$$
$$4k = 49$$
$$k = 12\tfrac{1}{4}$$

3 Find the set of values of k for which $kx^2 + x + k - 1 = 0$ has two distinct real roots.

Solution:

$$\therefore \quad b^2 - 4ac > 0$$
$$1 - (4 \times k \times (k - 1)) > 0$$
$$1 - (4k(k - 1)) > 0$$
$$1 - 4k^2 + 4k > 0$$
$$-4k^2 + 4k + 1 > 0$$
$$\therefore \quad 4k^2 - 4k - 1 < 0 \qquad (1)$$
$$4\left(k^2 - k - \frac{1}{4}\right) < 0 \qquad \text{Complete the square}$$
$$4\left[\left(k - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{4}\right] < 0$$
$$\left(k - \frac{1}{2}\right)^2 - \frac{1}{2} < 0$$

Min point of curve is at: $k = \frac{1}{2}$

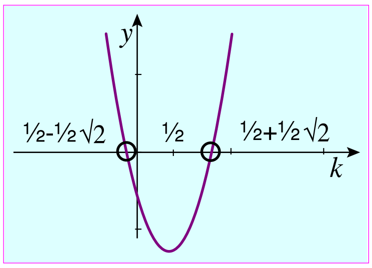
From (1) find the roots by formula:

$$k = \frac{4 \pm \sqrt{16 - (-16)}}{8} = \frac{4 \pm \sqrt{32}}{8} = \frac{4 \pm 4\sqrt{2}}{8} = \frac{1 \pm \sqrt{2}}{2}$$
$$\therefore \quad \frac{1 - \sqrt{2}}{2} < k < \frac{1 + \sqrt{2}}{2}$$

Set of boundary values: $-0{\cdot}2071 < k < 1{\cdot}2071$

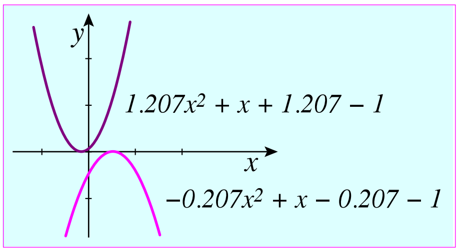
Solution for the discriminant quadratic:

$$4k^2 - 4k - 1 = 0$$



The original quadratic, with the two boundary values of k plotted.

$$-0{\cdot}2071 < k < 1{\cdot}2071$$



4 A line and curve intersect at two distinct points. The x-coordinate of the intersections can be found by the equation:

$$x^2 - 3kx + 7 - k = 0$$

Find the values of k that satisfy this equation.

Solution:

$$\therefore \quad b^2 - 4ac > 0$$

$$(-3k)^2 - [4 \times 1 \times (7 - k)] > 0$$

$$9k^2 - 28 + 4k > 0$$

$$9k^2 + 4k - 28 > 0$$

Factors of $9 \times 28 = 252 = 18 \times 14$

$$(k + 18/9)(k - 14/9) > 0$$

$$(k + 2)(9k - 14) > 0$$

$$\therefore \quad k < -2, \quad \text{and} \quad k > \frac{14}{9}$$

5 The equation $(k + 1)x^2 + 12x + (k - 4) = 0$ has real roots. Find the values of k.

Solution:

$$\therefore \quad b^2 - 4ac \geqslant 0$$

$$(12)^2 - [4(k + 1)(k - 4)] \geqslant 0$$

$$144 - [4(k^2 + k - 4k - 4)] \geqslant 0$$

$$144 - [4k^2 - 12k - 16] \geqslant 0$$

$$144 - 4k^2 + 12k + 16 \geqslant 0$$

$$-4k^2 + 12k + 160 \geqslant 0$$

$$4k^2 - 12k - 160 \leqslant 0 \qquad \text{multiply by } -1 \text{ and divide by } 4$$

$$k^2 - 3k - 40 \leqslant 0$$

$$(k + 5)(k - 8) \leqslant 0$$

$$-5 \leqslant k \leqslant 8$$

6 The equations $y = x^2 - 8x + 12$ and $2x - y = 13$ are given. Show that $x^2 - 10x + 25 = 0$. Find the value of the discriminant and what can you deduce about the first two equations.

Solution:

$$y = x^2 - 8x + 12$$

$$y = 2x - 13$$

$$\therefore \quad x^2 - 8x + 12 = 2x - 13$$

$$x^2 - 8x + 12 - 2x + 13 = 0$$

$$x^2 - 10x + 25 = 0$$

Now: $\quad b^2 - 4ac = (-10)^2 - 4 \times 25 = 100 - 100$

$\quad = 0 \quad$ i.e. one solution.

Deduction is that $2x - y = 13$ is tangent to $y = x^2 - 8x + 12$.

7 Find the discriminant of the equation $3x^2 - 4x + 2 = 0$ and show that the equation is always positive.

Solution:

$$\therefore \quad b^2 - 4ac = (-4)^2 - (4 \times 3 \times 2)$$
$$= 16 - 24$$
$$= -8$$

Therefore the equation has no real roots and does not cross the x-axis. Since the coefficient of the x^2 term is positive, the curve is ∪ shaped, and so the equation is always positive.

8 The equation $(2k - 6)x^2 + 4x + (k - 4) = 0$ has real roots. Show that $x^2 - 7x + 10 \leqslant 0$ and find the values of k.

Solution:

$$\therefore \quad b^2 - 4ac \geqslant 0$$
$$(4)^2 - [4(2k - 6)(k - 4)] \geqslant 0$$
$$16 - [4(k^2 - 14k + 24)] \geqslant 0$$
$$-8k^2 + 56k - 80 \geqslant 0$$
$$k^2 - 7k - 10 \leqslant 0 \qquad \text{multiply by } -1 \text{ \& divide by 8 (reverse the inequality)}$$
$$(k - 5)(k - 2) \leqslant 0$$
$$2 \leqslant k \leqslant 5$$

10.5 Complex & Imaginary Numbers (Extension)

For those doing science or going on to further maths, it should be pointed out that whilst it is true that there are no **real** solutions when $b^2 - 4a < 0$, there are in fact two imaginary solutions, that involve numbers with the square root of minus one.

An imaginary number is simply the square root of minus one, which has been given the letter i or j to identify it. Hence, if $\sqrt{-1} = i$, we can say that the solution to an equation such as $x^2 + 1 = 0$ is $x = \pm\sqrt{-1}$ or $x = \pm i$.

E.g. Solve $x^2 - 8x + 20 = 0$

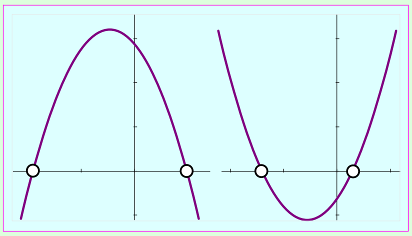
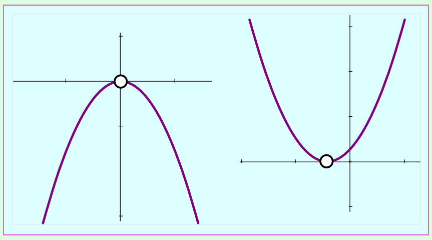
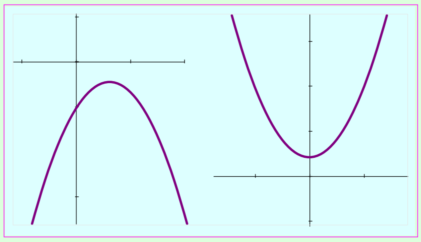
Solution:

$$x^2 - 8x + 20 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 20}}{2}$$
$$x = \frac{8 \pm \sqrt{64 - 80}}{2}$$
$$x = \frac{8 \pm \sqrt{-16}}{2}$$
$$x = \frac{8 \pm 4\sqrt{-1}}{2}$$
$$x = 4 \pm 2i$$

In the example above, $4 \pm 2i$ is called a complex number, as it it made up of the imaginary number i, and two real numbers 4, & 2.

In a complex number, such as: $p + qi$, the number p is called the real part and q the imaginary part.

10.6 Topic Digest

Case 1	Case 2	Case 3
$b^2 - 4ac > 0$ or $b^2 > 4ac$	$b^2 - 4ac = 0$ or $b^2 = 4ac$	$b^2 - 4ac < 0$ or $b^2 < 4ac$
Graph intersects the x-axis twice	Graph intersects the x-axis once	Graph does not intersect the x-axis
2 distinct real solutions	1 real solution $x = -\frac{b}{2a}$	No real solutions
If the **discriminant** is a perfect square, the solution is rational and can be factorised. If the **discriminant** is not a perfect square, the solution is irrational	Sometimes called repeated or coincident roots. The **quadratic** is a perfect square. The x-axis is a tangent to the curve.	Only complex roots, which involve imaginary numbers, $(\sqrt{-1})$.
		

11 • C1 • Sketching Quadratics

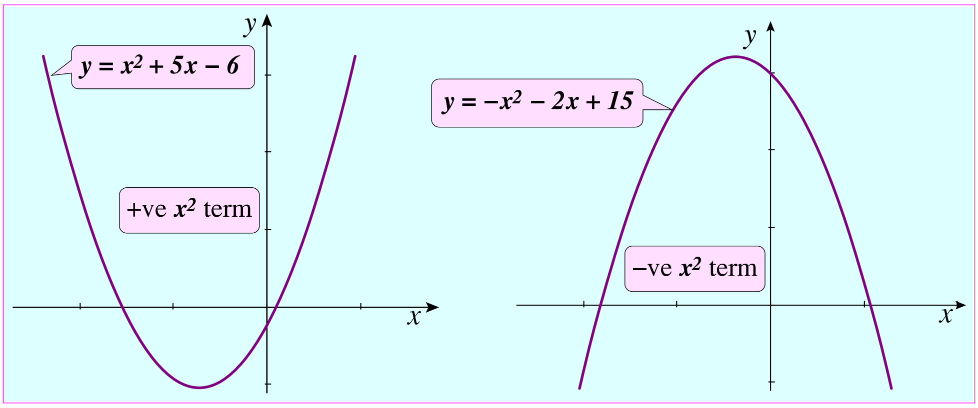
11.1 Basic Sketching Rules for any Polynomial Function

In order to sketch any graph you should know the following basic bits of information:

- The general shape of the graph according to the type of function, (∪ or '/\/' shape)
- The orientation of the graph, (∪ or ∩, '/\/' or '\/\' shape)
- The roots of the function, i.e. where it crosses the x-axis (if at all)
- Where the function crosses the y-axis, i.e. where $x = 0$
- The co-ordinates of the turning points or vertex, (max or minimum values)

11.2 General Shape & Orientation of a Quadratic

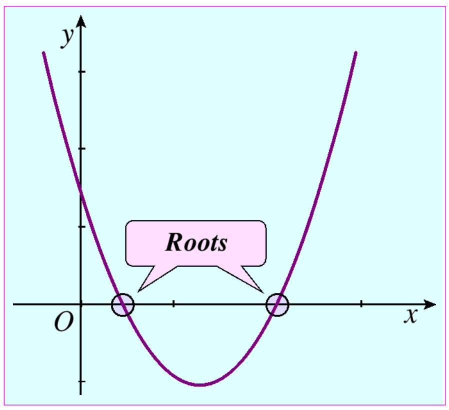
The general shape of a quadratic is a parabola. The orientation of the graph is determined by the sign of the x^2 term.



Orientation and shape of a quadratic function

11.3 Roots of a Quadratic

Using the techniques from the previous sections, find the roots of the quadratic. The discriminant can be used to find what sort of roots the quadratic has.



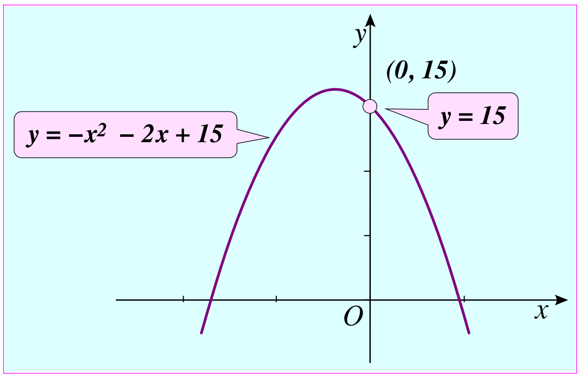
11.4 Crossing the y-axis

The function crosses the y-axis when $x = 0$

$\therefore \quad y = 15$

Co-ordinates are (0, 15)

For the standard function $ax^2 + bx + c$ the graph crosses the y-axis at c.



11.5 Turning Points (Max or Min Value)

By completing the square we can find the co-ordinates of the turning point directly:

$$y = x^2 + 4x - 6 = (x + 2)^2 - 2^2 - 6$$

$$= (x + 2)^2 - 10$$

Minimum value of the function occurs when $(x + 2)^2 = 0$, which is when $x = -2$. The quadratic is symmetrical about the line $x = -2$ and the vertex is at point $(-2, -10)$.

Note that for any other value of x then the $(x + 2)^2$ term is positive, so confirming that $x = -2$ represents a minimum.

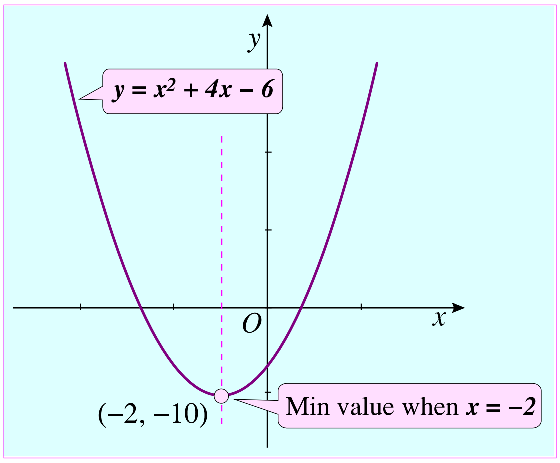
Alternatively, from the quadratic formula:

Min or max value of y is when $x = -\dfrac{b}{2a}$

For $y = x^2 + 4x - 6$

Min value of y is when $x = -\dfrac{4}{2} = -2$

$y = (-2)^2 + 4 \times (-2) - 6 = -10$



11.6 Sketching Examples

11.6.1 Example:

1 Sketch the following quadratic: $y = x^2 + 6x - 12$

Solution:

1) Note the shape of the graph: ∪
2) Crosses the y-axis at -12

This quadratic cannot be solved using basic factorisation so complete the square:

$$x^2 + 6x - 12 = \left(x + \frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 - 12$$
$$= (x + 3)^2 - 9 - 12$$
$$= (x + 3)^2 - 21$$

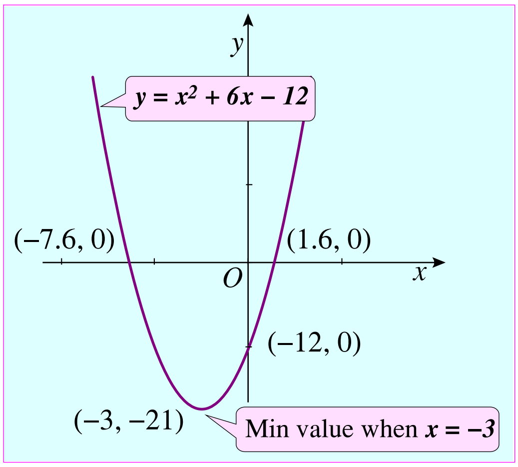
∴ A min is formed at $x = -3$

∴ Co-ordinates of the vertex is $(-3, -21)$

The curve crosses the x-axis at $(x + 3)^2 - 21 = 0$

$$(x + 3)^2 = 21$$
$$x + 3 = \pm\sqrt{21}$$
$$x = -3 + \sqrt{21} = 1{\cdot}58 \;\; (1.6 \text{ to } 2 \text{ sf})$$
$$x = -3 - \sqrt{21} = -7{\cdot}58 \;\; (-7.6 \text{ to } 2 \text{ sf})$$

Sketch and label, with all co-ordinates:



2 Sketch the following function: $f(x) = (x + 2)(x^2 - x - 6)$

Solution:

Factorise the quadratic part:

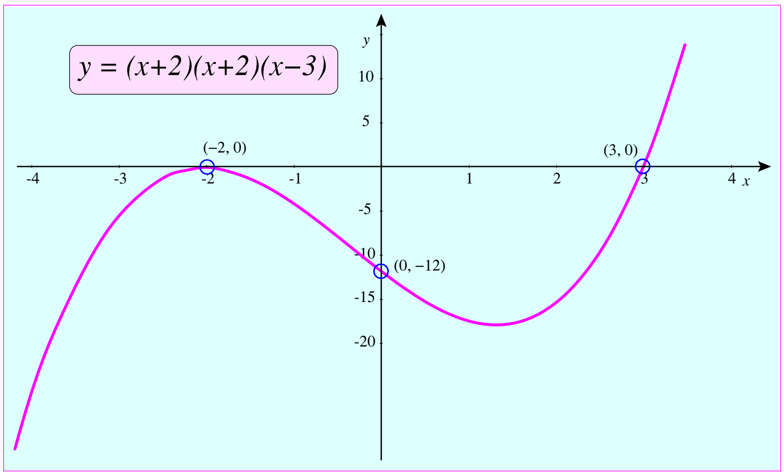
$$(x^2 - x - 6) = (x + 2)(x - 3)$$

$$\therefore \qquad f(x) = (x + 2)(x + 2)(x - 3)$$

Roots are at $x = -2, \; x = 3$

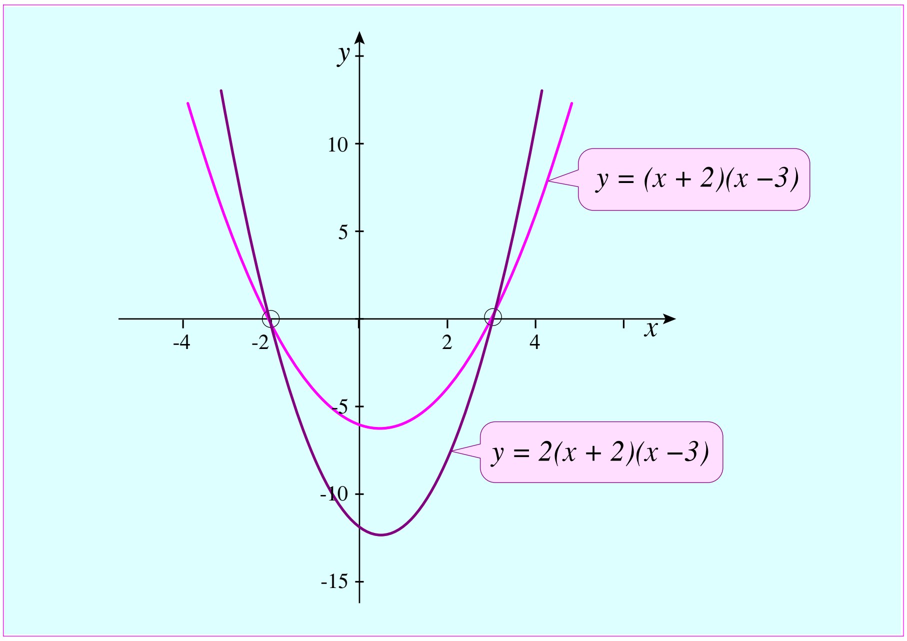
Note the double root and its effect on the sketch.

When $x = 0, \qquad f(0) = 2 \times 2 \times (-3) = -12$



11.7 Topical Tips

It is perhaps worth pointing out that a quadratic of the form $k(x + 2)(x - 3)$ will always have the same roots irrespective of the value of k.



12 • C1 • Further Quadratics

12.1 Reducing Other Equations to a Quadratic

You need to be able to recognise an equation that you can be convert to a standard quadratic form in order to solve. Just be aware that not all the solutions found may be valid.

Equations of the following forms can all be reduced to a simpler quadratic:

$$x^4 - 9x^2 + 18 = 0$$

$$\frac{8x}{x+3} = x - 3$$

$$x^{2/3} - x^{1/3} - 12 = 0$$

$$2\sqrt{x} + x - 24 = 0$$

$$\frac{18}{x^2} + \frac{3}{x} - 3 = 0$$

12.2 Reducing to Simpler Quadratics: Examples

12.2.1 Example:

1 Solve the equation: $x^4 - 9x^2 + 18 = 0$

Solution:

Let $u = x^2$

$$x^4 - 9x^2 + 18 \Rightarrow u^2 - 9u + 18 = 0$$

$$u^2 - 9u + 18 = 0$$

$$(u - 3)(u - 6) = 0$$

$$\therefore \quad u = 3 \quad or \quad u = 6$$

$$\therefore \quad x^2 = 3 \quad or \quad x^2 = 6$$

$$x = \pm\sqrt{3} \quad or \quad x = \pm\sqrt{6}$$

2 Solve $\dfrac{8x}{x+3} = x - 3$

Solution:

$$\frac{8x}{x+3} = x - 3$$

$$8x = (x - 3)(x + 3)$$

$$8x = x^2 - 9$$

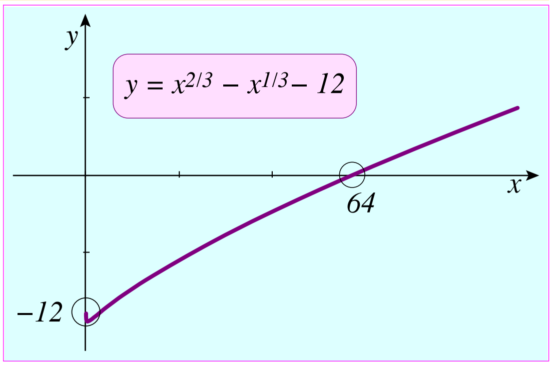
$$x^2 - 8x - 9 = 0$$

$$(x - 9)(x + 1) = 0$$

$$x = 9 \quad or \quad x = 1$$

3 Solve the equation: $x^{2/3} - x^{1/3} - 12 = 0$

(Graph produced for reference only - not usually given in the question)



Solution:

Let $u = x^{1/3}$

N.B. $x^{2/3} = \left(x^{1/3}\right)^2$

$$x^{2/3} - x^{1/3} - 12 \Rightarrow u^2 - u - 12 = 0$$

$$u^2 - u - 12 = 0$$

$$(u - 4)(u + 3) = 0$$

$\therefore \quad u = 4 \quad or \quad u = -3$

$\therefore \quad x^{1/3} = 4 \quad or \quad x^{1/3} = -3$

$\therefore \quad x = 4^3 \quad or \quad (-3)^3$

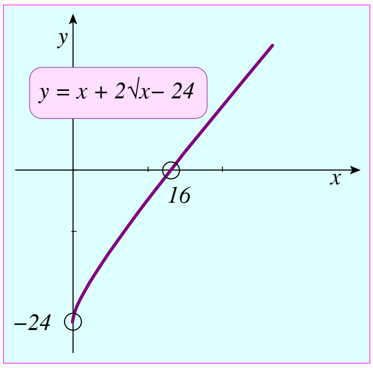
$\therefore \quad x = 64 \quad or \quad -27$

The only solution is $x = 64$

4

Solve $2\sqrt{x} + x - 24 = 0$

(Graph produced for reference only - not usually given in the question)



Solution:

Let $u^2 = x$

$\therefore \quad u = \sqrt{x}$

$$x + 2\sqrt{x} - 24 = 0$$

$$u^2 + 2u - 24 = 0$$

$$(u + 6)(x - 4) = 0$$

$\therefore \quad u = -6 \quad or \quad u = 4$

However, values of x less than zero are not allowed because of the square root term, therefore, a negative value of u is also not allowed.

$\therefore \quad x = u^2 \Rightarrow x = 16$

Substitute back into the original equation to check:

$$16 + 2 \times 4 - 24 = 0$$

Alternatively, rearrange, then square to remove the root

$$(2\sqrt{x})^2 = (24 - x)^2$$

$$4x = (24 - x)^2$$

$$4x = 24^2 - 48x + x^2$$

$$x^2 - 52x + 24^2 = 0$$

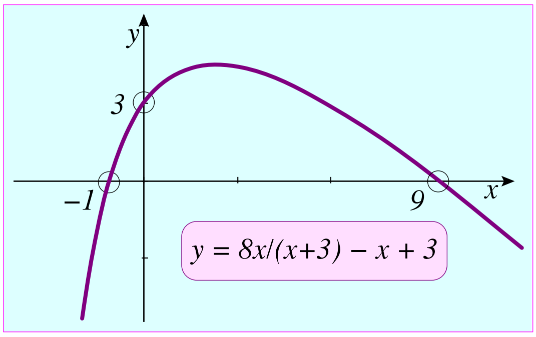
$$(x - 16)(x - 36) = 0$$

$$x = 16 \text{ or } 36$$

Substitute the values back into the original equation to test for a valid answer.

5

Solve $\dfrac{8x}{x + 3} = x - 3$



Solution:

$$\frac{8x}{x + 3} = x - 3$$

$$8x = (x - 3)(x + 3)$$

$$8x = x^2 - 9$$

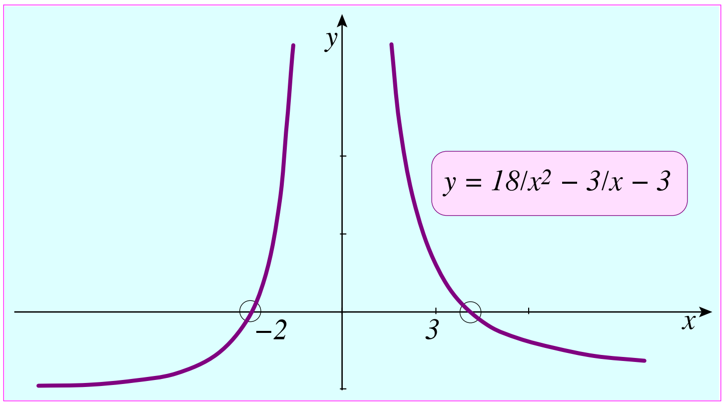
$$x^2 - 8x - 9 = 0$$

$$(x - 9)(x + 1) = 0$$

$$x = 9 \quad or \quad x = -1$$

6

Solve $\dfrac{18}{x^2} + \dfrac{3}{x} - 3 = 0$



Solution:
Multiply by x^2

$$\frac{18}{x^2} + \frac{3}{x} - 3 = 0$$

$$18 + 3x - 3x^2 = 0$$

$$3x^2 - 3x - 18 = 0$$

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3 \quad or \quad x = -2$$

7 Solve $8x^3 + \frac{1}{x^3} = -9$

Solution:

$$8x^3 + \frac{1}{x^3} = -9$$

$$8x^3 \times x^3 + \frac{x^3}{x^3} = -9 \times x^3$$

$$8x^6 + 1 = -9x^3$$

$$8x^6 + 9x^3 + 1 = 0$$

Let $u = x^3$

$$8u^2 + 9u + 1 = 0$$

$$(8u + 1)(u + 1) = 0$$

$$u = -\frac{1}{8} \quad or \quad u = -1$$

$$x^3 = -\frac{1}{8} \quad or \quad x^3 = -1$$

$$x = \sqrt[3]{-\frac{1}{8}} \quad or \quad x = \sqrt[3]{-1}$$

$$x = -\frac{1}{2} \quad or \quad x = -1$$

8 Given that $y = x^{1/3}$ show that $2x^{1/3} + 4x^{-1/3} = 9$ can be written as $2y^2 - 9y + 4 = 0$

Solution:

$$2x^{\frac{1}{3}} + \frac{4}{x^{\frac{1}{3}}} = 9 \qquad \text{Rewrite equation}$$

$$2y + \frac{4}{y} = 9 \qquad \text{Substitute}$$

$$2y^2 + 4 = 9y$$

$$2y^2 - 9y + 4 = 0 \qquad \textit{QED}$$

Solve for x:

$$(2y - 1)(y - 4) = 0$$

$$y = \frac{1}{2} \quad or \quad y = 4$$

but $y = x^{\frac{1}{3}}$

$$x^{\frac{1}{3}} = \frac{1}{2} \quad or \quad x^{\frac{1}{3}} = 4$$

Hence $x = \frac{1}{8} \quad or \quad x = 64$

9

10

12.3 Pairing Common Factors

For some expressions it is possible to find solutions by taking out common factors from pairs of terms.

12.3.1 Example:

1 Factorise: $st + 3t - 5s - 15$

$$st + 3t - 5s - 15$$

$$st - 5s + 3t - 15$$

$$s(t - 5) + 3(t - 5)$$

$$(t - 5)(s + 3)$$

2 Factorise: $3mn - 6m - n^2 + 2n$

$$3mn - 6m - n^2 + 2n$$

$$3m(n - 2) - (n^2 - 2n)$$

$$3m(n - 2) - n(n - 2)$$

$$(n - 2)(3m - n)$$

13 • C1 • Simultaneous Equations

13.1 Solving Simultaneous Equations

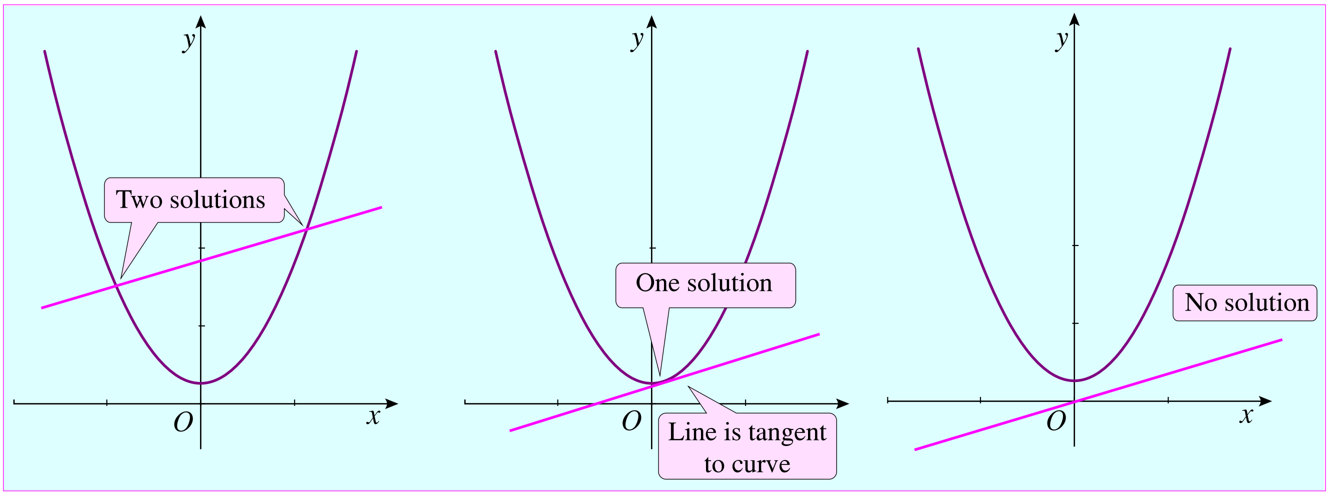
At GCSE level we learnt that there were three methods to solve linear simultaneous equations. These are the:

- Elimination method
- Substitution method
- Graphical method

At A level, simultaneous equations are extended to include solving a linear and a quadratic equation simultaneously. The substitution method is the method of choice, although a sketch of the functions involved is always helpful to ensure correct thinking.

With two linear simultaneous equations there can only be one solution at the intersection of the two lines, however, with a linear and a quadratic equation there may be two, one or no solution available.

In a sense, solving a normal quadratic for its roots is the same as solving for two equations, the given quadratic function and the linear equation of $y = 0$.



Solutions for a linear and a quadratic equation

13.2 Simultaneous Equations: Worked Examples

13.2.1 Example:

1 Find the co-ordinates where $y = 2x - 1$ meets $y = x^2$

$$2x - 1 = x^2 \quad \Rightarrow \quad x^2 - 2x + 1 = 0$$

$$(x - 1)(x - 1) = 0$$

$$\therefore \quad x = 1$$

$$y = 2 - 1$$

$$y = 1$$

Answer: $= (1,1) \leftarrow$ tangent

Since there is only one solution (or two equal solutions) then the line must be a tangent to the curve.

2 Find the co-ordinates of the points where $y = x^2 - 2x - 6$ meets $y = 12 + x - 2x^2$.

Solution:

$$12 + x - 2x^2 = x^2 - 2x - 6$$
$$3x^2 - 3x - 18 = 0$$
$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = -2, \text{ and } x = 3$$
$$y = 4 + 4 - 6 = 2$$
$$y = 9 - 6 - 6 = -3$$

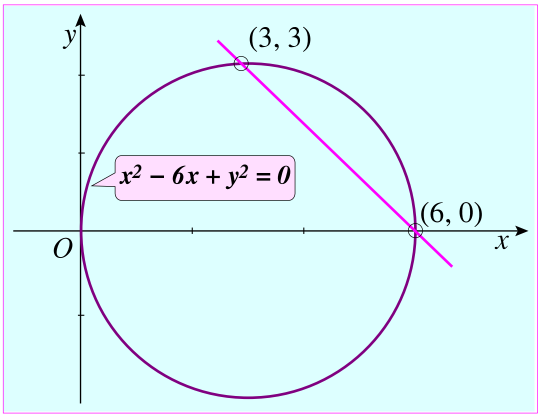
When $x = -2,\ y = 2,$ and when $x = 3,\ y = -3$

Answer: $= (-2,\ 2)$ and $(3, -3)$

3 Find the co-ordinates of the points where $x + y = 6$ meets $x^2 - 6x + y^2 = 0$.

$$x + y = 6 \quad \Rightarrow \quad y = 6 - x$$
$$x^2 - 6x + (6 - x)^2 = 0$$
$$x^2 - 6x + 36 - 12x + x^2 = 0$$
$$2x^2 - 18x - 36 = 0$$
$$x^2 - 9x - 18 = 0$$
$$(x - 3)(x - 6) = 0$$
$$\therefore \quad x = 3 \textit{ or } 6$$
$$y = 6 - x$$
$$\therefore \quad y = 3 \textit{ or } 0$$

Co-ordinates of intersection are $(6,\ 0)$ *and* $(3,\ 3)$



4 Prove that $y = 6x - 5$ is tangent to $y = x^2 + 2x - 1$

Let $x^2 + 2x - 1 = 6x - 5$

$$x^2 - 4x + 4 = 0$$
$$(x - 2)(x - 2) = 0$$
$$\therefore \quad x = 2 \quad \Rightarrow \quad y = 12 - 5 = 7$$

Only one solution, therefore tangent is at point (2, 7)

14 • C1 • Inequalities

14.1 Intro

An inequality compares two unequal quantities.

The method of solving inequalities varies depending on whether it is linear or not. All solutions of inequalities give rise to a range of solutions.

14.2 Rules of Inequalities

- Numbers can be added or subtracted to both side of the inequality as normal
- Both sides of the inequality can be multiplied or divided by a **positive** number, as normal
- If both sides are multiplied or divided by a **negative** number, the inequality is reversed
- If both sides of the inequality are transposed the inequality is also reversed

  e.g. $y < 6$ is the same as $6 > y$.

$$a + k > b + k \quad \text{for all values of } k$$

$$ak > bk \quad \text{for all +ve values of } k$$

$$ak < bk \quad \text{for all −ve values of } k$$

Note that the direction of the symbols indicates direction on the number line.

14.3 Linear Inequalities

For a linear inequality, the solution has only one range, and only one *boundary*.

14.3.1 Example: 1

1 Solve: $\dfrac{3 - 5x}{4} \geqslant -8$

$$3 - 5x \geqslant -32$$

$$-5x \geqslant -35$$

$$x \leqslant 7$$

2 Find the range of values for x that satisfy both the inequalities $7x - 4 \leqslant 8x - 8$ and $3x > 4x - 8$.

$$7x - 4 \leqslant 8x - 8 \qquad (1)$$

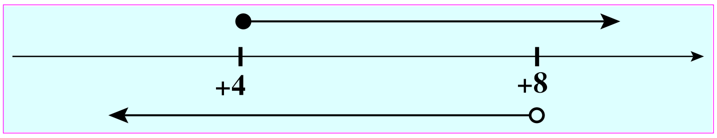
$$3x > 4x - 8 \qquad (2)$$

Evaluate (1) $7x - 8x \leqslant 4 - 8$

$$-x \leqslant -4$$

$$x \geqslant 4$$

Evaluate (2) $-x > -8 \quad \Rightarrow \quad x < 8$



Combine results from (1) & (2): $4 \leqslant x < 8$

14.4 Quadratic Inequalities

For a quadratic inequality, the solution has one or two ranges of solutions, with two *boundaries*.

There are two methods available for solving inequalities for quadratic or higher powers:

- Sketching: Factorise, sketch and read off the required regions
- Critical Values Table: Factorise, find critical values, construct table and read off the required regions

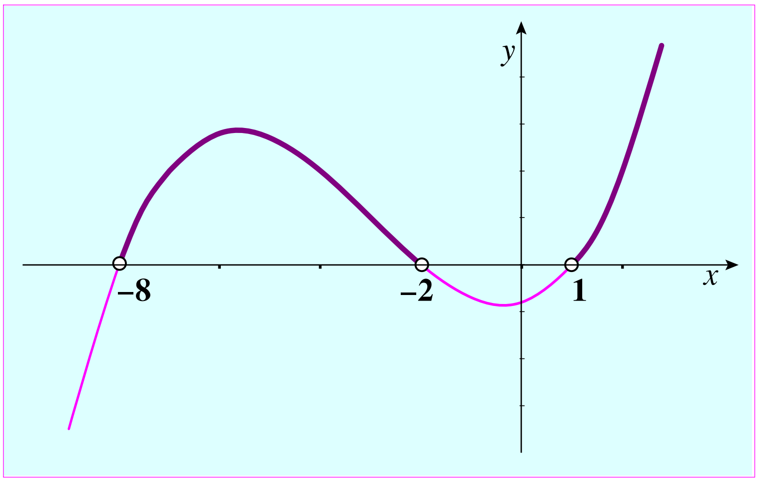
Note that if the quadratic has a positive x^2 term and arranged to be $<$ or $\leqslant 0$ then there is only one range for the solution. If the inequality is $>$ or $\geqslant 0$ then there are two ranges for the solution.

14.4.1 Sketching Method

14.4.1.1 Example:

1 e.g. Solve: $(x - 1)(x + 2)(x + 8) > 0$

This means we want any regions above the x-axis, (darker line and open points of intercept). Draw a sketch to indicate the correct regions.



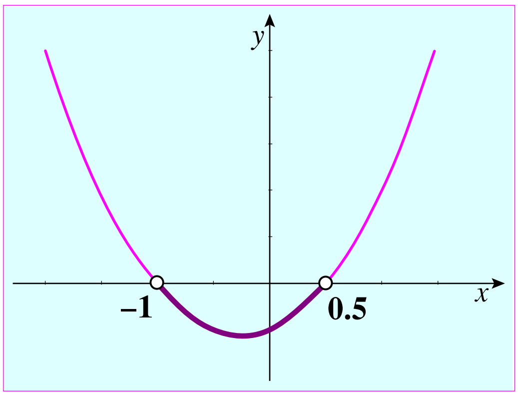
$$-8 < x < -2 \qquad \text{AND} \qquad x > 1$$

2 Solve $2x^2 + 3 - 1 < 0$

This means the region below the x-axis (darker line), not including the axes, (hence the open points at the intercept).

Factorising gives:

$$(2x - 1)(x + 1) < 0$$



$$\therefore \quad \textit{Answer} : -1 < x < \frac{1}{2}$$

14.4.2 Critical Values Table

This is a longer method, but one which is recommended when:

- you don't know what the sketch would look like or
- you are told to in the question.

Typical order of method:

- Rearrange for 0
- Factorise
- Critical values are where each factor = 0
- Arrange critical values in order (similar to a number line)
- Make table, marking positive and negative segments

14.4.2.1 Example:

1 Solve using table of critical values the inequality $x(x + 3)(x - 4) \geqslant 0$

Critical values are: 0, –3, and 4

Build table and note when expression is $\geqslant 0$:

Critical values:	–3	0	4	
	$x < -3$	$-3 < x < 0$	$0 < x < 4$	$x > 4$
x	–	–	+	+
$x + 3$	–	+	+	+
$x - 4$	–	–	–	+
$(x - 1)(x + 2)(x + 8)$	–	+	–	+

Answer: $-3 \leqslant x \leqslant 0$, AND $x \geqslant 4$

2 Find the values of k for which $kx^2 + 3kx + 5 = 0$ has two distinct roots:

$$b^2 - 4ac > 0 \quad \Rightarrow \quad (3k)^2 - 4 \times k \times 5 > 0$$

$$9k^2 - 20k > 0 \quad \Rightarrow \quad k(9k - 20) > 0$$

Critical values are 0, and $\frac{20}{9}$

	$k < 0$	$0 < k < \frac{20}{9}$	$k > \frac{20}{9}$
k	–	+	+
$9k - 20$	–	–	+
$k(9k - 20) > 0$	+	–	+

Answer: $k < 0$ AND $k > 20$

14.5 Inequality Examples

14.5.1 Example:

1 Find the values of k for which the quadratic $x^2 + (k - 1)x + k + 2 = 0$ has no roots.

Solution:

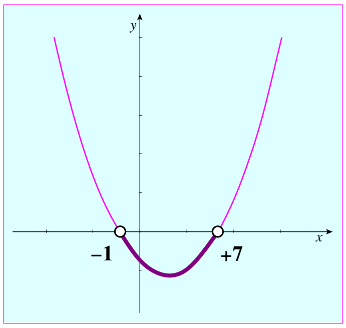
Consider the discriminant when less than 0:

$$b^2 - 4ac < 0 \quad \Rightarrow \quad (k-1)^2 - 4(k+2) < 0$$

$$k^2 - 2k + 1 - 4k - 8 < 0$$

$$k^2 - 6k - 7 < 0$$

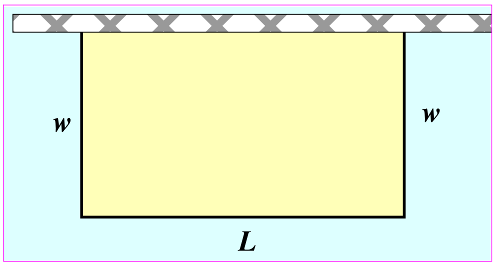
$$(k-7)(k+1) < 0$$



Answer: $-1 < k < 7$

2 A farmer has 90m of fencing and needs to construct a fence around a rectangular piece of ground, that is bounded by a stone wall. With a width of w and length L, what is the range of values that L can take if the area enclosed is a minimum of 1000m^2.

Solution:



Area: $Lw \geqslant 1000$

Length of fence: $2w + l = 90$

$\therefore \quad 2w = 90 - L$

$$w = 45 - \frac{L}{2}$$

$$\therefore \quad L\left(45 - \frac{L}{2}\right) \geqslant 1000$$

$$45L - \frac{L^2}{2} - 1000 \geqslant 0$$

$$90L - L^2 - 2000 \geqslant 0$$

$$L^2 - 90L + 2000 \leqslant 0$$

Critical values are:

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$L = \frac{-(-90) \pm \sqrt{90^2 - 4 \times 2000}}{2}$$

$$L = 45 \pm 5$$

Hence: $40 \leqslant L \leqslant 50$

3 Find the values of x for which: $5x + 1 > 7x - 7$ and $x^2 - 6x \leqslant 16$

Solution:

$$5x + 1 > 7x - 7$$
$$5x - 7x > -1 - 7$$
$$-2x > -8$$
$$x < 4$$

$$x^2 - 6x \leqslant 16$$
$$x^2 - 6x - 16 \leqslant 0$$
$$(x - 8)(x + 2) \leqslant 0$$
$$\therefore x \leqslant 8 \qquad x \geqslant -2$$
$$-2 \leqslant x \leqslant 8$$

Combining the two inequalities:

$$-2 \leqslant x < 4$$

4 This is an example that includes simultaneous equations, discriminants and inequalities.
A curve and a straight line have the following equations: $y = x^2 + 5$ and $y = k(3x + 2)$.
Find an equation in terms of x and k, that shows the x-coordinates of the points of intersection. If this equation has two distinct solutions, write an equation to show this and solve any inequality.

Solution:

$$x^2 + 5 = k(3x + 2)$$
$$x^2 - 3kx + 5 - 2k = 0$$

For 2 solutions $b^2 - 4ac > 0$

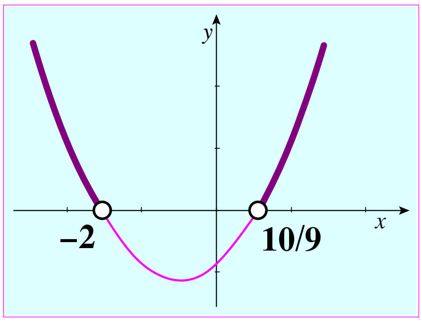
$$(-3k)^2 - 4(5 - 2k) > 0$$
$$9k^2 + 8k - 20 > 0$$

Solve to find the critical values:

$$9k^2 + 8k - 20 = 0$$
$$\left(x - \frac{10}{9}\right)\left(x + \frac{18}{9}\right) = 0$$
$$(9x - 10)(x + 2) = 0$$
$$x = \frac{10}{9} \qquad \text{AND} \qquad x = -2$$

$$\therefore \quad k < -2, \qquad \text{AND} \qquad k > \frac{10}{9}$$

$9 \times 20 = 180$
9×20
$-10 \times +18 \quad \Rightarrow +8 \qquad$ Factors
12×15



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14.6 Heinous Howlers

Do not cross multiply by $(x + a)$ since you don't know if x is −ve or +ve, however multiplying by $(x + a)^2$ will be fine, as it gives a positive result in each case.

14.6.1 Example:

$$\frac{x - 3}{x + 5} < 4$$

Not this: $x - 3 \nless 4(x + 5)$ ✘

But this: $\frac{x - 3}{x + 5} \times (x + 5)^2 < 4(x + 5)^2$

$$\therefore \quad (x - 3)(x + 5) < 4(x + 5)^2$$

$$(x - 3)(x + 5) - 4(x + 5)^2 < 0$$

$$(x + 5)\left[(x - 3) - 4(x + 5)\right] < 0$$

$$(x + 5)\left(x - 3 - 4x - 20\right) < 0$$

$$(x + 5)\left(-3x - 23\right) < 0$$

$$-(x + 5)\left(3x + 23\right) < 0$$

$$(x + 5)\left(3x + 23\right) > 0$$

$$x > -5 \quad \text{AND} \quad x < -7\frac{2}{3}$$

Do not get the inequality reversed.

So DO NOT put $3 < k < -2$ when you mean $-2 < k < 3$.

Think of the number line (or x-axis) when writing inequalities.

14.7 Topical Tips

For any quadratic with a +ve x^2 term, if the inequality is < 0 or $\leqslant 0$ then there is only one region of inequality e.g. $-1 < k < 7$

If the inequality is > 0 then there are two regions of inequality e.g. $k < -2$, AND $k > 10$

15 • C1 • Standard Graphs I

15.1 Standard Graphs

You must be familiar with all these basic graphs. From these basic graphs, other graphs can be deduced using transformations. In addition to direct questions on sketching graphs, it is well worth sketching a graph when answering a question, just to clarify your thinking. ('A picture is worth a 1000 words' as they say).

15.2 Asymptotes Intro

Asymptotes are straight lines on a graph that a curve approaches, but never quite reaches and does not cross. They represent values of x or y for which the function has no solution.

A vertical asymptote of $x = a$ is drawn when the function $f(x)$ approaches ± infinity, as x approaches a. Written as $f(x) \to \pm\infty$ as $x \to a$.

A horizontal asymptote of $y = b$ is drawn when the function $f(x)$ approaches b, as x approaches ± infinity. Written as $f(x) \to b$ as $x \to \pm\infty$.

Asymptotes are usually associated with rational functions (i.e. fractions) and exponentials.

Example:

Draw the asymptotes for $y = \dfrac{2x^2}{(x^2 - 1)}$

$$y = \frac{2x^2}{(x^2 - 1)} = \frac{2x^2}{(x - 1)(x + 1)}$$

Function has no solution (undetermined) when

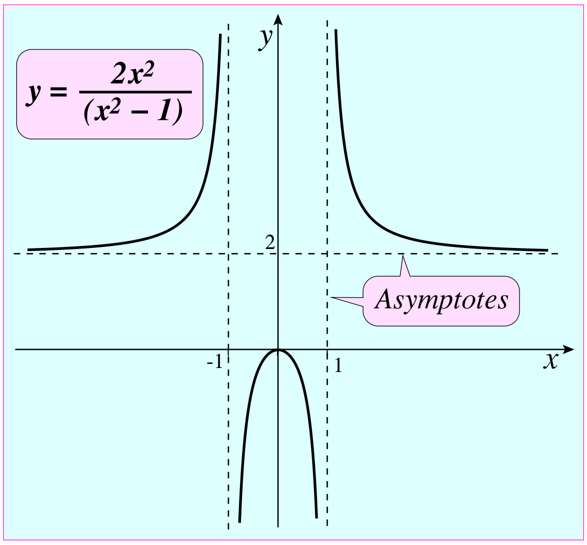
$x = 1$ or $x = -1$ and $y \to \pm\infty$

If $x \to \pm\infty$ then $y \to \dfrac{2 \times \infty}{\infty} \to 2$

Asymptotes appear at:

$x = 1, x = -1$ and $y = 2$

See later subsection on Finding Asymptotes.



15.3 Power Functions

Power functions of degree n have the general form of:

$$y = x^n$$

- All even-degree power functions ($y = x^{even}$) are classed as even functions, because the axis of line symmetry is the y-axis i.e. they are symmetrical about the y-axis. Curves pass through the origin and through the points $(-1, 1)$ and $(1, 1)$
- All odd-degree power functions ($y = x^{odd}$) are classed as odd functions, because they have rotational symmetrical about the origin. Curves pass through the origin and through the points $(-1, -1)$ and $(1, 1)$
- All even-degree polynomials behave like quadratics with the typical 'bucket' shape, and all odd-degree polynomials behave like cubics with a typical '/\/' shape. As the power increases, so the shape of the curve becomes steeper.
- The sign of the highest power determines the orientation of the graph:
  - For even-degree power functions, a positive coefficient gives a ∪ (upright bucket) shape whilst a negative coefficient gives a ∩ (empty bucket) shape.
  - For odd-degree power functions, a positive coefficient gives the typical '/\/' shape, whilst a negative coefficient gives a '\/\' shape.
- Note the starting points of the curves on the LHS.

15.3.1 Even Power Functions

A basic even power function function is given by:

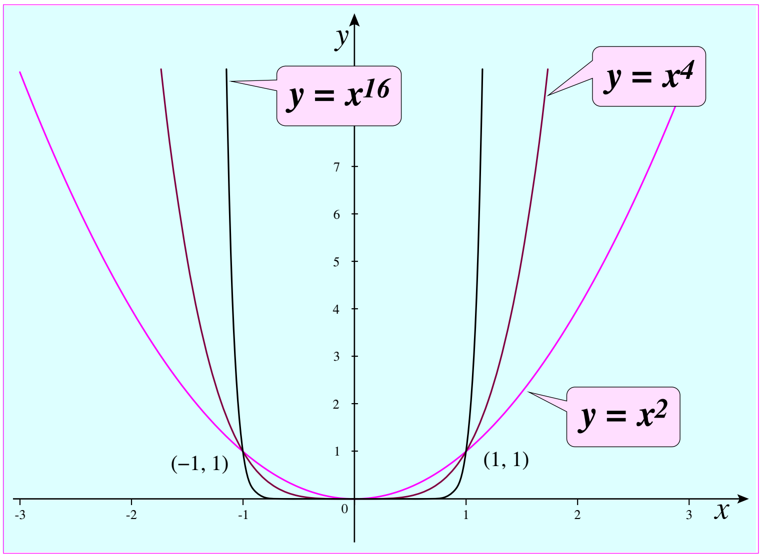
$$y = x^{even}$$

The function and graph is 'even' because the axis of line symmetry is the y-axis and:

$$f(x) = f(-x)$$

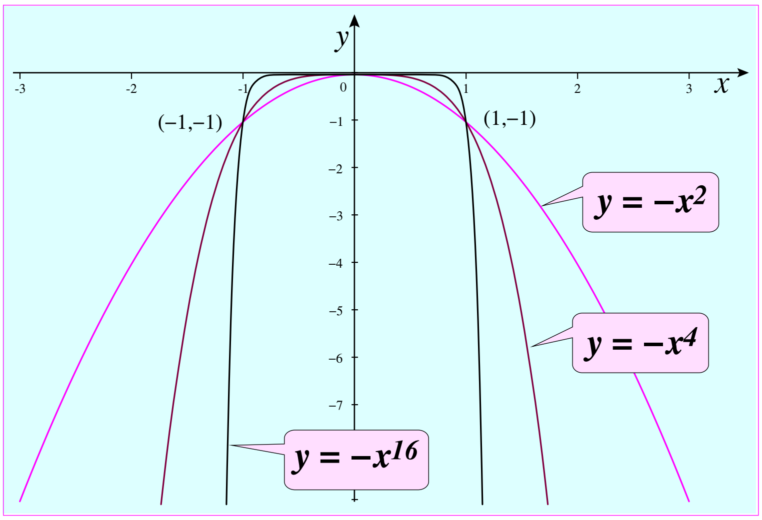
(This is a transformation with a reflection in the y-axis).

Curves with a +ve coefficient pass through the origin and through the points $(-1, 1)$ and $(1, 1)$



Even Power Functions – positive coefficient

Curves with a –ve coefficient pass through the origin and through the points $(-1, -1)$ and $(1, -1)$



Even Power Functions – negative coefficient

15.3.2 Odd Power Functions

A basic odd power function is given by:

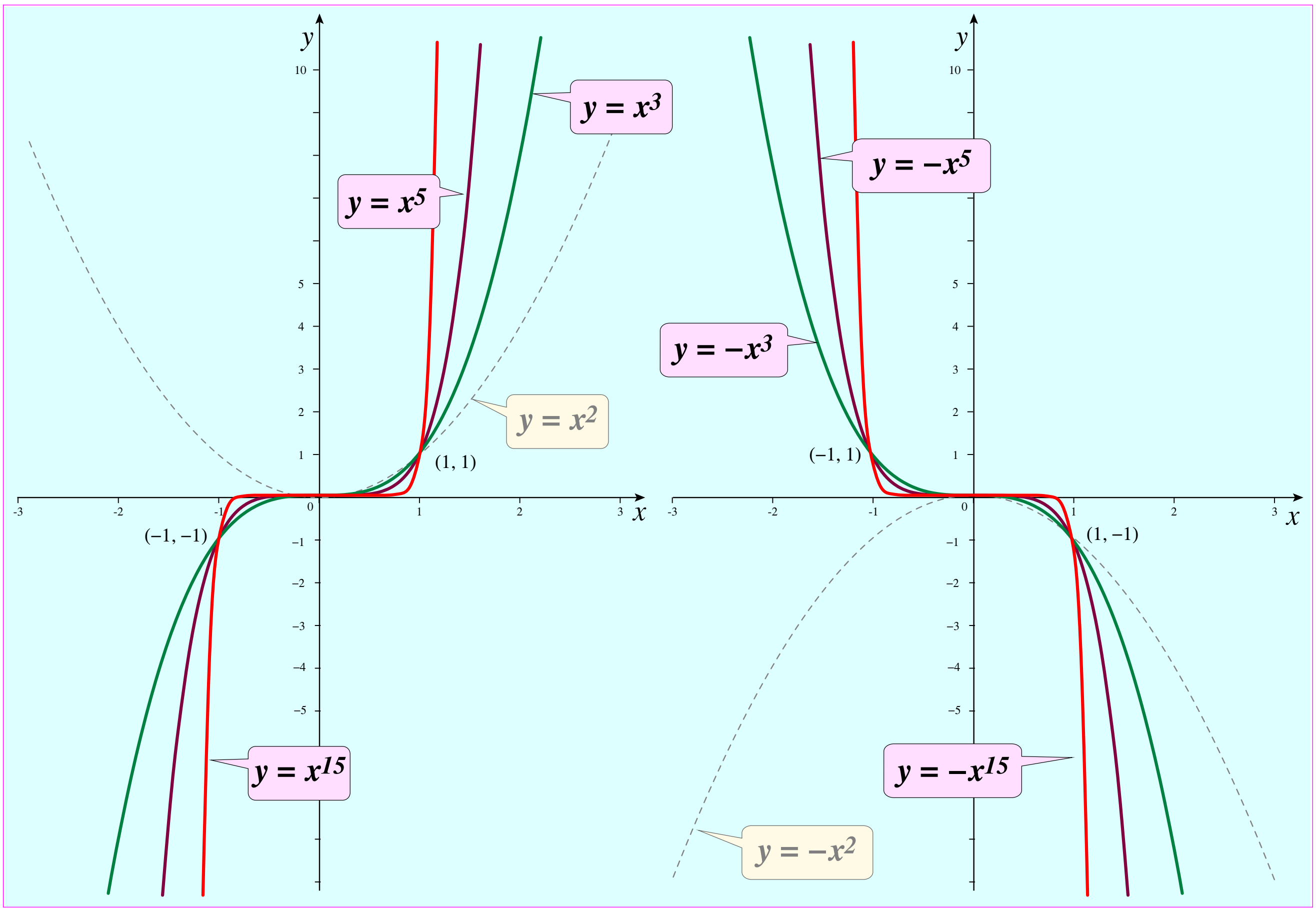
$$y = x^{odd}$$

Odd power functions have a familiar '/\/' shape or if x has a negative coefficient a '\/\' shape.
The function and graph is 'odd' because it has rotational symmetry about the origin and:

$$f(x) = -f(-x)$$

[This is equivalent to two transformations with a reflection in both the x-axis and y-axis].

Curves with a +ve coefficient pass through the origin and through the points (−1, −1) and (1, 1).
Curves with a −ve coefficient pass through the origin and through the points (−1, 1) and (1, −1).



Odd Power Functions

15.3.3 Quadratic Function

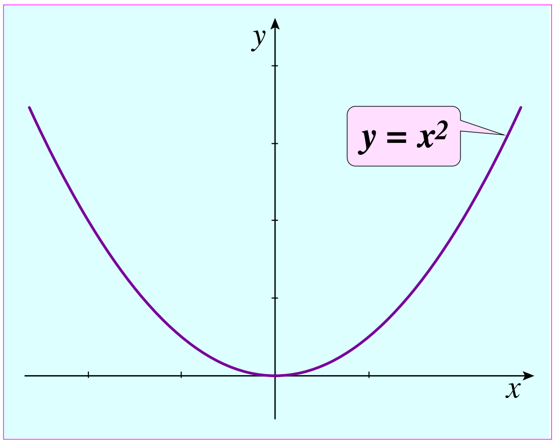
A basic quadratic function is given by:

$$y = x^2$$

This is a second order polynomial function, also called a parabola.
The function and graph is 'even' because the axis of line symmetry is the y-axis and:

$$f(x) = f(-x)$$

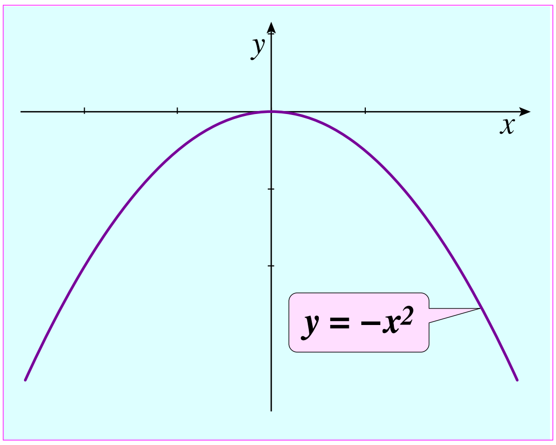
A +ve coefficient of x^2 gives the familiar ∪ shape, with one minimum value for the function.



The graph becomes an 'empty bucket' or ∩ shape when the squared term is negative:

$$y = -x^2$$

A −ve coefficient of x^2 gives one maximum value for the function.



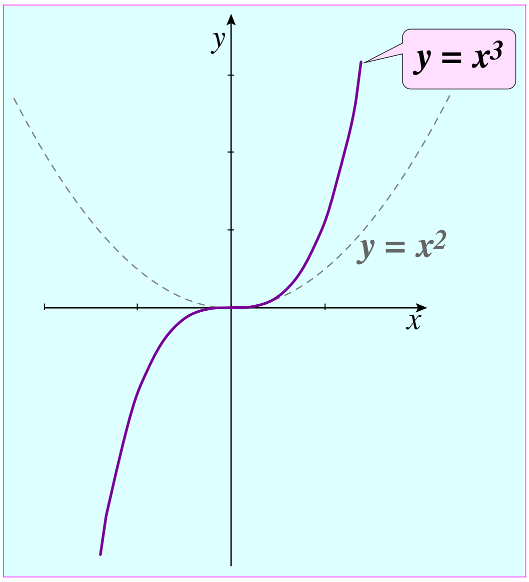
15.3.4 The Cubic Function

A basic cubic function is given by:

$$y = x^3$$

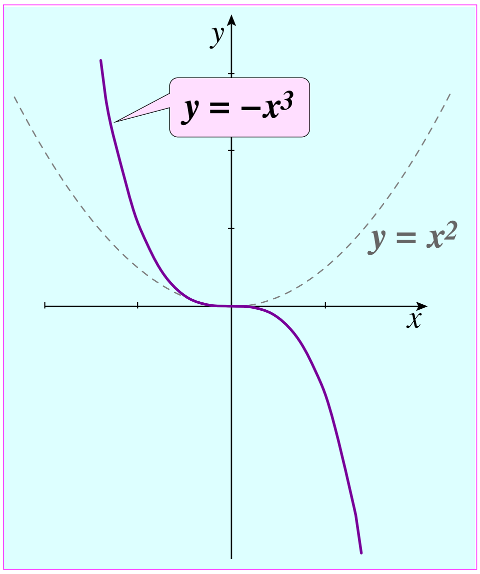
This is a third order polynomial function, with has a familiar '/\/' shape.
The function and graph is 'odd' and has rotational symmetry about the origin and:

$$f(x) = -f(-x)$$



The graph becomes an 'mirror image' with a '\/\' shape, when the cubed term is negative:

$$y = -x^3$$



15.3.5 The Quartic Function

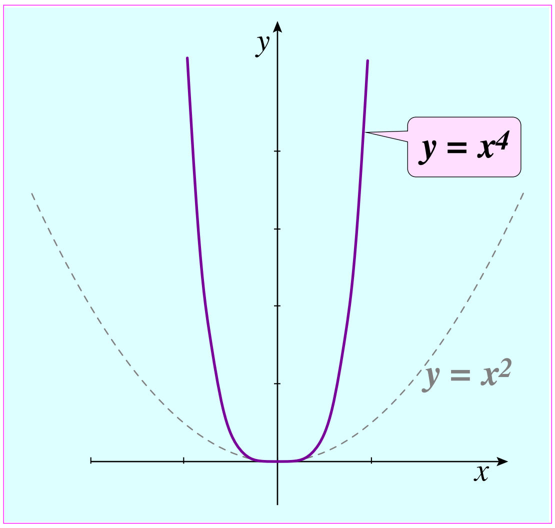
A basic Quartic function is given by:

$$y = x^4$$

This is a fourth order polynomial function.
The function and graph is 'even' because the axis of symmetry is the y-axis and:

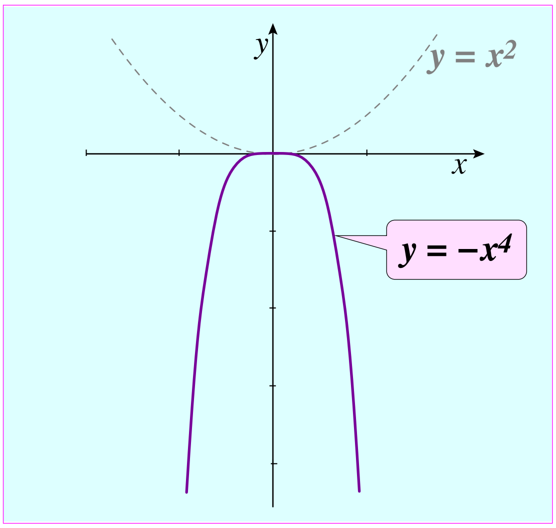
$$f(x) = f(-x)$$

A +ve coefficient of x^2 gives the familiar ∪ shape.



The graph becomes an 'empty bucket' or ∩ shape when the power term is negative:

$$y = -x^4$$



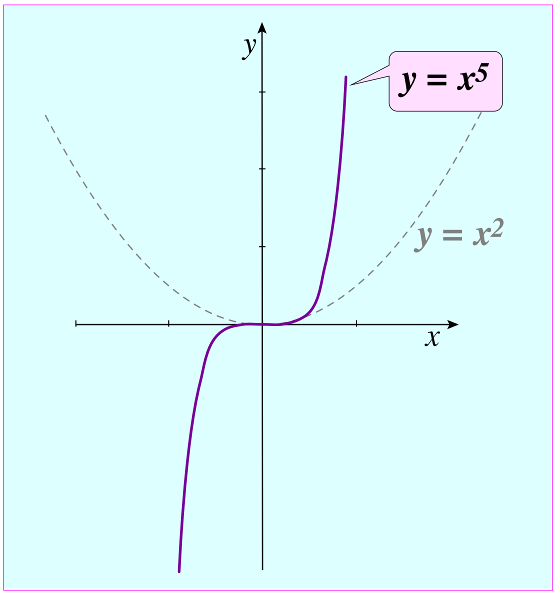
15.3.6 The Fifth Order Function

This is a fifth order polynomial function, with a familiar '⁄\/' shape. The basic function is:

$$y = x^5$$

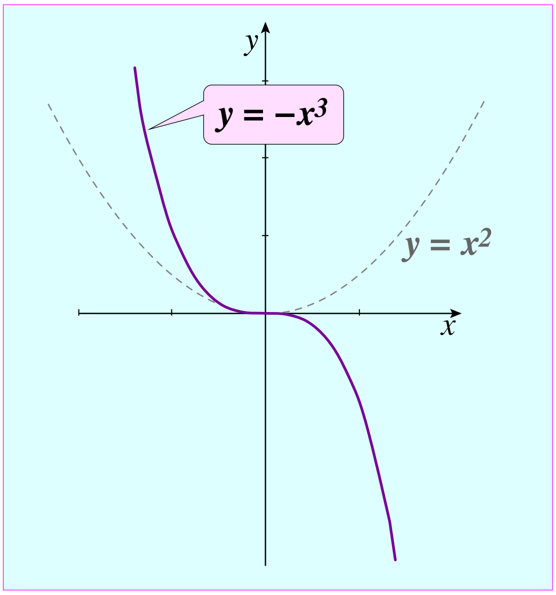
The function and graph is 'odd' and has rotational symmetry about the origin and:

$$f(x) = -f(-x)$$



The graph becomes an 'mirror image' with a '\/\' shape, when the power term is negative:

$$y = -x^5$$



15.3.7 General Polynomial Curves

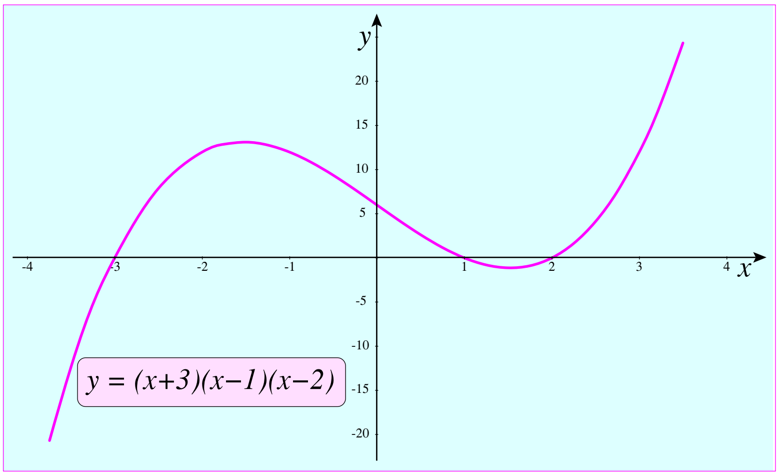
The previous graphs have been pure power functions with only one term. Adding more terms to the power function changes the shape of the curve somewhat, but the overall shape of the curve remains.

- The overall shape of a general polynomial graph is determined by the highest power less one:
- A cubic function will take a shape with two turning points '/\/', a fifth order function will have 4 turning points '/\/\/' etc.
- A quartic function will take on a typical '\/\/' shape with 3 turning points and so on.
- Note that some of these turning points may be disguised as inflection points or coincident roots, see the graph for $y = x^4$ for example (more in C2)

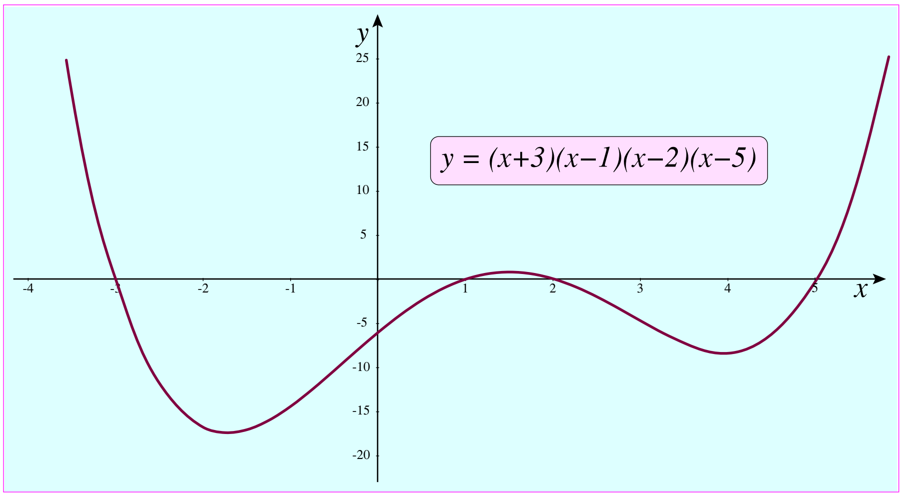
Function	*Order*	*Shape*	*Turning Points*
$ax^2 + bx + c$	*Second*	∪	1
$ax^3 + bx^2 + cx + d$	*Third*	/\/	2
$ax^4 + bx^3 + \ldots$	*Fourth*	\/\/	3
$ax^5 + bx^4 + \ldots$	*Fith*	/\/\/	4
etc			

A cubic equation has a rotational order of symmetry of 2, about the point where $x = -\dfrac{b}{3a}$

For a cubic equation with factors p, q, and r, i.e. $y = (x - p)(x - q)(x - r)$, and if any two factors are the same, then the x-axis will be tangent to the curve at that point.



Cubic or Third Order Polynomial



Fourth Order Polynomial

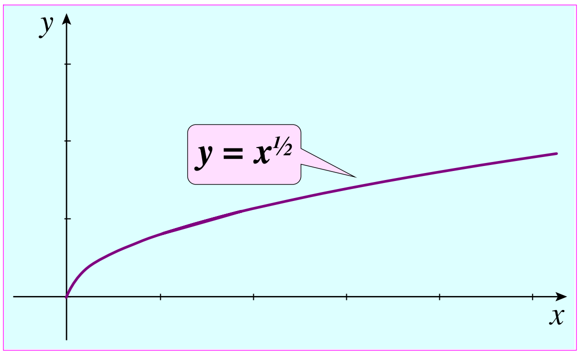
15.4 Roots and Reciprocal Curves

15.4.1 Square Root Function

The basic Square Root function is:

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

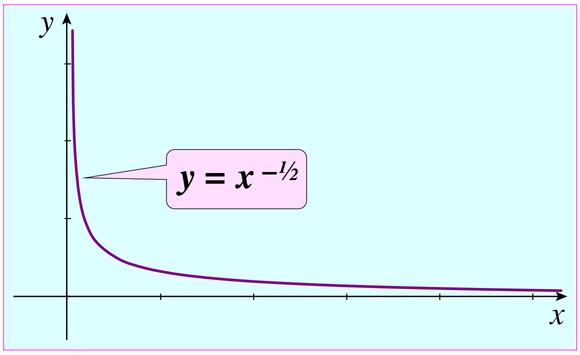
The square root function has no symmetry.
The function is neither even nor odd.



Inverse Square Root Function

$$y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

The graph has rotational symmetry about the origin, so this function is odd.
Asymptotes at the x-axis and y-axis.

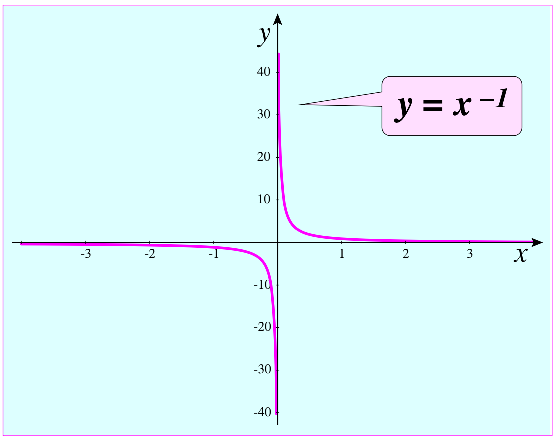


15.4.2 Reciprocal Functions

Inverse or reciprocal function:

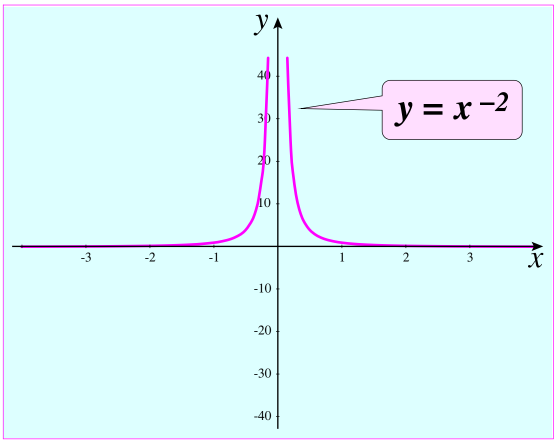
$$y = \frac{1}{x} = x^{-1}$$

The graph has rotational symmetry (order 2) about the origin, so this function is odd.
Asymptotes at the x-axis and y-axis.



$$y = \frac{1}{x^2} = x^{-2}$$

The function and graph is ‘even’ because the axis of line symmetry is the y-axis and so this function is odd.
Asymptotes at the x-axis and y-axis.

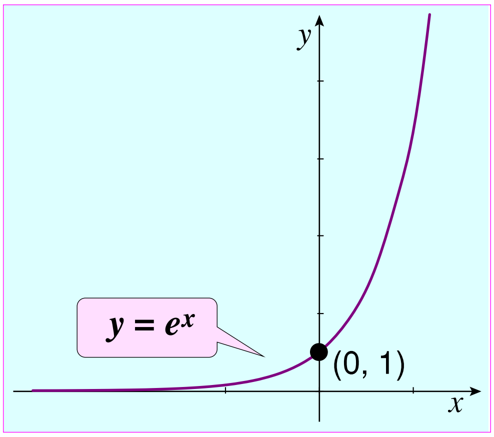


15.5 Exponential and Log Function Curves

15.5.1 Exponential Function

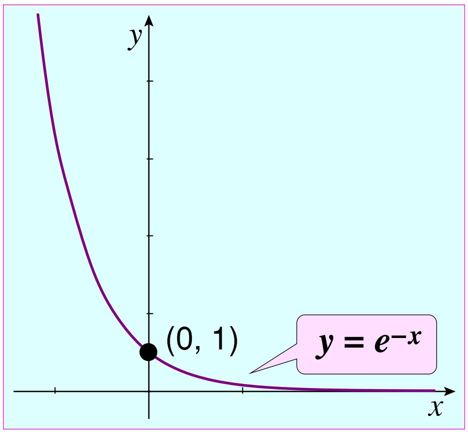
$$y = e^x$$

Asymptote at the x-axis. Intercept at (0, 1)



$$y = \frac{1}{e^x} = e^{-x}$$

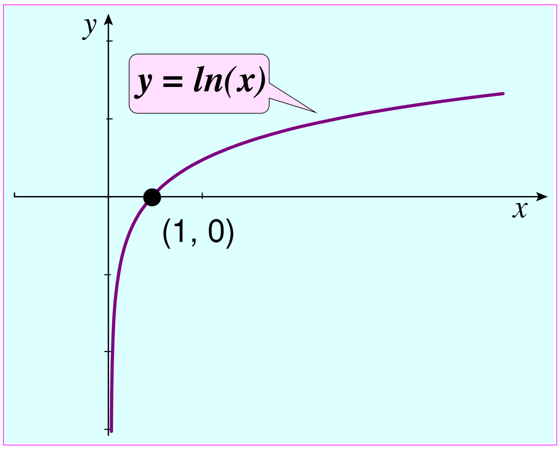
Asymptote at the x-axis. Intercept at (0, 1)



15.5.2 Log Function

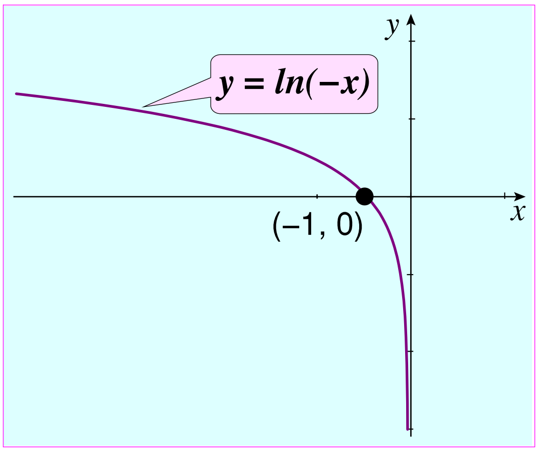
$$y = ln(x)$$

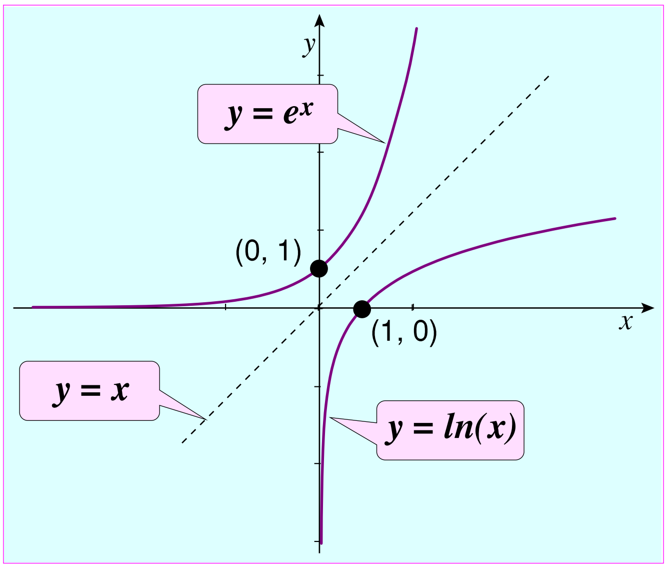
Asymptote at the y-axis. Intercept at (1, 0)



$$y = \frac{1}{ln(x)} = ln(-x)$$

Asymptote at the y-axis. Intercept at (−1, 0)

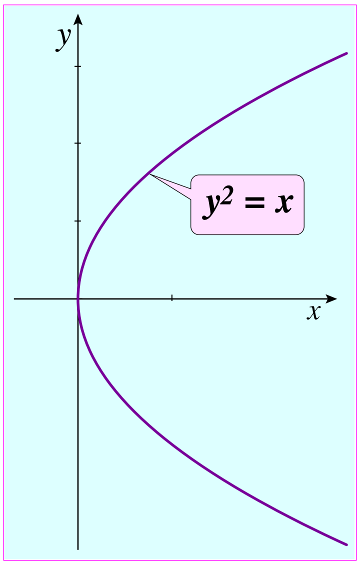




Inverse nature of e^x and ln x

15.6 Other Curves

Often appears in various questions, it is not a function in its own right as it is made up of a relation between $y = \sqrt{x}$ and $y = -\sqrt{x}$, joined at the origin.



15.7 Finding Asymptotes

I'm not sure if this is explicitly on the syllabus, but it is extremely useful stuff to know when sketching graphs.

An **Asymptote** is a line on a graph that the curve of a function approaches but never quite reaches. It is a limit beyond which the curve cannot pass.

Asymptotes are generally associated with rational functions (i.e. ratio, aka a fraction – geddit?)

There are three sorts of asymptote to consider, plus one associated part which is a 'hole':

- A vertical asymptote
- A horizontal asymptote
- A slanting asymptote
- A 'hole' in the function curve

For A-level purposes only the vertical & horizontal asymptotes probably need be considered, but you might as well learn the whole story.

15.7.1 The Vertical Asymptote

A vertical asymptote is one in which the function tends toward infinity for a particular value or values of x, which is why they are generally associated with rational functions. (Note that not all rational functions have asymptotes).

To find a vertical asymptote and its equation:

- Put the top and bottom expressions in factored form
- Cancel any common factors (but see later)
- Find the values of x for which the function becomes undefined, by setting the denominator (the bottom bit) to zero and solving for x.

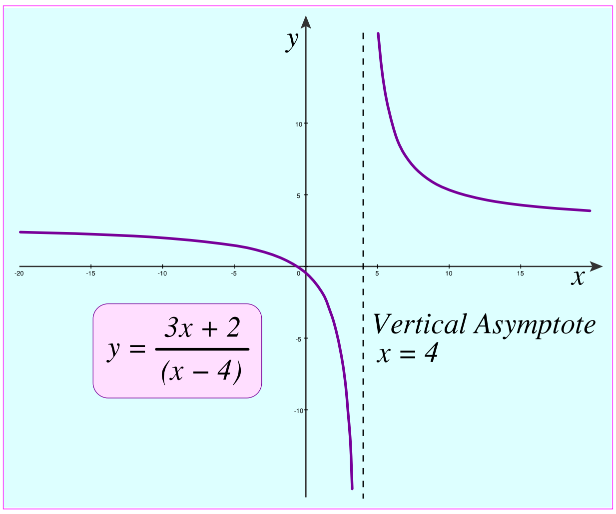
Example 1

$$f(x) = \frac{3x + 2}{x - 4} \qquad f(4) = \frac{14}{0} \to \infty$$

The function is undefined when $(x - 4) = 0$ or when $x = 4$

We can see that as $x \to 4$ then the denominator becomes very small, and therefore $f(x)$ will becomes very large and so $f(x) \to \infty$, as $x \to 4$.

This will give us a vertical asymptote at $x = 4$.



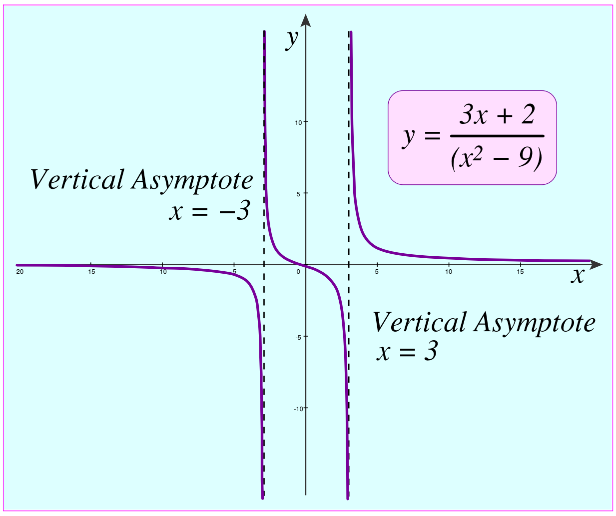
Example 2

$$f(x) = \frac{3x + 2}{x^2 - 9} = \frac{3x + 2}{(x - 3)(x + 3)}$$

$$f(3) = \frac{11}{0} \rightarrow \infty \qquad f(-3) = \frac{-7}{0} \rightarrow -\infty$$

This function is undefined when $(x^2 - 9) = 0$ or when $x = \pm\sqrt{9}$ or $x = \pm 3$
Factorising the denominator helps to visualise this.

This will give us a vertical asymptotes at $x = 3$ and $x = -3$.



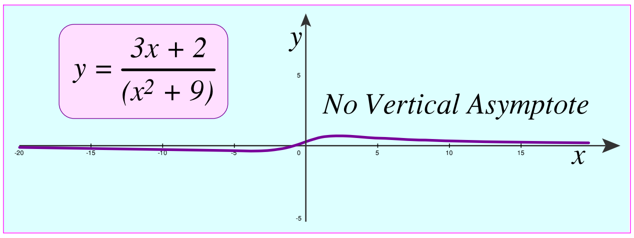
Example 3

$$f(x) = \frac{3x + 2}{x^2 + 9}$$

Setting the denominator to zero, means that:

$$x^2 + 9 = 0 \text{ and so } x = \pm\sqrt{-9} = \pm 3\sqrt{-1}$$

The denominator cannot be factorised as it has imaginary roots. As such, there are no asymptotes.



15.7.2 The Horizontal Asymptote

Horizontal Asymptotes are found by testing the function for very large values of x.

The position of the horizontal asymptote will depend on the degree of both the denominator and the numerator. It is recommended that the expressions should be in the standard unfactored form for this part.

There are three cases to look at:

- Degree of denominator and numerator are equal — Horizontal asymptote
- Degree of denominator > numerator — Horizontal asymptote of $y = 0$
- Degree of denominator < numerator — No Horizontal asymptote, but a slant one possible.

To find horizontal asymptotes:

- Put the top and bottom expressions in their standard unfactored form
- Test the function for very large values of x $(i.e.\ x \to \infty)$

Example 1

Take our previous first example:

$$f(x) = \frac{3x + 2}{x - 4}$$

The degree of denominator and numerator are equal.

If x is very large $(i.e.\ x \to \infty)$ then only the highest order terms in the numerator and denominator need to be considered, since lower order terms become irrelevant.

$$\therefore \quad f(x) = \frac{3x + 2}{x - 4} \qquad x \to \infty \qquad \therefore \quad f(x) \to \frac{3x}{x} = 3$$

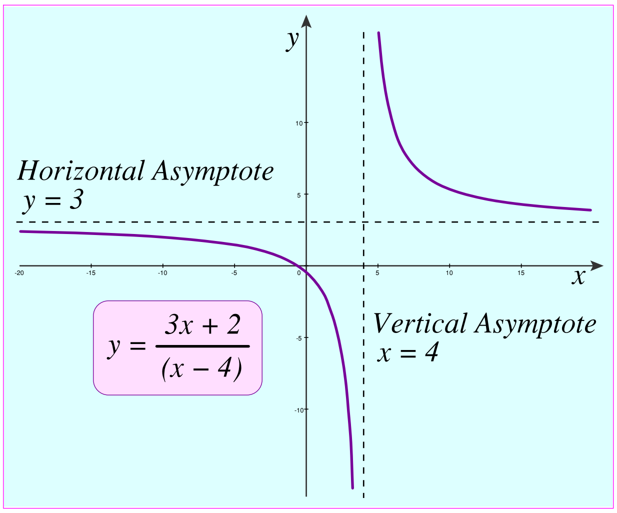
Alternatively, divide all the terms by the highest order x in the denominator:

$$f(x) = \frac{3x + 2}{x - 4} = \frac{\left(3 + \frac{2}{x}\right)}{\left(1 - \frac{4}{x}\right)} \qquad \therefore \text{ as } x \to \infty \qquad f(x) \to \frac{(3 + 0)}{(1 - 0)} = 3$$

We can say that as $x \to \infty$ $\quad \frac{2}{x}\ \&\ \frac{4}{x} \to 0 \quad \therefore \quad f(x) \to 3$

Similarly, as $x \to -\infty$ $\quad \frac{2}{x}\ \&\ \frac{4}{x} \to 0 \quad \therefore \quad f(x) \to 3$

So we have a horizontal asymptote at $y = 3$.



Example 2

In this example, the degree of denominator > numerator.
When $x \to \infty$ then consider only the higher order terms:

$$f(x) = \frac{3x + 2}{x^2 - 4} \qquad \text{as } x \to \infty \qquad f(x) \to \frac{3x}{x^2} = \frac{3}{x} = 0$$

Alternatively, divide all the terms by the highest order x in the denominator; x^2

$$f(x) = \frac{3x + 2}{x^2 - 4} = \frac{\left(\frac{3}{x} + \frac{2}{x^2}\right)}{\left(1 - \frac{4}{x^2}\right)} \qquad \text{As } x \to \pm\infty \qquad f(x) \to \frac{(0 + 0)}{(1 - 0)} = 0$$

We can say that as $x \to \infty$ $\quad \frac{3}{x};\ \frac{2}{x^2}\ \&\ \frac{4}{x^2} \to 0 \quad \therefore\ f(x) \to 0$

Similarly, as $x \to -\infty$ $\quad \frac{3}{x};\ \frac{2}{x^2}\ \&\ \frac{4}{x^2} \to 0 \quad \therefore\ f(x) \to 0$

So we have a horizontal asymptote at $y = 0$.

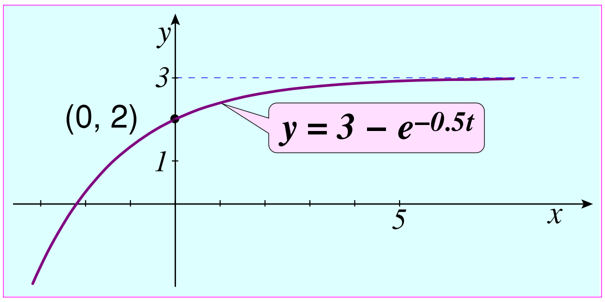
Example 3

This example looks at an exponential function. Plotting is made easy once the asymptote is found.

$$f(t) = 3 - e^{-0{\cdot}5t}$$

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty}\left(3 - \frac{1}{e^{0{\cdot}5t}}\right) = 3$$

Since $e^{-0{\cdot}5t}$ becomes close to zero when t increases to approximately 10, then the limit tends to 3 for values of $t > 10$.



15.7.3 The Slant or Oblique Asymptote

A slant or oblique asymptote may be found when the degree of denominator is one less than the numerator.

$$f(x) = \frac{ax^n + bx\ldots}{sx^{n-1} + tx\ldots}$$

In this case the function has to be rearranged by doing a partial long division.

Example

$$f(x) = \frac{3x^3 + 2x - 6}{x^2 - 4}$$

Solution:

Since the degree of denominator is one less than the numerator, do a partial long division. Division only has to be completed until the remainder is one degree less that the denominator.

$3x$	
$x^2 - 4 \;\big)\; 3x^3 + 0x^2 + 2x - 6$	Divide $3x^3$ *by* $x^2 = 3x$
$3x^3 + 0x^2 - 12x$	Multiply $(x^2 - 4)$ *by* $3x$
$14x - 6$	Subtract
$\frac{14x}{x^2}$	Dividing $14x$ *by* x^2 gives a small term
	Once the degree is small stop dividing

$$\therefore \quad f(x) = 3x - \frac{14x - 6}{x^2 - 4}$$

When $x \to \infty \qquad f(x) \to 3x$

The equation of the asymptote is $\qquad y = 3x$

Alternative Solution:

Alternatively, divide all the terms by x^2 :

$$f(x) = \frac{3x^3 + 2x - 6}{x^2 - 4} = \frac{3x + \frac{2x}{x^2} - \frac{6}{x^2}}{1 - \frac{4}{x^2}}$$

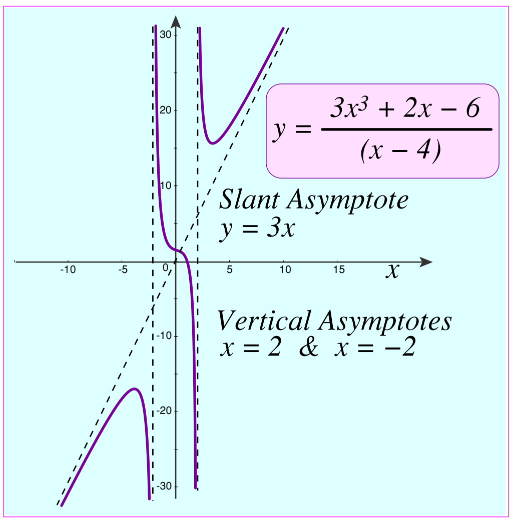
$$\therefore \text{ as } x \to \infty \qquad f(x) \to \frac{(3x + 0 - 0)}{(1 - 0)} = 3x$$

So we have a slant asymptote of $y = 3x$

Note also the vertical asymptotes at

$x = \pm 2$

since $(x^2 - 4) = (x - 2)(x + 2)$



15.7.4 Function is Undefined at a Point (a Hole)

This is really an extension to the rules discussed with regard to vertical asymptotes.

Writing the function in its factored form will show if there are any common terms in the denominator and numerator. Although these factors cancel out and there is no vertical asymptote with these factors, they still produce a point that is undefined.

Example

$$f(x) = \frac{(3x + 2)(x + 3)}{(x - 4)(x + 3)}$$

The function is undefined when $(x - 4) = 0$ or when $(x + 3) = 0$.

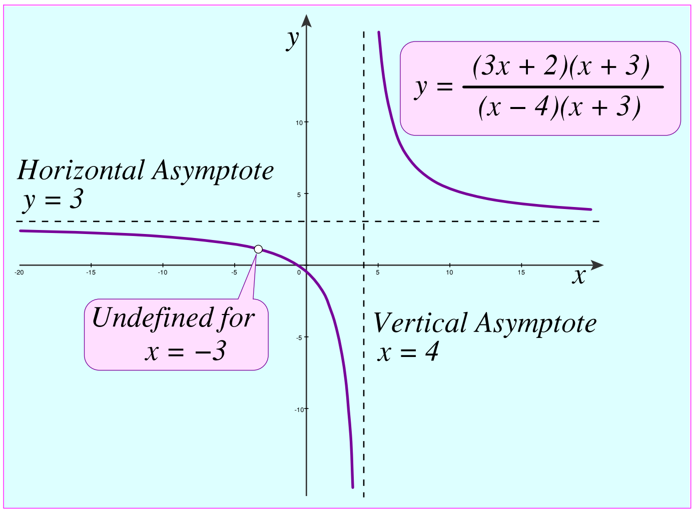
Evaluating the function for these two values gives:

$$f(4) = \frac{14 \times 7}{0} \to \infty \qquad f(-3) = \frac{-7 \times 0}{0} = \frac{0}{0}$$

This will give us a vertical asymptote at $x = 4$, however there is no asymptote at $x = -3$.

$f(-3)$ is undefined at that point, but values either side are unaffected, thus creating a hole.

(Note: most graphing apps will not show this).



15.8 Worked Examples

15.8.1 Example:

1

2

16 • C1 • Graph Transformations

C1 / C3 Combined

16.1 Transformations of Graphs

A transformation refers to how shapes or graphs change position or shape. Knowing how transformations take place allows for the mapping of a standard function (see previous section) to a more complex function.

Transformations considered here consists of :

- Translations parallel to the x-axis or y-axis
- One way stretches parallel to the x-axis or y-axis
- Reflections in both the x-axis & y-axis

Other transformations include enlargements, rotations and shears, but these are not covered specifically.

Using the equation for a semicircle to illustrate the various transformations will give a good grounding on how to apply them. The equation of a semicircle, radius 3, centred at (0, 0) is:

$$y = \sqrt{9 - x^2}$$

It is important to become familiar with function notation, as questions are often couched in these terms.

For example: 'The function $f(x)$ maps to $f(x) + 2$. Describe the transformation.' It is also important to learn the correct phraseology of the answers required, (see later).

In function notation, our equation above can be written as:

$$f(x) = \sqrt{9 - x^2}$$

where $f(x)$ represents the output of the function and x the input of the function.

Any changes to the input, represent changes that are with respect to the x-axis, whilst any changes that affect the whole function represent changes that are with respect to the y-axis.

It is also useful to think of the function as:

$$f(x) = \sqrt{9 - (x)^2}$$

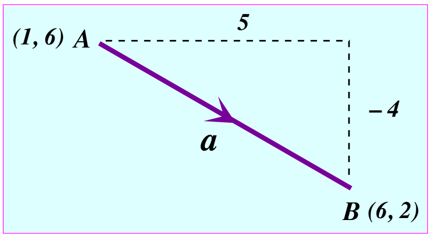
The addition of the brackets, reminds us that changes to the input must be applied as a substitution. Thus, if we want to map $f(x)$ to $f(x + 2)$ then the function becomes:

$$f(x) = \sqrt{9 - (x + 2)^2}$$

16.2 Vector Notation

Vectors are covered in greater detail in C4, but for now you need to know how to write a displacement of an object or point in vector notation.

Moving from point A to point B requires a move of 5 units in the x direction followed by -4 units in the y direction, and is written thus:



$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \equiv \begin{pmatrix} 5 \\ -4 \end{pmatrix} = \text{5 across; 4 down}$$

where 5 & -4 are the components in the x & y direction.

16.3 Translations Parallel to the y-axis

Translations are just movements in the x-y plane without any rotation. enlargement or reflections. The movement can be described as a vector.

The simplest translation to get your head around is the movement in the y-axis. Recall that a straight line, in the form of $y = mx + c$, will cross the y-axis at point $(0, c)$. It should be no surprise that if c is varied, the graph will move (translate) in a vertical direction parallel to the y-axis.

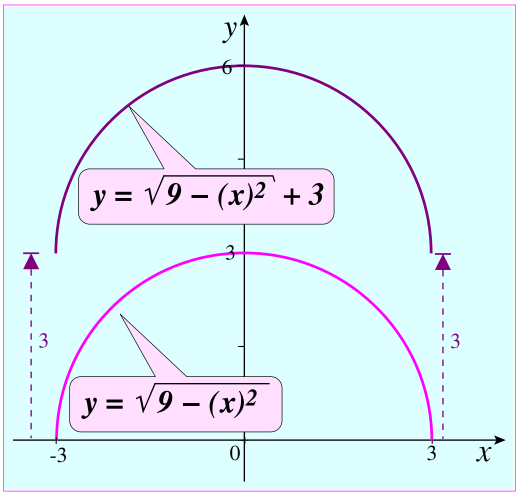
In general, the function $f(x)$ maps to $f(x) + a$ by translating $f(x)$ parallel to the y-axis, in the positive direction by a units.

Map $y = \sqrt{9 - (x)^2}$ to $y = \sqrt{9 - (x)^2} + 3$

i.e. map: $f(x)$ to $f(x) + 3$

The graph is translated in the vertical direction, parallel to the y-axis, by 3 units.

This is represented by the vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$



16.4 Translations Parallel to the x-axis

Translation along the x-axis is not immediately intuitive.

In general, the function $f(x)$ maps to $f(x - a)$ by translating $f(x)$ parallel to the x-axis, in the positive direction by a units.

Note that the value of a is negative. This is explained by the fact that in order for $f(x - a)$ to have the same value as $f(x)$ then the value of x must be correspondingly larger in $f(x - a)$, hence it must be moved in the positive direction.

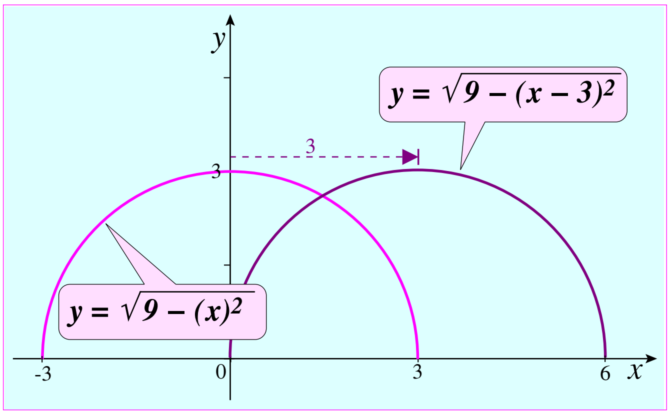
Map $y = \sqrt{9 - (x)^2}$ to $y = \sqrt{9 - (x - 3)^2}$

i.e. map: $f(x)$ to $f(x - 3)$

The graph is translated in the horizontal direction, parallel to the x-axis, by 3 units.

This is represented by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

Note how the -3 appears inside the squared bracket. i.e. replace x with $(x - 3)$

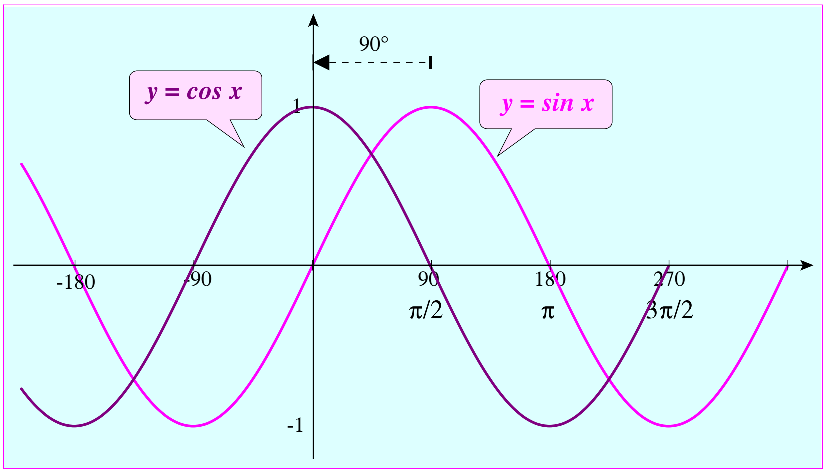


Translating a sine graph 90° to the left gives a cosine graph.

The vector is $\begin{pmatrix} -90° \\ 0 \end{pmatrix}$

$y = f(\theta)$ maps to $y = f(\theta + 90)$

Hence $\cos\theta = \sin(\theta + 90)$

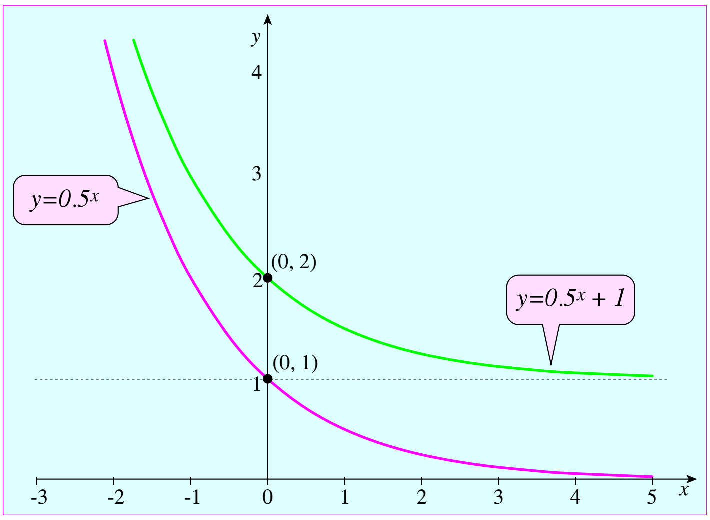


Translation of an exponential:

1) Map $y = 0.5^x$ to $y = 0.5^x + 1$

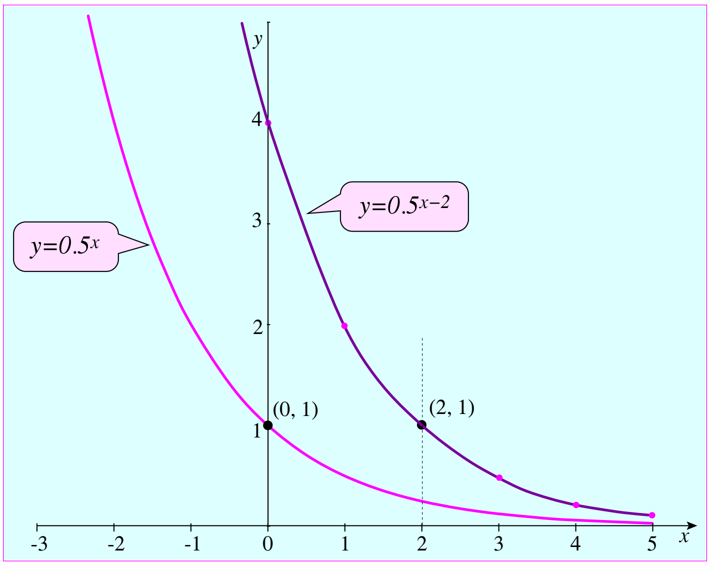
The graph is translated parallel to the y-axis, in the positive direction by 1 unit.

This is represented by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the graph now passes through the point (0, 2), with the asymptote now at $y = 1$.



2) Map $y = 0.5^x$ to $y = 0.5^{x-2}$

The graph is translated parallel to the x-axis, in the positive direction by 2 units, such that it now passes through the point (2, 1), however since the exponent is smaller, the graph is steeper.



16.5 One Way Stretches Parallel to the y-axis

Once again, the simplest axis is the y-axis.

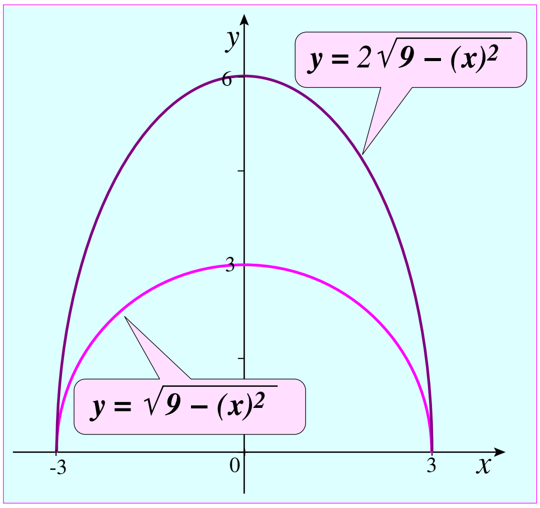
In general, the function $f(x)$ maps to $k f(x)$ by stretching $f(x)$ parallel to the y-axis, by a scale factor of k.

Note that parts of the curve that cross the x-axis, do not change position.

Map $y = \sqrt{9 - (x)^2}$ to $y = 2\sqrt{9 - (x)^2}$

i.e. map: $f(x)$ to $2f(x)$

The graph is stretched parallel with the y-axis with a scale factor of 2.

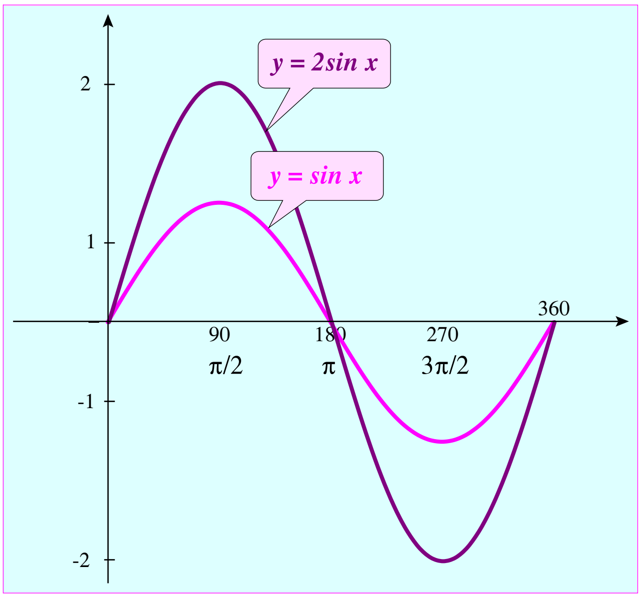


Map $y = \sin x$ to $y = 2\sin x$

i.e. map: $f(x)$ to $2f(x)$

The graph is stretched parallel with the y-axis with a scale factor of 2.

Note the stretch extends in both the positive and negative y directions.



16.6 One Way Stretches Parallel to the x-axis

In general, the function $f(x)$ maps to $f(kx)$ by stretching $f(x)$ parallel to the x-axis, by a scale factor of $\frac{1}{k}$.

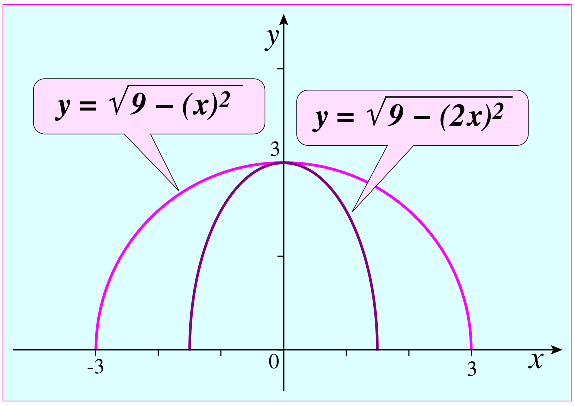
Again, not very intuitive. [You can view this as a compression of scale factor a, but keeping to the idea of stretches is probably simpler].

If $k > 1$, the scale factor will be < 1, if $k < 1$ the scale factor will be > 1.

Map $y = \sqrt{9 - (x)^2}$ to $y = \sqrt{9 - (2x)^2}$

i.e. map: $f(x)$ to $f(2x)$

The graph is stretched parallel to the x-axis with a scale factor of ½.

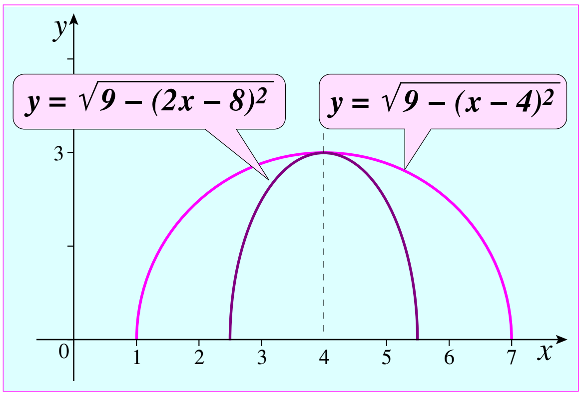


Care must be taken when applying changes to the mapping. The next two examples show how the changes might not be so obvious.

Map: $f(x - 4)$ to $f[2(x - 4)]$

i.e. map $y = \sqrt{9 - (x - 4)^2}$ to $y = \sqrt{9 - (2x - 8)^2}$

As in the example above, the graph is stretched parallel to the x-axis with a scale factor of ½.

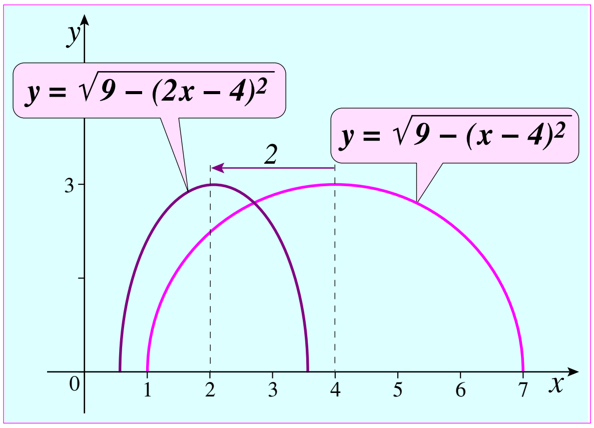


Note what happens in this case. Suppose we get the transformation wrong and apply the scaling incorrectly:

We map $y = \sqrt{9 - (x - 4)^2}$ to $y = \sqrt{9 - (2x - 4)^2}$

This is the same as saying:
Map: $f(x - 4)$ to $f[2(x - 4 + 2)]$

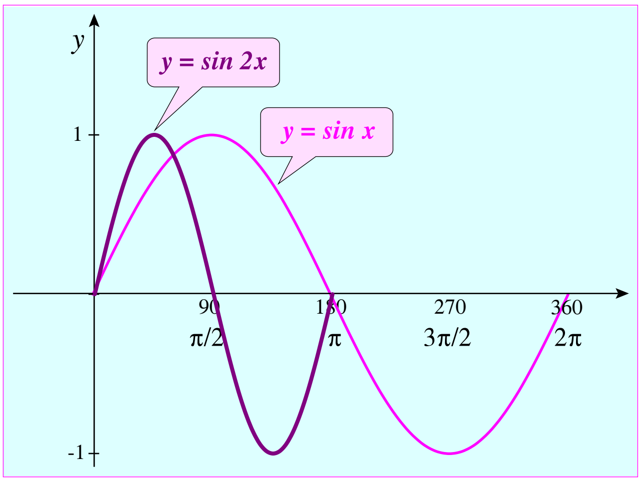
As in the example above, the graph is stretched parallel to the x-axis with a scale factor of ½, but the function has been translated to the left by 2 units. Hence the vertex is now centred on $x = 2$. In effect,- the translation is done first, and the scaling second.



Map $y = \sin x$ to $y = \sin 2x$

i.e. map: $f(x)$ to $f(2x)$

The graph is stretched parallel to the x-axis with a scale factor of ½.

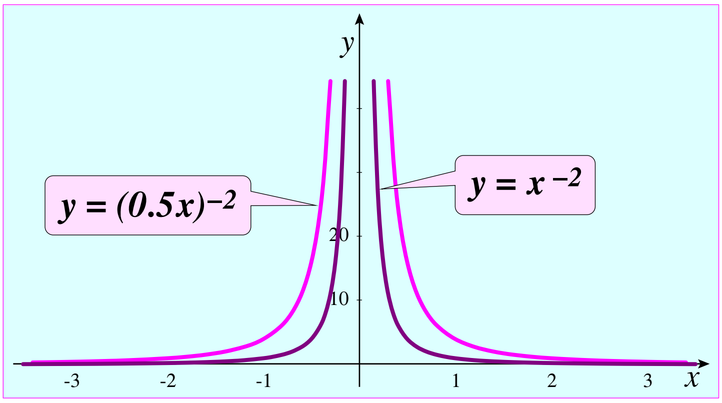


Map $y = \frac{1}{x^2}$ to $y = \frac{4}{x^2}$

Now $y = \frac{4}{x^2} \Rightarrow \frac{1}{\frac{1}{4}x^2} \Rightarrow \frac{1}{\left(\frac{1}{2}x\right)^2}$

i.e. map: $f(x)$ to $f(0{\cdot}5x)$

The graph is stretched parallel to the x-axis with a scale factor of 2.

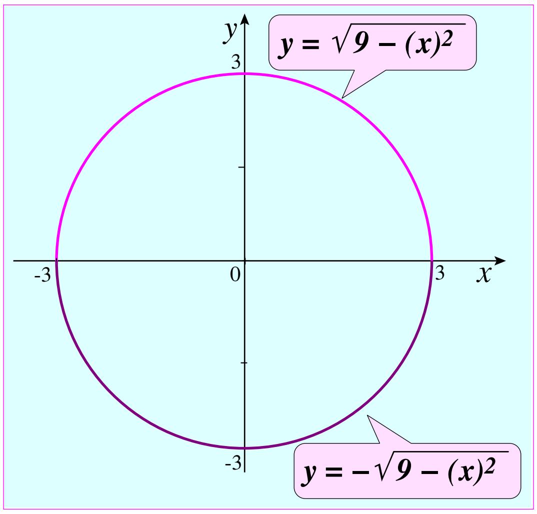


16.7 Reflections in both the x-axis & y-axis

In general, the function $f(x)$ maps to $-f(x)$ by a reflection of $f(x)$ in the x-axis.

Map $y = \sqrt{9-(x)^2}$ to $y = -\sqrt{9-(x)^2}$

The graph is reflected in the x-axis.

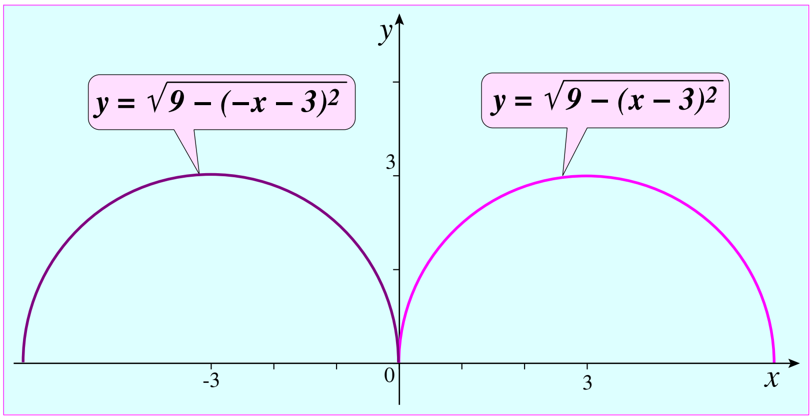


In general, the function $f(x)$ maps to $f(-x)$ by a reflection of $f(x)$ in the y-axis.

Map $y = \sqrt{9-(x-3)^2}$ to

$y = \sqrt{9-(-x-3)^2}$

The graph is reflected in the y-axis.



16.8 Translating Quadratic Functions

In translating any quadratic function, the technique of completing the square can be used to find the translations required in the x and y axes. The standard form of a completed square is:

$$y = a(x+k)^2 + q$$

To map $y = x^2$ to the completed square of a quadratic, the required vector is $\begin{pmatrix} -k \\ q \end{pmatrix}$

Note the sign of the constant k.

16.9 Translating a Circle Function

The basic equation of a circle, with radius r, centred on the origin is:

$$x^2 + y^2 = r^2$$

This can be mapped to a circle, radius r, centred at the point (a, b), by a vector $\begin{pmatrix} a \\ b \end{pmatrix}$. The equation then becomes:

$$(x-a)^2 + (y-b)^2 = r^2$$

16.10 Transformations Summary

Given Function	Map to this Function	Transformation required (note the phraseology):
$y = f(x)$	$y = f(x) + a$	Translate parallel to the y-axis, by a units, in the positive direction $\begin{pmatrix} 0 \\ a \end{pmatrix}$
	$y = f(x - a)$	Translate parallel to the x-axis, by a units, in the positive direction $\begin{pmatrix} a \\ 0 \end{pmatrix}$
	$y = k f(x)$	Vertical one way stretch, parallel to the y-axis, by a scale factor k
	$y = f(kx)$	Horizontal one way stretch, parallel to the x-axis, by a scale factor $\frac{1}{k}$
	$y = f\left(\frac{x}{k}\right)$	Horizontal one way stretch, parallel to the x-axis, by a scale factor k
	$y = -f(x)$	A reflection of $f(x)$ in the x-axis.
	$y = f(-x)$	A reflection of $f(x)$ in the y-axis.
	$y = -f(-x)$	A rotation of 180°

16.11 Recommended Order of Transformations

Very often the order of applying multiple transformations does not make any difference, but occasionally it can. There are two ways of going about choosing the order. The first is by the checklist below or the other is to look at the order of calculation in the function.

In general, the transformations of $y = f(x)$ to $y = k f(x)$ and $y = f(x) + a$ both operate outside the function $f(x)$, and can be done at anytime.

The problems start when transformations mess with the function $f(x)$ such as $y = f(kx)$ or $y = f(x - a)$. It is important to recognise that in shifting from $f(x)$ to, say, $y = f(x - a)$ you are replacing (x) in the original function by $(x - a)$. Similarly for $y = f(kx)$, (x) is replaced by (kx) or even $(-kx)$.

A good general order to apply the transformations are:

- Apply any horizontal translations parallel to the x-axis
- Apply any stretching
- Carry out any reflections
- Apply any vertical translations parallel to the y-axis

Looking at the order of calculation should give you a good idea of the order required. Start by looking at the function $f(x)$ first by replacing the x part and then work outwards. (A bit like doing function of function sums really).

E.g. 1 Transform the graph of $y = x^3$ to $y = 3(x + 2)^3 - 5$
Starting on the inside, $x^3 \Rightarrow (x + 2)^3 \Rightarrow 3(x + 2)^3 \Rightarrow 3(x + 2)^3 - 5$.
So the sequence is a horizontal translation of −2, a vertical stretch of scale factor 3, and a vertical translation of −2. A total vector move of $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$.

E.g. 2 Transform the graph of $y = \sqrt{x}$ to $y = 2\sqrt{(3 - x)}$
Starting on the inside, $\sqrt{x} \Rightarrow \sqrt{(x + 3)} \Rightarrow \sqrt{(-x + 3)} \Rightarrow 2\sqrt{(3 - x)}$
So the sequence is a horizontal translations of −3, a reflection in the y-axis, and a vertical stretch of scale factor 2.

16.12 Example Transformations

16.12.1 Example:

1 Map $y = x^2$ to $y = x^2 - 5x + 6$

Solution:

Complete the square:

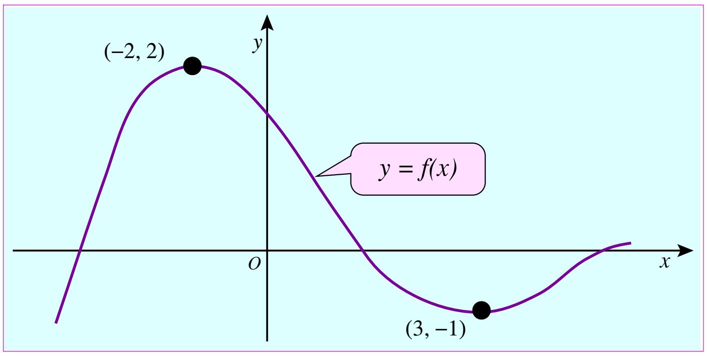
$$y = x^2 - 5x + 6 = \left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6$$

$$y = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 6 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{24}{4}$$

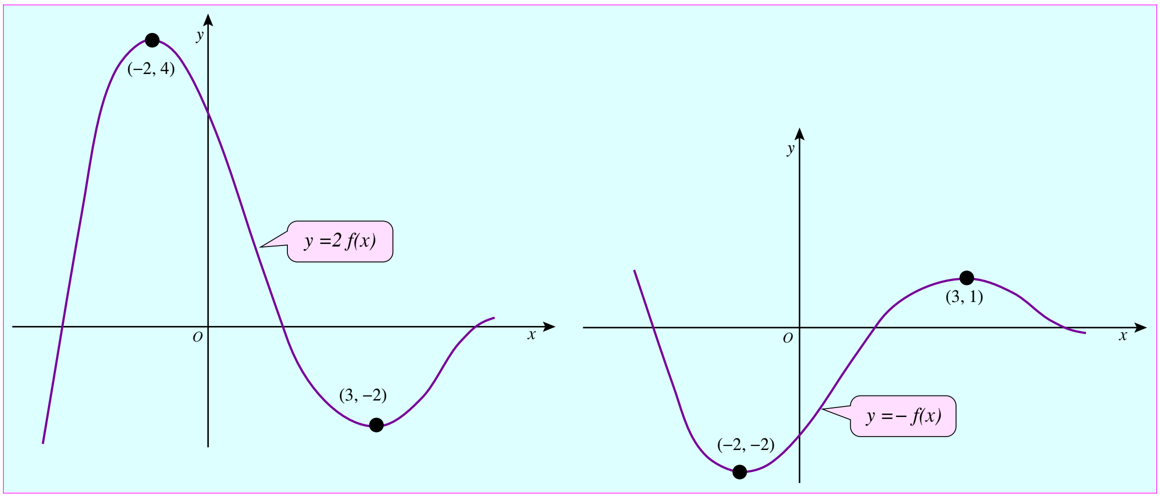
$$y = \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$$

Translation required is: $\begin{pmatrix} 5/2 \\ -1/4 \end{pmatrix}$

2 The function $y = f(x)$ is shown, with turning points as marked.
Sketch the curves for $y = 2f(x)$ and $y = -f(x)$, marking the coordinates of the turning points.



Solution:



3 Describe the mapping of $y = 3\sqrt{x}$ to $y = 9\sqrt{x}$

Solution:

This is a vertical one way stretch, parallel to the y-axis, by a scale factor $9/3$ or 3.

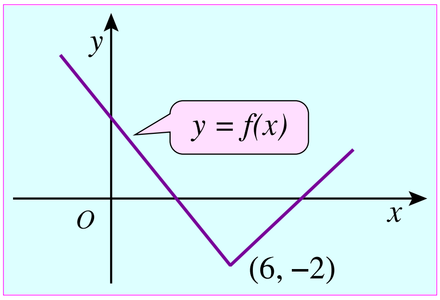
4 Describe the mapping of $y = x^3$ to $y = 10 - x^3$

Solution:

$x^3 \Rightarrow -x^3 \Rightarrow 10 - x^3$

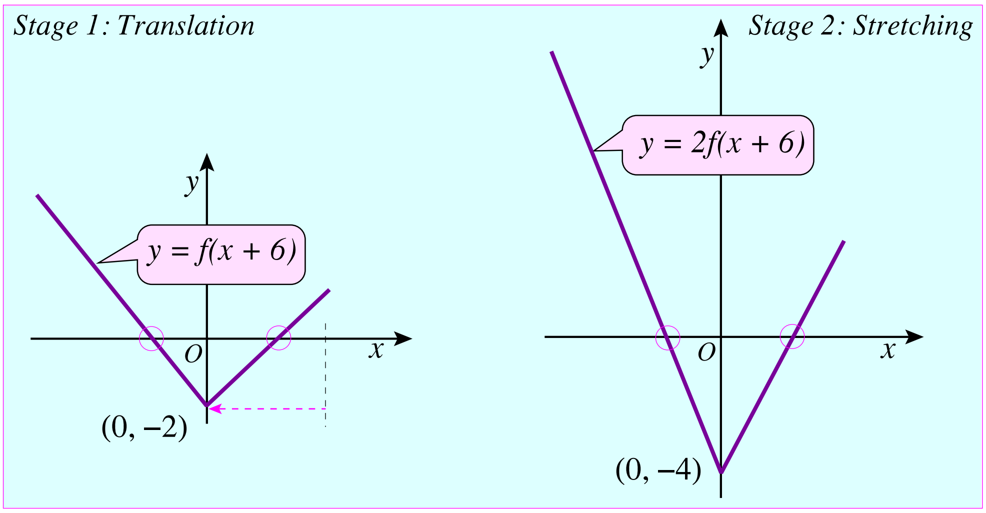
This is a reflection in the y-axis, followed by a translation parallel to the y-axis of 10 units or vector $\begin{pmatrix} 0 \\ 10 \end{pmatrix}$

5 The function $y = f(x)$ is shown, with a turning point as marked.
Sketch the curve for $y = 2f(x - 6)$, marking the coordinate of the turning point.



Solution:

This has to be tackled in two stages. Do the translation first then the stretch. Note how the points of intersection with the x-axis remain unchanged after the stretch.



16.13 Topical Tips

In the exam you need to tick off the following points:

- Type of transformation: Translation, stretch, or reflection?
- The direction of transformation: Translation parallel to x or y axis, or reflected in which axis?
- Magnitude of transformation: number of units moved or scale factor?
- A vector quantity can be used to describe the translation.
- Use the correct terminology, e.g. Translation instead of 'move' or 'shift'. 'Reflection' - not 'mirror'

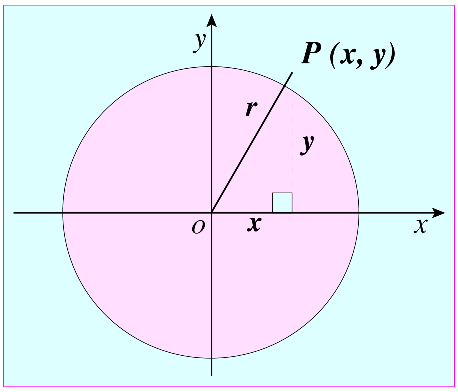
[Note: although this is in the C1 section, this chapter also includes topics more suited to C3. You will need to refer back to this section then.]

17 • C1 • Circle Geometry

17.1 Equation of a Circle

17.1.1 Centre (0, 0)

A circle, centre (0, 0), radius r.



From Pythagoras' theorem:

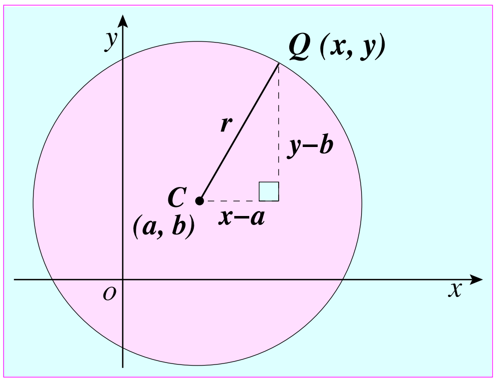
$$x^2 + y^2 = r^2$$

Therefore, a circle, centre (0, 0), radius r has an equation of $x^2 + y^2 = r^2$

$$x^2 + y^2 = r^2$$

17.1.2 Centre (a, b)

If you translate the circle, $x^2 + y^2 = r^2$, by $\begin{pmatrix} a \\ b \end{pmatrix}$



the centre of the circle becomes (a, b), with radius r.

From Pythagoras' theorem:

$$(x - a)^2 + (y - b)^2 = r^2$$

Expand the brackets & simplify to give:

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

$$x^2 + y^2 - 2ax - 2by + c = 0$$

where $c = a^2 + b^2 - r^2$

$$(x - a)^2 + (y - b)^2 = r^2$$

$$x^2 + y^2 - 2ax - 2by + c = 0$$

Note that:

- the coefficients of x^2 and y^2 are equal to 1, all other terms are linear
- There is no xy term
- The coefficients of $x = -2 \times x$ co-ordinate of the centre
- The coefficients of $y = -2 \times y$ co-ordinate of the centre

17.2 Equation of a Circle Examples

17.2.1 Example:

1 Find the centre and radius of the circle $x^2 + (y + 3)^2 = 25$

Solution:

Compare with the general equation of a circle:

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x - 0)^2 + (y - (-3))^2 = 5^2$$

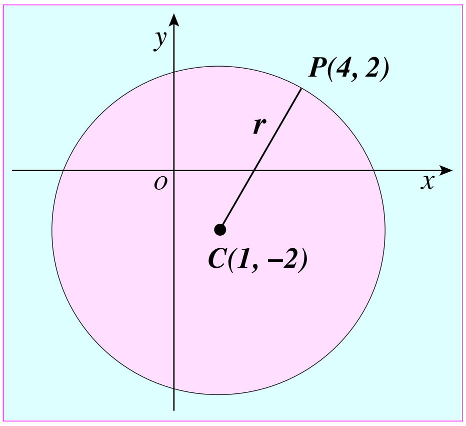
$$\therefore \quad a = 0, \; b = -3 \text{ and } r = 5$$

Centre is (0, –3), radius is 5

2 A circle with centre (1, –2), which passes through point (4, 2)

Find the radius of the circle

Find the equation of the circle



Solution:

Use pythag and the points given to find radius:

$$r^2 = (4 - 1)^2 + (2 - (-2))^2$$

$$= 25$$

$$\therefore \quad r = 5$$

Using the standard form: the equation of this circle is:

$$(x - 1)^2 + (y + 2)^2 = 25$$

3 Show that the equation $x^2 + y^2 + 4x - 6y - 3 = 0$ represents a circle. Give the co-ordinates of the centre and radius of the circle.

Solution:

Compare with the standard form of $x^2 + y^2 - 2ax - 2by + c = 0$

$$-2a = 4 \quad \Rightarrow \quad a = -2$$

$$-2b = -6 \quad \Rightarrow \quad b = 3$$

$$\therefore \quad \text{Centre of circle} = (-2, \; 3)$$

$$r = \sqrt{a^2 + b^2 - c}$$

$$= \sqrt{(-2)^2 + 3^2 - (-3)}$$

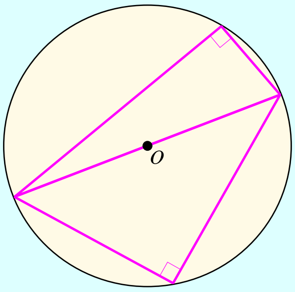
$$= \sqrt{16} = 4$$

Note:

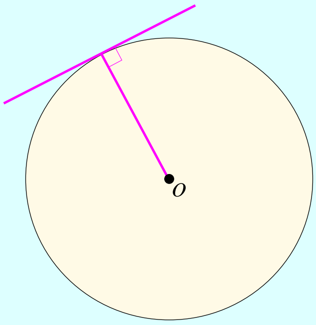
An alternative and simpler method is to complete the square to put the equation into the standard form of $(x - a)^2 + (y - b)^2 = r^2$ (See later).

17.3 Properties of a Circle

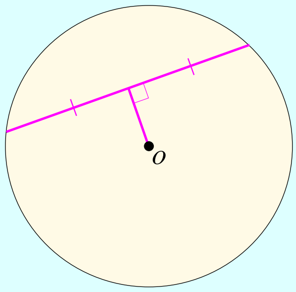
A reminder: the first three properties shown are of interest in the core modules.



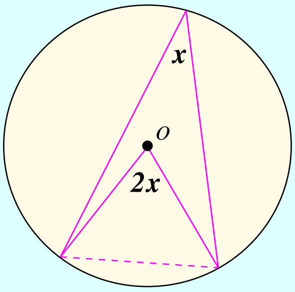
The angle in a semicircle is a right angle

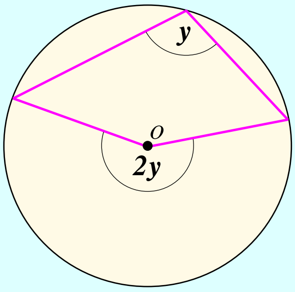


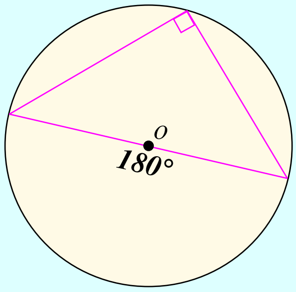
A radius and a tangent to a point form a right angle



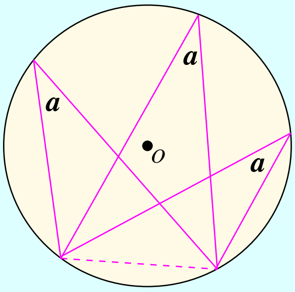
A perpendicular line to the chord which passes through the center, 'bisects' the chord



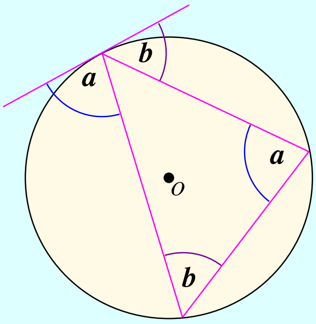




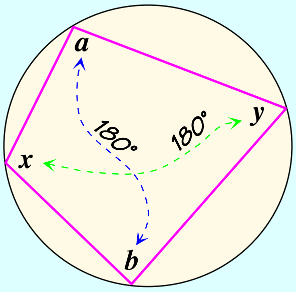
Angle at the centre = 2 × angle at the circumference (2 triangles with common chord as a base)



Angles in the same segment are equal. (Triangles with a common chord as a base)



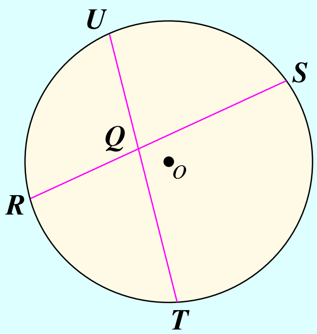
Alternate Segment Theorem Chord meets tangent, angle between them = angle in alternate segment

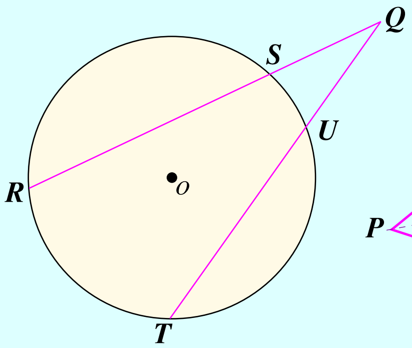


Opposite angles in a cyclic quadrilateral = 180°

$a + b = 180$

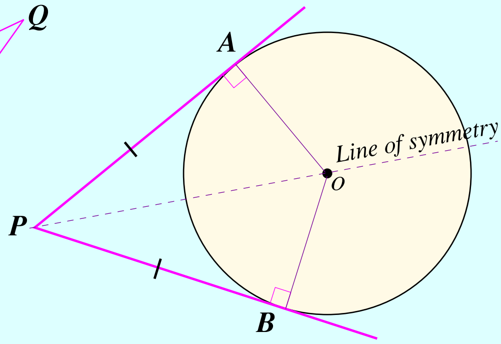
$x + y = 180$





Intersecting chords

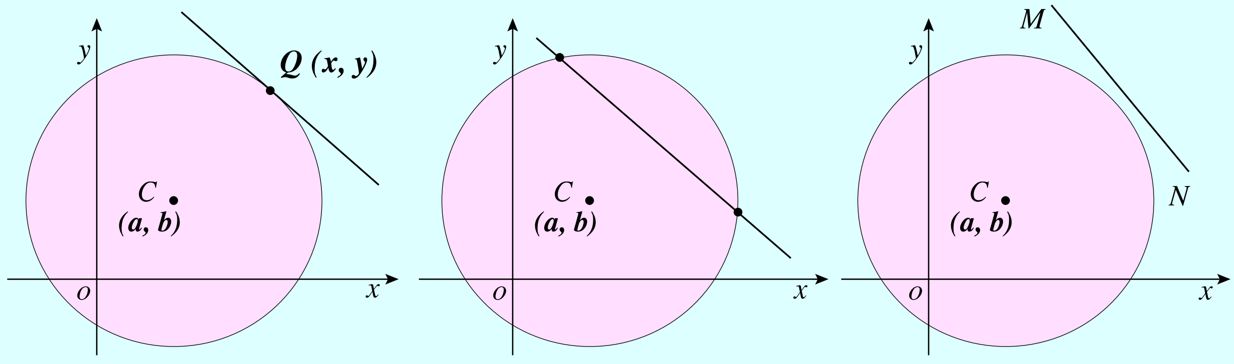
RQ × QS = UQ × QT



Tangents that meet at an external point (P) are equal in length

17.4 Intersection of a Line and a Circle

There are three scenarios that can describe the intersection of a straight line and a circle:



1 (repeated) solution (tangent) — 2 distinct solutions (chord) — No solutions

- A straight line may just touch the circle at one point, in which case it becomes a tangent to the circle
- A line can cut the circle in two places and part of the line will form a chord
- Option 3 is for the line to make no contact and miss the circle altogether, (see line *MN* in the diagram).

There are two methods of solving these problems:

a) by using simultaneous equations. If the resulting quadratic equation has repeated roots, there is only one solution and the line is tangent to the circle. If there are two roots, then the line cuts the circle in two places. If there are no solutions, then there is no interception of the circle by the line.

b) by comparing the perpendicular distance from the line to the centre of the circle and comparing the result with the radius.

17.4.1 Example:

1 Show that the line $2y = 5 - x$ is a tangent to the circle $x^2 + y^2 = 5$.

Solution:

If the line is a tangent, there should be only one solution from the simultaneous equations of:

$$x = 5 - 2y \qquad (1)$$

$$x^2 + y^2 = 5 \qquad (2)$$

$$(5 - 2y)^2 + y^2 = 5 \qquad \text{Substitute (1) into (2)}$$

$$4y^2 - 20y + 25 + y^2 - 5 = 0 \qquad \text{expand}$$

$$5y^2 - 20y + 20 = 0 \qquad \text{simplify}$$

$$y^2 - 4y + 4 = 0 \qquad (3)$$

If there is only one solution, then $b^2 - 4ac = 0$ and hence $b^2 = 4ac$

$$(-4)^2 = 4 \times 1 \times 4$$

$$\text{LHS} \quad 16 = 16 \quad \text{RHS}$$

Alternatively, find the roots of equation (3)

$$y^2 - 4y + 4 = 0$$

$$(y - 2)(y - 2) = 0$$

$$y = 2 \text{ (coincident roots i.e. only 1 solution)}$$

Hence, the line is a tangent to the circle.

2 A line, $y = kx$, is tangent to the circle $x^2 + y^2 + 10x - 20y + 25 = 0$.
Show that the x-coordinates of the intersection points are given by:
$(1 + k^2)x^2 + (10 - 20k)x + 25 = 0$, and find the x-coordinates of the tangents to the circle.

Solution:

$$x^2 + y^2 + 10x - 20y + 25 = 0 \quad \text{given}$$

$$x^2 + (kx)^2 + 10x - 20(kx) + 25 = 0 \quad \text{substitute}$$

$$x^2 + k^2x^2 + 10x - 20kx + 25 = 0 \quad \text{simplify}$$

$$\therefore \quad (1 + k^2)x^2 + (10 - 20k)x + 25 = 0 \quad \text{QED}$$

The tangents to the circle are given when the discriminant $= 0$

$$b^2 - 4ac = 0$$

$$(10 - 20k)^2 - 4(1 + k^2) \times 25 = 0$$

$$(100 - 400k + 400k^2) - 100 - 100k^2 = 0$$

$$300k^2 - 400k = 0$$

$$\therefore \quad k(300k - 400) = 0$$

$$k = 0 \quad or \quad k = \frac{400}{300} = \frac{4}{3}$$

Note: the line $y = kx$ passes through the origin, so we are looking for the two tangents that pass through the origin.
The x-coordinates of the tangents to the circle are found by substituting the values of k just found into the given equation:

$$(1 + k^2)x^2 + (10 - 20k)x + 25 = 0$$

$$k = 0 \quad \Rightarrow \quad x^2 + 10x + 25 = 0$$

$$(x + 5)(x + 5) = 0$$

$$\therefore \quad x = -5$$

$$k = \frac{4}{3} \quad \Rightarrow \quad \left(1 + \frac{16}{9}\right)x^2 + \left(10 - 20 \times \frac{4}{3}\right)x + 25 = 0$$

$$\frac{25}{9}x^2 - \frac{50}{3}x + 25 = 0$$

$$25x^2 - \frac{50 \times 9}{3}x + 25 \times 9 = 0$$

$$x^2 - 6x + 9 = 0$$

$$(x - 3)(x - 3) = 0$$

$$\therefore \quad x = 3$$

The x-coordinates of the tangents to the circle are $x = -5$ and $x = 3$.

17.5 Completing the Square to find the Centre of the Circle

Completing the square puts the equation into the standard form of $(x - a)^2 + (y - b)^2 = r^2$ from which you can read off the co-ordinates of the centre of the circle and its radius.

17.5.1 Example:

1 Find the radius and the centre of the circle for the equation $x^2 + y^2 - 6x - 8y + 9 = 0$

Solution:

Complete the square for both the x and y terms.

$$x^2 - 6x + y^2 - 8y + 9 = 0$$

$$(x - 3)^2 - 3^2 + (y - 4)^2 - 4^2 + 9 = 0$$

$$(x - 3)^2 + (y - 4)^2 - 9 - 16 + 9 = 0$$

$$(x - 3)^2 + (y - 4)^2 = 16$$

Centre of circle is $(3, 4)$, radius $= 4$

2 Find the radius and the centre of the circle for the equation $3x^2 + 3y^2 + 12x - 24y + 12 = 0$

Solution:

Divide through by 3 to ensure the coefficients of the squared terms are 1, then complete the square for both the x and y terms.

$$3x^2 + 3y^2 + 12x - 24y + 12 = 0$$

$$x^2 + y^2 + 4x - 8y + 4 = 0$$

$$x^2 + 4x + y^2 - 8y + 4 = 0$$

$$(x + 2)^2 - 2^2 + (y - 4)^2 - 4^2 + 4 = 0$$

$$(x + 2)^2 + (y - 4)^2 - 4 - 16 + 4 = 0$$

$$(x + 2)^2 + (y - 4)^2 = 16$$

Centre of circle is $(-2, 4)$, radius $= 4$

3 Find the centre of the circle and value of k for the equation $x^2 + y^2 - 6x - k = 0$ when the radius is 5.

Solution:

Complete the square for both the x and y terms.

$$x^2 - 6x + y^2 - k = 0$$

$$(x - 3)^2 - 3^2 + (y - 0)^2 - k = 0$$

$$(x - 3)^2 - 9 + y^2 - k = 0$$

$$(x - 3)^2 + y^2 = 9 + k$$

But radius: $r^2 = 9 + k = 5^2$

$$k = 25 - 9 = 16$$

Centre of circle is $(3, 0)$, $k = 16$

17.6 Tangent to a Circle

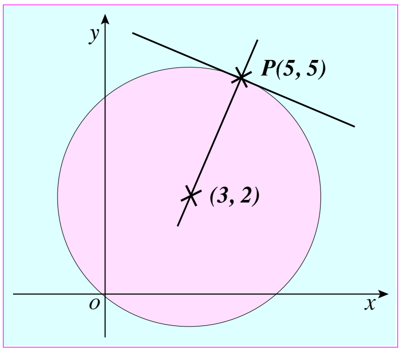
To find a tangent to circle, we use the property that a tangent to a circle is at right angles to the radius at that point. (This is because we have not learnt how to differentiate an equation with the same form as the equation of a circle).

17.6.1 Example:

1 Show that the point P (5, 5) lies on the circle

$x^2 + y^2 - 6x - 4y = 0$

and find the equation of the tangent at P.



Solution:

Substituting (5, 5) into the given equation:

LHS = $25 + 25 - 30 - 20 = 0$ = RHS

$\therefore$ P (5, 5) does lie on the circle

To find gradient of the tangent, first find the gradient of a line from P to the centre. Therefore, find the co-ordinates of the centre.

Match the given equation with the standard form:

$$x^2 + y^2 - 2ax - 2by + c = 0$$

$$x^2 + y^2 - 6x - 4y = 0$$

$$\therefore \quad -2a = -6 \quad \Rightarrow \quad a = 3$$

$$\therefore \quad -2b = -4 \quad \Rightarrow \quad b = 2$$

Centre = (3, 2)

Gradient of the radius through P is

$$\frac{5-2}{5-3} = \frac{3}{2}$$

Gradient of the tangent $= -\frac{2}{3}$

Formula for a straight line is $y - y_1 = m(x - x_1)$

Equation of the tangent is $\quad y - 5 = -\frac{2}{3}(x - 5)$

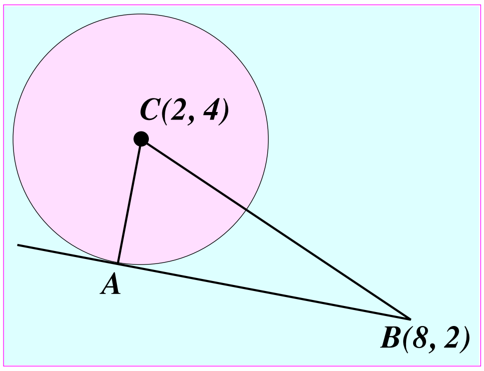
$$\Rightarrow \quad 3y - 15 = -2x + 10$$

$$\Rightarrow \quad 3y + 2x - 15 = 0$$

2 A line is drawn from point B (8, 2) to be a tangent to the circle $x^2 + y^2 - 4x - 8y - 5 = 0$.

Find the length of the tangent.

Sketch a diagram!



Solution:

Match the given equation with the standard form:

$$x^2 + y^2 - 2ax - 2by + c = 0$$

$$x^2 + y^2 - 4x - 8y - 5 = 0$$

$\therefore \quad -2a = -4 \quad \Rightarrow \quad a = 2$

$\therefore \quad -2b = -8 \quad \Rightarrow \quad b = 4$

Centre $= (2, \ 4)$

$$r = \sqrt{a^2 + b^2 - c}$$

$$r = \sqrt{(2)^2 + 4^2 - (-5)}$$

$$r = \sqrt{25}$$

$$r = 5$$

Using pythag:

$$BC^2 = AC^2 + AB^2 \qquad (1)$$

but $\quad BC^2 = (2 - 8)^2 + (4 - 2)^2 = 36 + 4 = 40$

Sub in (1) $\quad 40 = r^2 + AB^2$

$\therefore \quad AB^2 = 40 - r^2 = 40 - 25 = 15$

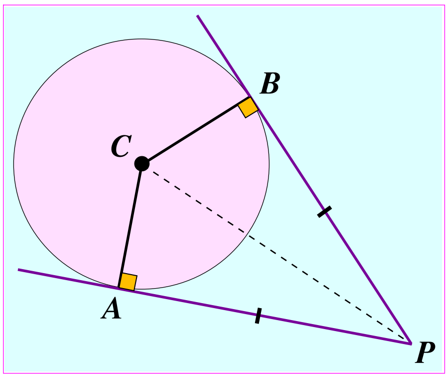
$$AB = \sqrt{15}$$

Length of tangent $= \sqrt{15}$

17.7 Tangent to a Circle from Exterior Point

From any external point outside a circle, you can draw two tangents, and the lengths of both these tangents will be equal.

i.e. $AP = BP$



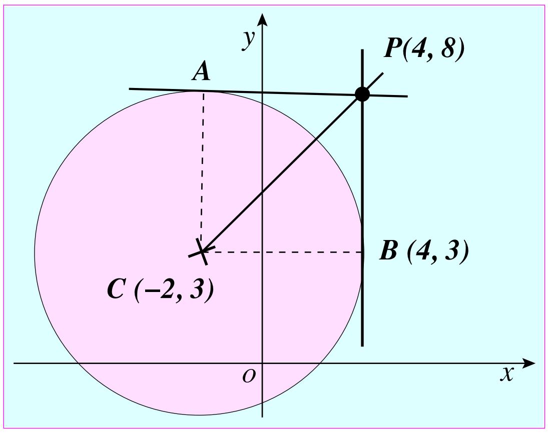
17.7.1 Example:

1

A circle $(x + 2)^2 + (y - 3)^2 = 36$ has a line drawn from its centre to a point P (4, 8).

What is the length of the line CP and the length of the tangent from P to the circle?

Sketch the circle.



Solution:

The length of the line CP can be found from Pythagoras.

From the co-ordinates of the points C & P, the differences in x and y positions are used thus:

$$CP^2 = (x_p - x_c)^2 + (y_p - y_c)^2$$

$$= (4 - (-2))^2 + (8 - 3)^2$$

$$= 6^2 + 5^2$$

$$\therefore \quad CP = \sqrt{61}$$

A tangent can be drawn from P to A and P to B.

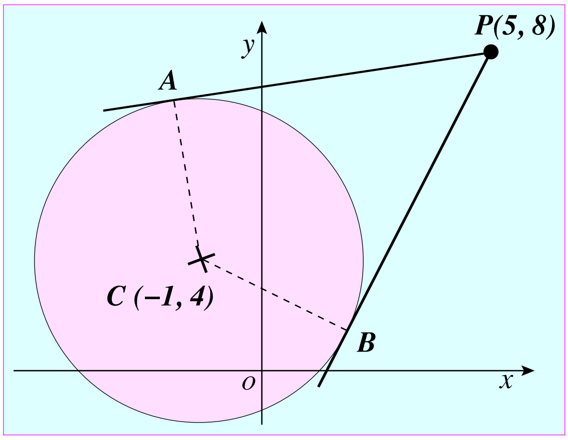
From the equation the radius is 6, (CB) and the since P is vertically above B, we can see that the length of the line BP is 5.

The length of the both tangents is therefore 5.

2 A circle $(x + 1)^2 + (y - 4)^2 = 9$ has a line drawn from its centre to a point P (5, 8).

What is the length of the tangent from P to the circle?

Sketch the circle.



Solution:

The length of the line CP can be found from Pythagoras.

From the co-ordinates of the points C & P, the differences in x and y positions are used thus:

$$CP^2 = (x_p - x_c)^2 + (y_p - y_c)^2$$
$$= (5 - (-1))^2 + (8 - 4)^2$$
$$= 6^2 + 4^2 = 52$$
$$\therefore \quad CP = \sqrt{52}$$

From the equation the radius is 3, (CB)

$$CP^2 = CB^2 + BP^2$$
$$\therefore \quad BP^2 = CP^2 - CB^2$$
$$= 52 - 3^2$$
$$= 43$$
$$BP = \sqrt{43}$$

17.8 Points On or Off a Circle

The general principle of proving that a given point lies on the circle is to show that the equation of the circle is satisfied when the co-ordinates of the point are substituted into the equation, and compare the LHS and RHS side of the equation.

To see if a given point lies inside or outside the circle, you need to compare the radius of the circle to the distance from the point to the centre of the circle. This can be done, either directly with pythag, using the co-ordinates of the point and the centre, or by substitution into the equation of the circle.

Having the equation in the form of $(x - a)^2 + (y - b)^2 = r^2$ is ideal, and means that after substituting the point co-ordinates into the LHS, a direct comparison can be made to the radius squared on the RHS.

If the equation is of the form $x^2 + y^2 - 2ax - 2by + c = 0$ then if the LHS equals zero, then the point is on the circle, if less than 1, inside the circle, or if greater than 1, outside the circle.

17.8.1 Example:

1 A circle has the equation $(x + 3)^2 + (y - 5)^2 = 5^2$. Show that point P (1, 2) lies on the circle and calculate whether point Q (−1, 2) is inside or outside the circle.

Solution:

For point P (1, 2), evaluate the LHS and compare with RHS:

Given $(x + 3)^2 + (y - 5)^2 = 25$

$$(1 + 3)^2 + (2 - 5)^2$$

$$(4)^2 + (-3)^2$$

$$16 + 9$$

$$\text{LHS} \quad 25 = 25 \quad \text{RHS}$$

$\therefore$ point P lies on the circle.

Point Q (−1, 2)

Given $(x + 3)^2 + (y - 5)^2 = 25$

$$(-1 + 3)^2 + (2 - 5)^2$$

$$(2)^2 + (-3)^2$$

$$4 + 9$$

$$\text{LHS} \quad 13 < 25 \quad \text{RHS}$$

$\therefore$ point Q lies inside the circle.

2 A circle has the equation $(x - 5)^2 + (y + 2)^2 = 5^2$. Establish if the line $y = 2x$ meets the circle in any way or lies outside the circle.

Solution:

If the line and the circle meet there should be a solution if $y = 2x$ is substituted into the equation of the circle:

Given $(x - 5)^2 + (y + 2)^2 = 25$

and $y = 2x$

$$(x - 5)^2 + (2x + 2)^2 = 25$$

$$x^2 - 10x + 25 + 4x^2 + 8x + 4 = 25$$

$$5x^2 - 2x + 4 = 0$$

To test for a solution, find the discriminant:

$$D = b^2 - 4ac$$

$$= 4 - 4 \times 5 \times 4$$

$$= -76$$

Hence, there is no solution, as the discriminant is negative.

3 Find the coordinates of the points where the circle $(x - 5)^2 + (y - 3)^2 = 90$ crosses the x-axis.

Solution:

If the line and the circle meet there should be a solution when $y = 0$ is substituted into the equation of the circle:

Given $(x - 5)^2 + (y - 3)^2 = 90$

and $y = 0$

$$(x - 5)^2 + (0 - 3)^2 = 90$$

$$(x - 5)^2 + 9 = 90$$

$$(x - 5)^2 = 81$$

$$(x - 5) = \pm\sqrt{81}$$

$$x = 5 \pm \sqrt{81}$$

$$x = 5 \pm 9$$

$$x = -4 \quad \text{and} \quad x = 14$$

17.9 Worked Examples

17.9.1 Example:

1 A circle has centre, C (7, 13), radius 13, with the equation:

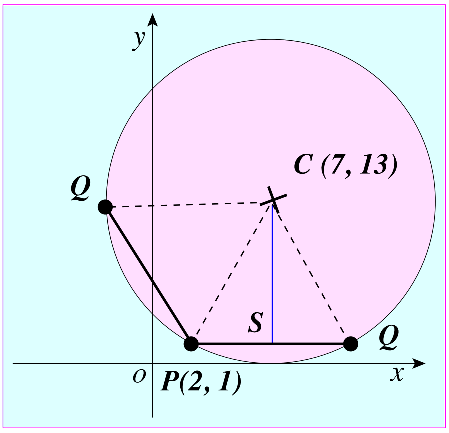
$$(x - 7)^2 + (y - 13)^2 = 13^2$$

A point P, lies on the circle at (2,1). Another point Q also lies on the circle and the length of the chord PQ is 10. What is the shortest distance from the centre C, to the chord PQ.
Prove point M (4, 10) is inside the circle.

Solution:

The point Q can take one of two positions, both will give the correct solution.

The shortest distance from C to the chord PQ is when the line CS is at a right angle to the chord and bisects the chord.



Using pythag on the triangle CSQ we have:

$$CS^2 + 5^2 = 13^2$$
$$CS^2 = 13^2 - 5^2 = 144$$
$$CS = \sqrt{144}$$

$\therefore$ Shortest distance from C to $PQ = 12$

If point M is inside the circle, then the distance CM must be less that the original radius.

Using pythag: $CM^2 = 3^2 + 3^2 = 54$

$CM = 7{\cdot}348$

2

17.10 Circle Digest

Equations of a Circle	Centre	Radius
$x^2 + y^2 = r^2$	$(0,\ 0)$	r
$(x - a)^2 + (y - b)^2 = r^2$	$(a,\ b)$	r
$(x - x_1)^2 + (y - y_1)^2 = r^2$	$(x_1,\ y_1)$	r
$x^2 + y^2 - 2ax - 2by + c = 0$ where $c = a^2 + b^2 - r^2$	$(a,\ b)$	$r = \sqrt{a^2 + b^2 - c}$

Other useful equations:

$$y = mx + c$$

$$y - y_1 = m(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$$

$$m_1 m_2 = -1$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Length of line between 2 points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Co-ordinate of the Mid point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

18 • C1 • Calculus 101

18.1 Calculus Intro

Anyone going on to study maths, science or engineering as a career will need to be adept at using calculus. It's almost as if everything you have ever done in maths so far, has been a preparation for learning about this new subject.

If you find it difficult to grasp at first, don't despair. It took two brilliant minds, in the form of Isaac Newton and Gottfried Leibniz, to discover the techniques and a further 100 years before it finally became of age, in the form that we now know it.

Calculus is divided, like Gaul, into two parts, differential calculus and integral calculus. Differential calculus and integral calculus are inverse operations so it is relatively easy to move from one to the other.

18.2 Historical Background

The two men attributed with the discovery of calculus are Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716), who developed their ideas independently of each other. Both these mathematicians looked at the problem in different ways, and indeed, their old methods have been reformulated into the more rigorous approach then we know today.

Newton started work during the Great Plague of 1664 and developed his differential calculus as the "method of fluxions" and integral calculus as the "inverse method of fluxions". It was many years later that he published his methods, by which time Leibniz was becoming well known from his own publications. Hence the controversy of who developed calculus first.

Leibniz was very careful about choosing his terminology and symbols, and it it mainly his notation that is used today. It is Leibniz that gave us dy/dx and the integral sign $\int$ which is a script form of the letter S, from the initial letter of the Latin word *summa* (sum).

18.3 What's it all about then?

In the next section we will learn the techniques required, but it is important to understand what calculus can do for us.

First, differential calculus.

In simple terms, differentiating a function will give us another function, called the gradient function, from which we can calculate the gradient of the curve at any given point. This gradient is defined as the gradient of the tangent to the curve at that point.

If you know the gradient you have a measure of how y is changing with respect to x. i.e. the rate of change.

For example if y is distance and x is time, the gradient gives $\dfrac{distance}{time}$ which is velocity.

Second, integral calculus.

As stated above, differential calculus and integral calculus are inverse operations. If you know the differential, or gradient function, you can get back to the original function by using integration, and visa versa. (Some simple caveats apply, but see later).

If you integrate a function you find you are actually measuring the area under the curve of the gradient function. However, it is not very intuitive to see how these two branches of calculus are connected. Perhaps the way to look at it is this: to integrate a gradient function in order to get back to the original function, you need a add up all the gradients defined by the gradient function. To do that, take a bacon slicer and slice the gradient function up into incredible small slices and sum the slices together. In doing so, you end up with the area under the curve. Remember, when integrating a function, you need to think of that function as a gradient function.

differentiation $\Rightarrow$ gradient of a curve

integration $\Rightarrow$ area under a curve

18.4 A Note on OCR/AQA Syllabus Differences

In C1 & C2 only functions of the form $y = ax^n$ are considered.

OCR splits calculus with rational functions into two parts: differentiation in C1 & integration in C2.

AQA takes a different approach. C1 contains differentiation & integration, but only for positive integers of n, whilst C2 then considers differentiation & integration for all rational values of n.

19 • C1 • Differentiation I

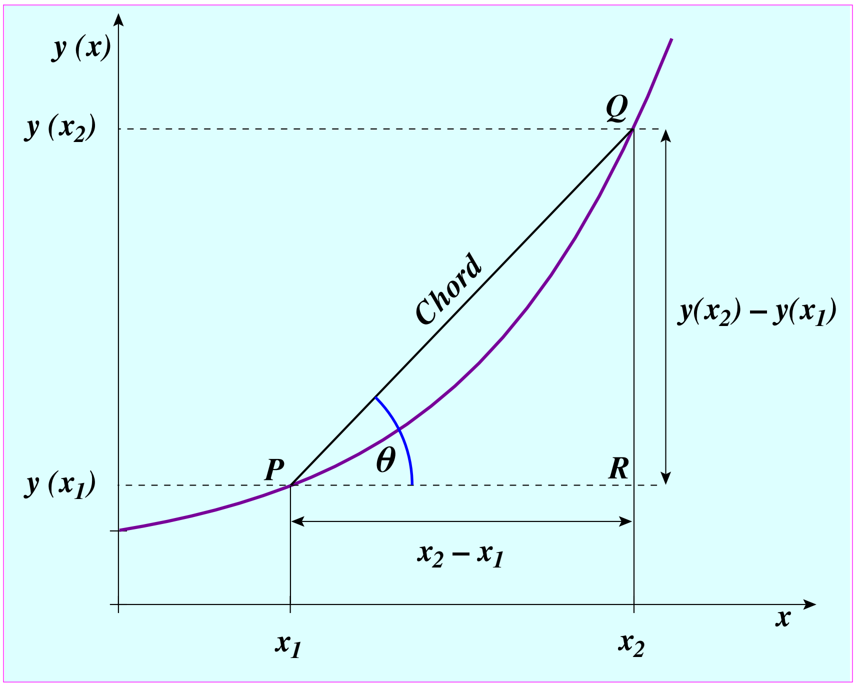
OCR C1 / AQA C1 / C2

Differentiation is a major branch of maths that explores the way in which functions change with respect to a given variable. In particular, it is concerned with the **rate** at which a function changes at any given point. In practise this means measuring the gradient of the curve at that given point and this has been defined as the gradient of the tangent at that point.

To find this gradient we derive a special Gradient function that will give the gradient at any point on the curve. This is called differentiation.

Differentiation also allows us to find any local maximum or minimum values in a function, which has many practical uses in engineering etc.

19.1 Average Gradient of a Function



Average Gradient of a function

The average gradient of a curve or function between two points is given by the gradient of the chord connecting the points. As illustrated, the chord PQ represents the average gradient for the interval x_1 to x_2.

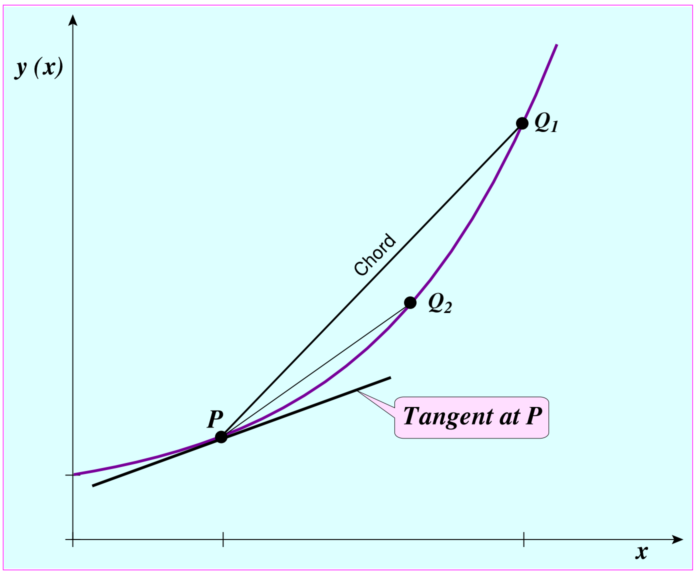
$$\text{Gradient} = \frac{rise}{run} = \frac{QR}{PR} = \frac{y(x_2) - y(x_1)}{x_2 - x_1}$$

Note that: $\tan\theta = \dfrac{QR}{PR}$

The gradient represents the rate of change of the function.

We can see this by looking at the units of the gradient. If the y-axis represents, say, distance and the x-axis represents time, then the units of the gradient would be $distance / time = speed$.

So far so good, but we really need the rate of change at a given point, say P. The average gradient is only an approximation to the actual gradient at P, but this can be improved if we move point Q closer to point P. As Q get closer to P, the straight line of the chord becomes the tangent to the curve at P. See illustration below.



Gradient of a function at P

19.2 Limits

The concept of limits is absolutely fundamental to calculus and many other branches of maths. The idea is simple enough: we ask what happens to a function when a variable approaches a particular value.

If the variable is x and it approaches, (or tends towards), the value k, we write $x \to k$. Beware, this is not the same as saying that $x = k$, as the function might not be defined at k. We have to sneak up on the solution:-)

As $x \to k$, we can find the value that our function approaches, and this is called the **limit** of the function.

This is expressed with the following notation:

$$\lim_{x \to k} f(x) = L$$

This is read as "the limit of f of x, as x approaches k, is L". This does not mean that $f(k) = L$, only that the **limit** of the function is equal to L.

From the graph above, we can see that as the interval between P & Q gets smaller, then the gradient of the chord tends toward the gradient of the tangent. The gradient of the tangent is the limit.

19.2.1 Example:

Find the limit of the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ as x approaches 1.

Solution:

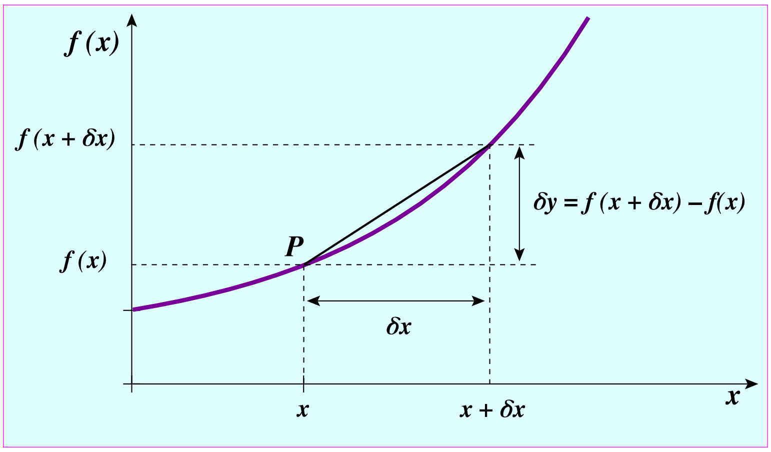
Note that $f(1)$ is not defined, (the denominator would be 0 in this case).

$f(0{\cdot}9)$	$f(0{\cdot}99)$	$f(0{\cdot}999)$	$f(1{\cdot}0)$	$f(1{\cdot}001)$	$f(1{\cdot}01)$	$f(1{\cdot}1)$
1·900	1·990	1·999	*undefined*	2·001	2·010	2·100

$$\therefore \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

$$\text{or} \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

19.3 Differentiation from First Principles



Differentiation from First Principles

Starting with the average rate of change as before, using an interval from x to $x + \delta x$, where δx is a very small increment. The value of our function $f(x)$ will range from $f(x)$ to $f(x + \delta x)$.

$$\text{Gradient} = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{\delta y}{\delta x}$$

$$= \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{(x + \delta x) - x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Now let $\delta x \to 0$. In other words, let the interval shrink to a point, at P:

$$\text{Gradient} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

This limit function is denoted by the symbols:

$$\frac{dy}{dx} \quad \text{or} \quad f'(x)$$

Thus: $$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

This is called the gradient function, the first derivative or differential of y with respect to (w.r.t) x.

19.3.1 Example:

An example of differentiating from first principles:

If $f(x) = 2x^2 + 3x + 4$

then $f(x + \delta x) = 2(x + \delta x)^2 + 3(x + \delta x) + 4$

$\therefore \quad \delta y = f(x + \delta x) - f(x)$

$= 2(x + \delta x)^2 + 3(x + \delta x) + 4 - (2x^2 + 3x + 4)$

$= (4x + 3)\delta x + 2(\delta x^2)$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{(4x + 3)\delta x + 2(\delta x^2)}{\delta x} = 4x + 3 + 2(\delta x)$$

As $\delta x \to 0$ then:

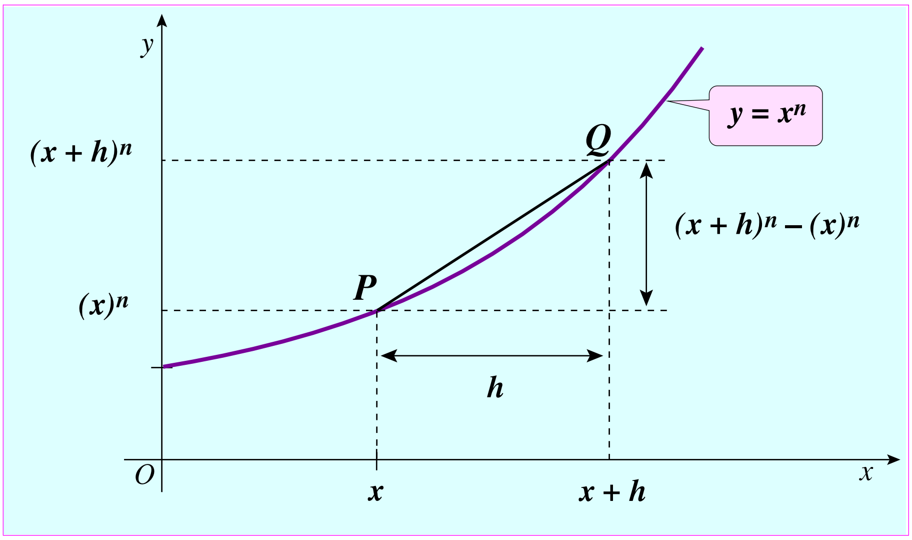
$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 4x + 3$$

Hence $4x + 3$ is the limiting value as δx approaches zero, and is called the differential of $f(x)$.

19.4 Deriving the Gradient Function

Traditionally, the symbols δx & δy have been used to denote the very small increments in x & y. The increments, δx & δy, should not be confused with dx & dy.

The gradient for any function can be found using the above example, but differentiation from first principles is rather long winded. A more practical method is derived next. and we use h instead of δx.



Deriving the Gradient Function

$$\text{Gradient of } PQ = \frac{\text{change in } y}{\text{change in } x} = \frac{(x+h)^n - x^n}{h}$$

Use the binomial theorem to expand $(x+h)^n$

$$= \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n\right) - x^n}{h}$$

$$= \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n}{h}$$

$$= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}$$

Now let $h \to 0$. In other words, let the interval h shrink to a point, at P and chord PQ tends to the tangent at P:

$$\text{Gradient at } P = \frac{dy}{dx} = \lim_{h \to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}\right]$$

$$= nx^{n-1} + 0 + \ldots + 0$$

$$\therefore \qquad \frac{dy}{dx} = nx^{n-1}$$

Hence, the general term for the gradient function of x^n *is* nx^{n-1}, which applies for all real numbers of n.

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

19.5 Derivative of a Constant

We can use the normal rules derived in the last section for finding the derivative of a constant, C.

For $y = C$:

$$\begin{aligned} y &= C \\ &= Cx^0 \\ \therefore \quad \frac{dy}{dx} &= C \times 0 \times x^{-1} \\ &= 0 \end{aligned}$$

This makes sense as $y = c$ represents a horizontal straight line, which has a gradient of zero. In addition adding a constant to a function only changes its position vertically and does not change the gradient at any point.

19.6 Notation for the Gradient Function

If the equation is given in the form $y = ax\ldots$ then the gradient function is written $\frac{dy}{dx}$.

Similarly, for an equation such as $s = t^2 - 4t$ the gradient function is written as $\frac{ds}{dt}$.

For an equation in the form $f(x) = \ldots$ then the gradient function is written $f'(x)$.

It should be understood that $\frac{dy}{dx}$ is not a fraction, but is rather an operator $\frac{d}{dx}$ on y. Perhaps $\frac{d(y)}{dx}$ *or* $\frac{d}{dx}(y)$ is a better way to write the gradient function.

Later on, in C3, we will see that :

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

19.7 Differentiating Multiple Terms

Using function notation; the following is true:

$$\text{If } y = f(x) \pm g(x) \text{ then } \frac{dy}{dx} = f'(x) \pm g'(x)$$

In other words, we differentiate each term individually. When differentiating, you will need to put the function in the right form.

- Differentiating $af(x) \Rightarrow af'(x)$
- Terms have to be written as a power function before differentiating, e.g. $\sqrt{x} = x^{\frac{1}{2}}$
- Brackets must be removed to provide separate terms before differentiating, e.g. $(x - 4)(x - 1) \Rightarrow x^2 - 5x + 4$
- An algebraic division must be put into the form $ax^n + bx^{n-1}\ldots c$ e.g. $y = \frac{x^4 + 7}{x^2} = x^2 + 7x^{-2}$
- Differentiating a constant term results in a zero
- Recall $x^0 = 1$

19.8 Differentiation: Worked Examples

19.8.1 Example:

1 Differentiate the equation $y = x^6 - 2x^4 + 3x^2 - 4x + 5$ and find the gradient at the point (1, 3).

$$y = x^6 - 2x^4 + 3x^2 - 4x + 5$$

$$\frac{dy}{dx} = 6x^5 - 8x^3 + 6x - 4$$

When $x = 1$ $\quad \frac{dy}{dx} = 6 - 8 + 6 - 4 = 0$

2 Differentiate the equation $y = 2x^4\sqrt{x}$ and find the gradient at the point (1, 2).

$$y = 2x^4\sqrt{x} = 2x^4x^{0.5} = 2x^{4.5}$$

$$\frac{dy}{dx} = 9x^{3.5}$$

When $x = 1$ $\quad \frac{dy}{dx} = 9$

3 Differentiate the equation $y = \frac{2}{\sqrt[3]{x}}$

$$y = \frac{2}{\sqrt[3]{x}} = 2x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{4}{3}} = -\frac{2}{3x^{\frac{4}{3}}} = -\frac{2}{3\left(x^{\frac{1}{3}}\right)^4} - \frac{2}{3\left(\sqrt[3]{x}\right)^4}$$

19.9 Rates of Change

Differentiation is all about rates of change. In other words, how much does y change with respect to x. Thinking back to the definition of a straight line, the gradient of a line is given by the change in y co-ordinates divided by the change in x co-ordinates. So it should come as no surprise that differentiation also gives the gradient of a curve at any given point.

Perhaps the most obvious example of rates of change is that of changing distance with time which we call speed. This can be taken further, and if the rate of change of speed with respect to time is measured we get acceleration.

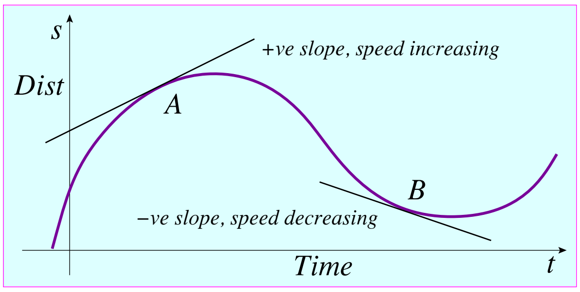
In terms of differentiation this can be written as:

$$\frac{ds}{dt} = v \qquad \text{where } s = \text{distance, } t = \text{time \& } v = \text{velocity}$$

$$\frac{dv}{dt} = a \qquad \text{where } s = \text{distance, } t = \text{time \& } a = \text{acceleration}$$

$$\frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}\cdot\frac{ds}{dt} = \frac{d^2s}{dt^2}$$

The gradient at A is the rate at which distance is changing w.r.t time. i.e. speed. A +ve slope means speed is increasing and a −ve slope means it is decreasing.



19.9.1 Example:

1 An inert body is fired from a catapult, at time $t = 0$, and moves such that the height above sea level, y m, at t secs, is given by:

$$y = \frac{1}{5}t^5 - 16t^2 + 56t + 3$$

a) Find $\frac{dy}{dt}$ and the rate of change of height w.r.t time when $t = 1$

b) When $t = 2$, determine if the height is increasing or decreasing.

Solution:

$$\frac{dy}{dt} = t^4 - 32t + 56$$

When $t = 1$ $\quad \frac{dy}{dt} = 1 - 32 + 56 = 25\, m/sec$

When $t = 2$ $\quad \frac{dy}{dt} = 2^4 - 64 + 56 = 8\, m/sec$

Since the differential is positive the height must be increasing.

19.10 Second Order Differentials

So far we have differentiated the function $y = f(x)$ and found the first derivative $\frac{dy}{dx}$ or $f'(x)$.

If we differentiate this first derivative, we obtain the second derivative written as $\frac{d^2y}{dx^2}$ or $f''(x)$.

We can then use this second derivative to classify parts of the curve, (see later). Do not confuse the notation as a squared term. It simply means the function has been differentiated twice. This is the conventional way of writing the 2nd, 3rd, or more orders of differential.

19.10.1 Example:

1 Find the second derivative of $y = 2x^4 - 3x^2 + 4x - 5$

$$\frac{dy}{dx} = 8x^3 - 6x + 4$$

$$\frac{d^2y}{dx^2} = 24x^2 - 6$$

2 Find the second derivative of $y = 4\sqrt{x}$

$$y = 4\sqrt{x} = 4x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -x^{-\frac{3}{2}}$$

3 Find the second derivative of $y = 3x^5 - \sqrt{x} + 15$

$$y = 3x^5 - x^{\frac{1}{2}} + 15$$

$$\frac{dy}{dx} = 15x^4 - \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 60x^3 + \frac{1}{4}x^{-\frac{3}{2}}$$

19.11 Increasing & Decreasing Functions

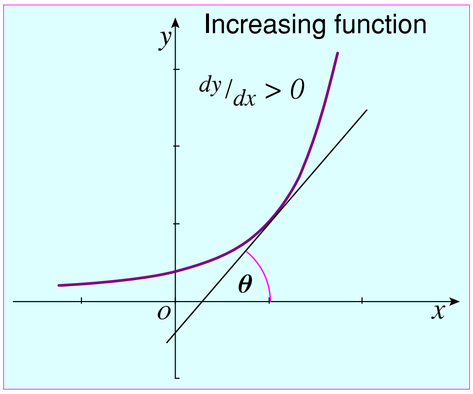
When moving left to right on the x-axis, if the gradient of the curve is positive then the function is said to be an increasing function, and if the gradient is negative then the function is said to be a decreasing function.

Increasing Function

As x increases y increases and $\frac{dy}{dx}$ is positive

$Tan\,\theta > 0$ for range of $0 < \theta < \frac{\pi}{2}$

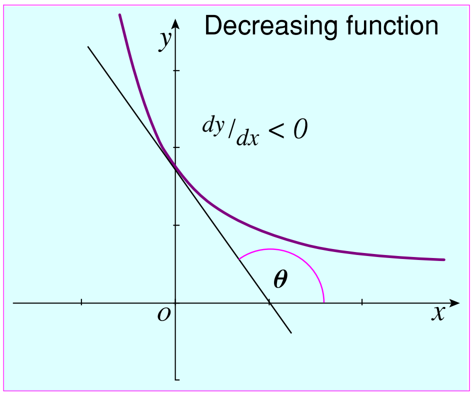
Note: $Tan\,\theta = \frac{dy}{dx}$



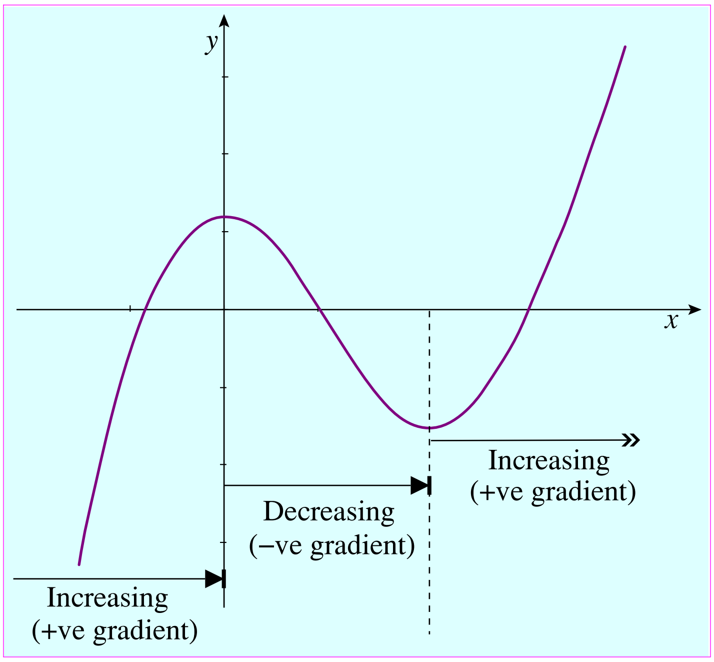
Decreasing Function

As x increases y decreases and $\frac{dy}{dx}$ is negative

$Tan\,\theta < 0$ for range of $\frac{\pi}{2} < \theta < \pi$



In this following example, the function has increasing and decreasing parts to the curve and the values of x must be stated when describing these parts. Note that at the change over from an increasing to a decreasing function and visa versa, the gradient is momentarily zero. These points are called stationary points – more later.



Increasing and Decreasing Function

To find the values of x for which the function is either increasing and decreasing, differentiate the function and set the gradient function to > 0, or < 0 accordingly. Then solve the inequality. It is instructive to see the both the function and gradient function plotted on the same graph, as in the first example below.

19.11.1 Example:

1 For what values of x does $x^3 - 3x^2 + 4$ become an increasing function?

Solution:

$$y = x^3 - 3x^2 + 4$$

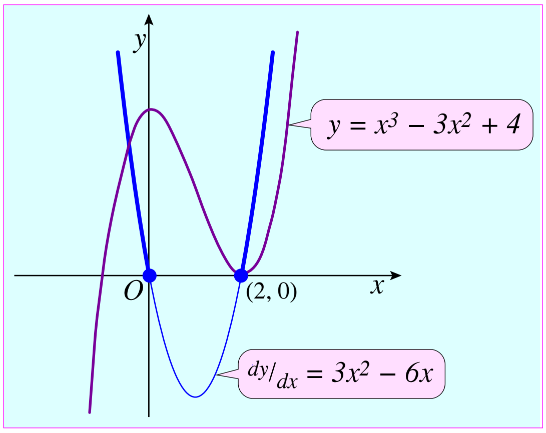
$$\frac{dy}{dx} = 3x^2 - 6x$$

For increasing y $\quad \frac{dy}{dx} > 0 \qquad \therefore \qquad 3x^2 - 6x > 0$

$$x(3x - 6) > 0$$

$\therefore \qquad x < 0 \quad or \quad x > 2$

Note how the curve of the gradient function which is above the x-axis matches the parts of original function that are increasing.



2 Show that the function $f(x) = x^5 - x^{-3}$ is an increasing function for $x > 1$.

Solution:

$$f(x) = x^5 - x^{-3}$$

$$f'(x) = 5x^4 + 3x^{-4}$$

For an increasing function, $f'(x)$ must be > 0.

$\therefore \qquad 5x^4 + 3x^{-4} > 0$

$\therefore \qquad 5x^4 + \frac{3}{x^4} > 0$

Any value of $x > 1$ will give a positive result.

3 Find the values of x for which the function $f(x) = x^3 + 3x^2 - 9x + 6$ is decreasing.

Solution:

$$f(x) = x^3 + 3x^2 - 9x + 6$$

$$f'(x) = 3x^2 + 6x - 9$$

For an decreasing function, $f'(x)$ must be < 0.

$$3x^2 + 6x - 9 < 0$$

$$3(x^2 + 2x - 3) < 0$$

$$3(x + 3)(x - 1) < 0$$

$\therefore \qquad -3 < x < 1$

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OCR C1 / AQA C1 / C2

20.1 Tangent & Normals

Recall that once the gradient of a line has been found, then the gradient of the normal to the line can also be found using the key equation below:

$$m_1 \,.\, m_2 = -1$$

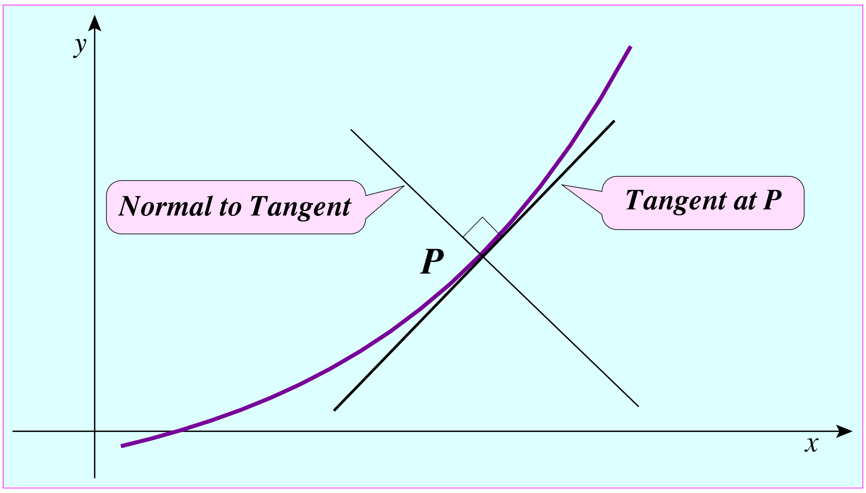
where m_1 = gradient of the tangent and m_2 = gradient of the normal.

Remember that the equation of a straight line that passes through the point (x_1, y_1) is given by

$$y - y_1 = m_1(x - x_1)$$

and the equation of the normal is given by:

$$y - y_1 = -\frac{1}{m_1}(x - x_1)$$



Tangents and normals to a curve

20.1.1 Example:

1 Find the equation of the tangent to the curve $y = 2x^2 - 3x + 4$ at the point (2, 6):

Solution:

$$\frac{dy}{dx} = 4x - 3$$

At (2, 6) $x = 2$, $\quad \frac{dy}{dx} = 8 - 3 = 5$

Equation of tangent:

$$y - 6 = 5(x - 2)$$

$$y = 5x + 4$$

2 Show that there are 2 points on the curve $y = x^2(x - 2)$ at which the gradient is 2, and find the equations of the tangent at these points.

Solution:

$$y = x^2(x - 2)$$
$$= x^3 - 2x^2$$
$$\therefore \quad \frac{dy}{dx} = 3x^2 - 4x$$
$$\text{Now:} \quad \frac{dy}{dx} = 2$$
$$\therefore \quad 3x^2 - 4x = 2$$
$$3x^2 - 4x - 2 = 0$$
$$x = \frac{4 \pm \sqrt{16 - 4 \times 3 \times (-2)}}{6}$$
$$x = \frac{4 \pm 2\sqrt{10}}{6} = \frac{2 \pm \sqrt{10}}{3}$$

$\therefore$ there are 2 points at which the gradient is 2 etc…

3 Find the equation of the tangent to the curve $y = (x - 2)(x + 6)$ at the points where the curve cuts the x-axis, and find the co-ordinates of the point where the tangents intersect.

Solution:

Function cuts the x-axis at (2, 0) and (−6, 0)

$$y = (x - 2)(x + 6) \Rightarrow x^2 + 4x - 12$$
$$\frac{dy}{dx} = 2x + 4$$

At point (2, 0), $x = 2$ and $\therefore$ gradient = 8

Hence equation of tangent is:

$$y - 0 = 8(x - 2)$$
$$y = 8x - 16 \qquad (1)$$

At point (−6, 0), $x = -6$ and $\therefore$ gradient = −8

Hence equation of tangent is:

$$y - 0 = -8(x + 6)$$
$$y = -8x - 48 \qquad (2)$$

Solve simultaneous equations (1) & (2) to find intersection at:

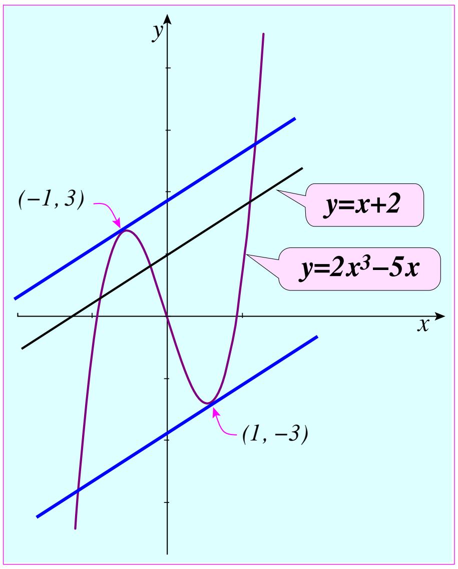
$$8x - 16 = -8x - 48$$
$$16x = -32$$
$$x = -2$$
$$\therefore \quad y = -16 - 16 = -32$$

Co-ordinate of intersection of tangents is (−2, −32)

4 Find the equation of the tangents to the curve $y = 2x^3 - 5x$ which are parallel to the line $y = x + 2$.

Solution:

Gradient of tangents have the same gradients as the line $y = x + 2$ which has a gradient of 1.
Therefore, find the points on the curve where the gradients are 1.



$$\frac{dy}{dx} = 6x^2 - 5 = 1$$

$$\Rightarrow 6x^2 = 6 \quad \Rightarrow \quad x^2 = 1$$

$$\therefore x = \pm 1$$

Hence when:
$x = 1 \qquad y = 2 - 5 = -3$
$x = -1 \qquad y = -2 + 5 = 3$

The two points are: $(1, -3)$ & $(-1, 3)$
Tangent at $(1, -3)$: $y + 3 = 1(x - 1)$
$y = x - 4$

Tangent at $(-1, 3)$: $y - 3 = 1(x + 1)$
$y = x + 4$

5 Find the equation of the normal to the curve $y = 2x - x^3$ at the point where $x = -1$. Find the co-ordinates of the points at which this normal meets the curve again.

Solution:

At the point where $x = -1$, $y = -1$.

$$\frac{dy}{dx} = 2 - 3x^2$$

At $(-1, -1)$ $\quad \frac{dy}{dx} = 2 - 3(-1)^2 = -1$

$\therefore$ gradient of normal $= 1$

Equation of normal: $y + 1 = 1(x + 1)$

$$\Rightarrow \quad y = x$$

Solve for x & y in (1) & (2):

$$y = x \qquad (1)$$

$$y = 2x - x^3 \qquad (2)$$

Substitution into (2) $\therefore$ $x = 2x - x^3$

$$\Rightarrow \quad x^3 - 2x + x = 0$$

$$\Rightarrow \quad x(x^2 - 1) = 0$$

$$x(1 + x)(1 - x) = 0$$

$\therefore \quad x = 0, \ -1, \text{ and } 1$

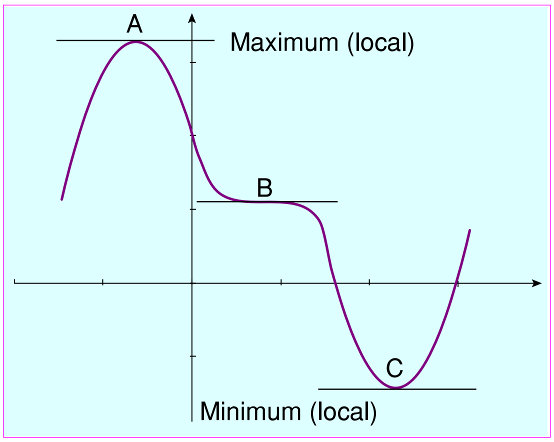
$\therefore$ Normal meets curve at $(-1, -1)$ (given) and also $(0, 0)$ & $(1, 1)$

20.2 Stationary Points

This is one of the most important applications of differentiation. Stationary points often relate to a maximum or minimum of an area, volume or rate of change or even a profit/loss in a business. Here, the rate of change is momentarily nil and the gradient is zero, hence they are called stationary points. A stationary point is one where the function stops increasing or decreasing.

There are two types of stationary point; a turning point and an inflection point.

- **Turning Points** are points where a graph changes direction and the gradient changes sign, they can be either a maximum or a minimum point. (see point A & C on diagram below)
- An **Inflection point** changes its sense of direction, but the gradient does not change sign, (see point B on diagram below)
- At all these points, the gradient of the tangent is 0
- To find a turning point, let $\frac{dy}{dx} = 0$
- Curves can have more than one max or min point, hence these may be named as a **Local** max or min.



Turning points illustrated at point A and C

20.2.1 Example: Stationary Points

1 Find the co-ordinates for the two stationary points of the equation $2x^2 + xy + y^2 = 64$:

Solution

Differentiating each term to find the dy/dx:

$$4x + \frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}(1 + 2y) = -4x$$

$$\frac{dy}{dx} = -\frac{4x}{(1 + 2y)}$$

Stationary point when gradient $= 0$

$$\frac{dy}{dx} = -\frac{4x}{(1 + 2y)} = 0$$

$$\therefore \quad 4x = 0 \quad \Rightarrow x = 0$$

Now solve original equation for y

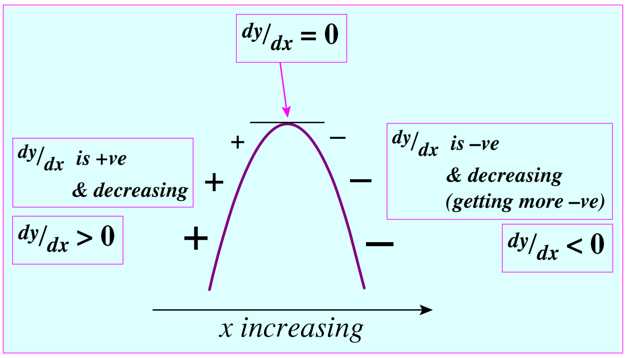
$$0 + 0 + y^2 = 64$$

$$\therefore \quad y = \sqrt{64} = \pm 8$$

Co-ordinates are: $(0, 8)$ & $(0, -8)$

20.3 Maximum & Minimum Turning Points

From the diagram, moving left to right, as x increases, $\frac{dy}{dx}$ is positive, but decreasing.
The gradient decreases to 0 at the local maximum, then becomes negative.
The gradient continues to become more negative as x increases, i.e. $\frac{dy}{dx}$ continues to decrease.



There are three ways of distinguishing between max & min points:

- by testing the value of ***y*** either side of the turning point
- by the Second Derivative Test
- by testing the **gradient** either side of the turning point

20.3.1 Testing the value of y

Finding the value of y either side of the turning point is one method of finding a max and min. However, most questions expect you to find the derivative of the function and then solve to give the co-ordinates of the turning points. In which case the other two methods are preferred.

20.3.2 Second Derivative Test for Max or Min

The derivative $\frac{dy}{dx}$ represents how y changes w.r.t x. We need an expression to show how $\frac{dy}{dx}$ changes w.r.t to x.
Differentiating $\frac{dy}{dx}$ to find $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ will give us the required expression.
This is called the second derivative and is written $\frac{d^2y}{dx^2}$.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ then the point must be a maximum, because $\frac{dy}{dx}$ is decreasing as x increases.

Similar arguments exist for the minimum case, so:

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ then the point must be a minimum, because $\frac{dy}{dx}$ is increasing as x increases.

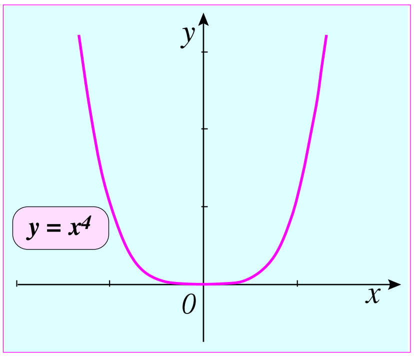
There is an exception to these rules, which is when $\frac{dy}{dx} = 0$ **and** $\frac{d^2y}{dx^2} = 0$.

In this example $y = x^4$.

$$\frac{dy}{dx} = 4x^3 \Rightarrow 0 \quad \text{when } x = 0$$

$$\frac{d^2y}{dx^2} = 12x^2 \Rightarrow 0 \quad \text{when } x = 0$$

(Note: a similar graph is produced when $y = x^n$ and n is even and $\geqslant 4$).



When $\frac{d^2y}{dx^2} = 0$, we have either a maximum, a minimum or some other arrangement. So the 2nd derivative test does not always reveal the solution and the third method should be used.

20.3.2.1 Example:

1 Find the turning points of $y = x^2(6 - x)$ and distinguish between them.

Solution:

$$y = 6x^2 - x^3 \qquad \therefore \quad \frac{dy}{dx} = 12x - 3x^2$$

Let $\quad 12x - 3x^2 = 0$

$3x(4 - x) = 0$

$\therefore \quad x = 0 \text{ and } 4$

$$\frac{d^2y}{dx^2} = 12 - 6x$$

when $x = 4 \quad \dfrac{d^2y}{dx^2} = 12 - 24 = -12 \quad$ is negative

when $x = 0 \quad \dfrac{d^2y}{dx^2} = 12 - 0 = 12 \quad$ is positive

$\therefore$ (4, 32) is a maximum and (0, 0) is therefore a minimum.

2 Find the co-ordinates for the turning points of $y = x^3 - 3x^2 + 4$ and identify the max and min points.

Solution:

$$y = x^3 - 3x^2 + 4$$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 6x$$

Let $\quad 3x^2 - 6x = 0$

$3x(x - 2) = 0$

$\therefore \quad x = 0 \text{ and } 2$

When $\quad x = 0, \quad y = 4$

When $\quad x = 2, \quad y = 0$

Co-ordinates of the turning points are: $\quad (0,\ 4) \ (2,\ 0)$

$$\frac{d^2y}{dx^2} = 6x - 6$$

when $x = 0 \quad \dfrac{d^2y}{dx^2} = -6 \quad$ is negative i.e. maximum

when $x = 2 \quad \dfrac{d^2y}{dx^2} = 6 \quad$ is positive i.e. minimum

$\therefore$ (0, 4) is a maximum and (2, 0) is therefore a minimum.

20.3.3 Gradient Test for Max or Min

With this method you need to test the gradient either side of the turning point. An example will illustrate the method:

20.3.3.1 Example:

Find the co-ordinates of the stationary points of the function $y = 2x^3 + 3x^2 - 72x + 5$

Solution:

$$\frac{dy}{dx} = 6x^2 + 6x - 72$$

Let: $6x^2 + 6x - 72 = 0$

$\div 6$ $x^2 + x - 12 = 0$

$(x + 4)(x - 3) = 0$

$\therefore$ $x = -4, \ or\ 3$

Examine the gradients either side of the solutions:

Use values of $x = -4 \pm 1, \text{ and } 3 \pm 1$

If $x = -5$, $\frac{dy}{dx} = 150 - 30 - 72 = 48$ i.e. positive

If $x = -3$, $\frac{dy}{dx} = 54 - 18 - 72 = -36$ i.e. negative

If $x = 2$, $\frac{dy}{dx} = 24 + 18 - 72 = -30$ i.e. negative

If $x = 4$, $\frac{dy}{dx} = 96 + 24 - 72 = 48$ i.e. positive

	−5	(−4)	−3
$\frac{dy}{dx}$	+		−

	2	(3)	4
$\frac{dy}{dx}$	−		+

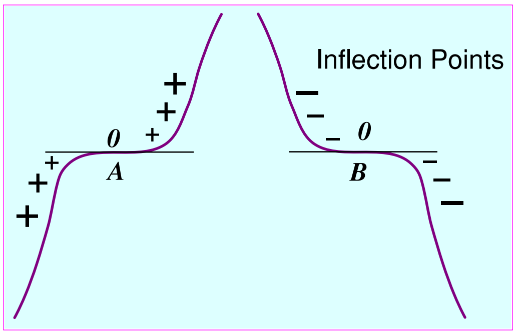
$\therefore$ $(-4, 213)$ *is a max*, $(3, -13)$ *is a min*

20.4 Points of Inflection & Stationary Points (Not in Syllabus)

A point of inflection is one in which the function changes its sense of direction. It changes from a clockwise direction to an anti-clockwise direction, or from concave downward to being concave upward, or visa-versa.

Inflection points can be either stationary or non-stationary.

Stationary Inflection Points



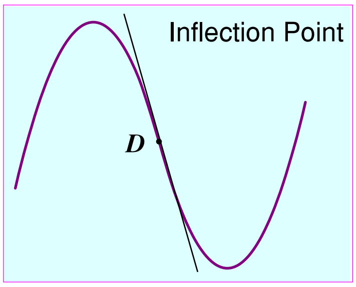
The gradient of the curve leading to point A is positive, decreases to 0, and then increases again, but remains positive.
Similarly, the gradient leading to point B is negative, decreases to 0, then becomes negative again.

The tangent at the inflection point is parallel with the x axis and so $\frac{dy}{dx} = 0$, hence it is a stationary point.
The second derivative is also 0 i.e. $\frac{d^2y}{dx^2} = 0$

At an inflection point, $\frac{d^3y}{dx^3} \neq 0$.

The tangent crosses the curve at the inflection point.

Non-stationary Inflection Points



In this case, $\frac{dy}{dx} \neq 0$, but $\frac{d^2y}{dx^2} = 0$

The tangent crosses the curve at the inflection point.

20.5 Classifying Types of Stationary Points

20.6 Max & Min Problems (Optimisation)

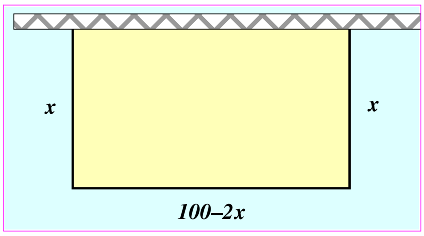
- In any max/min question differentiating twice will generally be needed. Once to solve $dy/dx = 0$ to find the x values of the max/min points and a second time to determine if they are maxima or minima. ($d^2y/dx^2 = +ve$ is a minima).
- An area problem will differentiate to a linear equation, with only one solution for the optimum point.
- A volume problem will differentiate to a quadratic equation, and a choice of a max and min will appear.
- There will be often a question asking for an explanation of why one of the answers is a valid answer and the other is not. In this case, plug the values found back into the original equations and see if they make sense.

20.6.1 Example:

1 A rectangular piece of ground is to be fenced off with 100m of fencing, where one side of the area is bounded by a wall currently in place. What is the maximum area that can be fenced in?

Solution:

Method of attack:
a) What is max/min (area in this case)
b) Find a formula for this (one variable)
c) Differentiate formula and solve for $dy/dx = 0$
d) Substitute answer back into the original formula



$$Area = x(100 - 2x)$$
$$= 100x - 2x^2$$
$$\frac{dA}{dx} = 100 - 4x$$

For a Max/Min $\quad 100 - 4x = 0 \qquad \left(\frac{d^2y}{dx^2} = -4, \quad \therefore \text{a maximum}\right)$

$$\therefore \quad x = 25$$

Max Area $= 25(100 - 50) = 1250\ m^2$

2 A cuboid has a square base, side x cm. The volume of the cuboid is 27 cm^3.
Given that the surface area $A = 2x^2 + 108x^{-1}$ find the value of x for the minimum surface area.

$$A = 2x^2 + 108x^{-1}$$
$$\frac{dA}{dx} = 4x - 108x^{-2}$$

For min/max $\quad \frac{dA}{dx} = 4x - 108x^{-2} = 0$

$$\therefore \quad 4x = \frac{108}{x^2}$$
$$4x^3 = 108$$
$$x^3 = 27 \qquad \left(\frac{dA}{dx} = 4 + 216x^{-3}\right)$$
$$x = 3 \qquad \left(\Rightarrow 4 + 216 \times 3^{-3} \Rightarrow +ve \therefore \text{a min}\right)$$

Minimum surface area is when the cuboid is a cube with all sides equal to x.

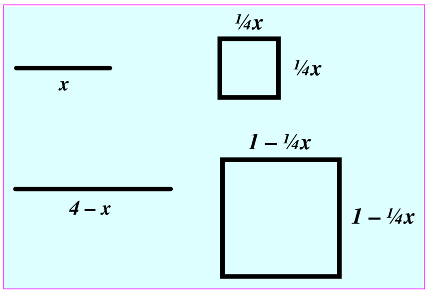
3 A piece of wire, length 4m, is cut into 2 pieces (not necessarily equal), and each piece is bent into a square. How should this be done to have:

a) the smallest total area from both squares?

b) the largest total area from both squares?

Solution:

Draw a sketch!



Total area is:

$$Area = \left(\frac{1}{4}x\right)^2 + \left(1 - \frac{1}{4}x\right)^2$$

$$= \frac{1}{16}x^2 + \left(1 - \frac{1}{4}x\right)\left(1 - \frac{1}{4}x\right)$$

$$= \frac{1}{16}x^2 + 1 - \frac{1}{4}x - \frac{1}{4}x + \frac{1}{16}x^2$$

$$= \frac{2}{16}x^2 - \frac{1}{2}x + 1$$

$$= \frac{1}{8}x^2 - \frac{1}{2}x + 1$$

$$\frac{dy}{dx} = \frac{1}{4}x - \frac{1}{2}$$

Let $\frac{1}{4}x - \frac{1}{2} = 0$ (for max/min)

$\frac{1}{4}x = \frac{1}{2}$

$\therefore$ $x = 2$

Now $\frac{d^2y}{dx^2} = \frac{1}{4}$ i.e. positive, $\therefore$ a minimum

a) smallest area is therefore when $x = 2$, (i.e when wire cut in half)

$$Area = \frac{1}{8}(2^2) - \frac{1}{2}(2) + 1$$

$$= \frac{1}{2}m^2$$

b) biggest area must be when $x = 0, \Rightarrow 1\ m^2$

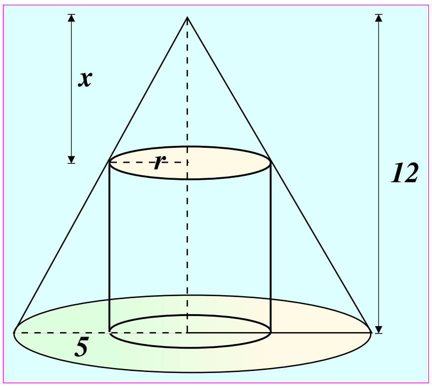
4 A hollow cone of radius 5cm and height 12 cm, is placed on a table. What is the largest cylinder that can be hidden underneath it?

Solution:

Recall that:

Volume of cone = $\frac{1}{3}\pi r^2 h$

Volume of cylinder = $\pi r^2 h$



Consider the cone split into two cones…

Ratio of radius/height of large cone to small cone:

$$5 : 12 = r : x$$

$$\therefore \quad x = \frac{12}{5}r$$

$$\therefore \text{ Height of cylinder } = 12 - \frac{12}{5}r$$

$$\text{Volume of cylinder } = \pi r^2\left(12 - \frac{12}{5}r\right)$$

$$= 12\pi r^2 - \frac{12}{5}\pi r^3$$

$$\frac{dV}{dx} = 24\pi r - \frac{36}{5}\pi r^2$$

Min or max: $\quad 24\pi r - \frac{36}{5}\pi r^2 = 0$

$\times 5 \quad 120\pi r - 36\pi r^2 = 0$

$$\pi r(10 - 3r) = 0$$

$$r = 0 \textbf{ or } \frac{10}{3}$$ $r = 0$ means no cylinder - reject soln

$$\therefore \quad \text{Max Volume } = \pi\left(\frac{10}{3}\right)^2\left(12 - \frac{12}{5} \times \frac{10}{3}\right)$$

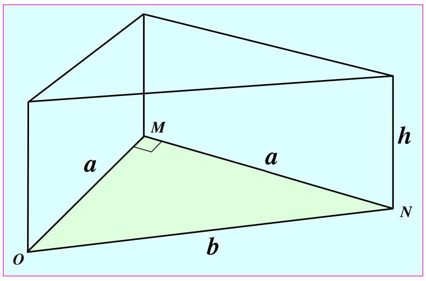
$$= \pi\left(\frac{10}{3}\right)^2(12 - 8) = \pi\frac{100}{9} \times 4$$

$$= \frac{400}{9}\pi$$

5 A right angled ‘cheese’ style wedge has two sides a, and one side b and angle ∠OMN 90°, with height h.

The perimeter of the base is 72 cm and the height is 1/16th of side b.

Find the value of a to maximise the volume.



Solution:

Need to find a formula for the volume in terms of a, so that this can be differentiated to show the change of volume w.r.t to a. Need to also find h in terms of a.

Using the perimeter to relate volume and a:

$$P = 2a + b = 72$$

$$\therefore \quad b = 72 - 2a$$

$$Area\ of\ base = \frac{1}{2}a \times a$$

$$h = \frac{b}{16} = \left(\frac{72 - 2a}{16}\right)$$

$$Volume = \frac{1}{2}a^2 h = \frac{1}{2}a^2\left(\frac{72 - 2a}{16}\right)$$

$$= \frac{1}{32}a^2(72 - 2a)$$

$$= \frac{72}{32}a^2 - \frac{2}{32}a^3$$

$$V = \frac{9}{4}a^2 - \frac{1}{16}a^3$$

$$\frac{dV}{dr} = \frac{18}{4}a - \frac{3}{16}a^2$$

For max $\quad \frac{dV}{dr} = \frac{9}{2}a - \frac{3}{16}a^2 = 0$

$$= \frac{72}{16}a - \frac{3}{16}a^2 = 0 \qquad \text{(Common denominator)}$$

$$= a(72 - 3a) = 0$$

$$a = 0 \text{ or } 24$$

$$\frac{d^2V}{da^2} = \frac{72}{16} - \frac{6}{16}a \qquad \text{(test for max/min)}$$

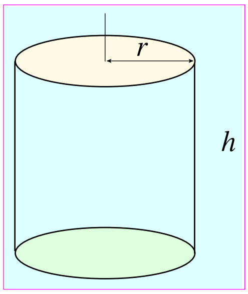
For $a = 24$ $\quad \frac{d^2V}{dr^2} = \frac{72}{16} - \frac{144}{16} = -\frac{72}{16}$ $\quad$ i.e. –ve result hence a maximum

$\therefore$ Maximum volume when $a = 24\ cm$

$$b = 72 - 48 = 24$$

$$h = \frac{24}{16} = 1\frac{1}{2}$$

6 A cylinder has height h and radius r. The volume of the cylinder is 250 cm^3.
Find the optimum value of r to ensure the surface area is a minimum.



Solution:

Need to find a formula for the volume and surface area in terms of h & r. Then eliminate one of the variables to give a function that can be differentiated.

Surface area: $A = 2\pi r^2 + 2\pi rh$ (1)

Volume of cylinder $V = \pi r^2 h$ (2)

Eliminate h to give V in terms of r: $h = \dfrac{V}{\pi r^2}$ from (2)

Surface area: $A = 2\pi r^2 + 2\pi r \times \dfrac{V}{\pi r^2}$ Substitute 2 into 1

$$= 2\pi r^2 + 2Vr^{-1}$$

$$\frac{dA}{dr} = 4\pi r - 2Vr^{-2} \quad (3)$$

For max/min: $\dfrac{dA}{dr} = 0$

$\therefore$ $4\pi r - \dfrac{2V}{r^2} = 0$

$$4\pi r^3 - 2V = 0$$

$$r^3 = \frac{2V}{4\pi}$$

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

$$r = \sqrt[3]{\frac{250}{2\pi}} = \frac{5}{\sqrt[3]{\pi}}$$ (exact answer)

$r = 3{\cdot}414$ (4sf)

To determine if this is a max or min, find the second derivative:

$$\frac{dA}{dr} = 4\pi r - 2Vr^{-2}$$

$$\frac{d^2A}{dr^2} = 4\pi + 4Vr^{-3}$$

$$\frac{d^2A}{dr^2} > 0 \qquad r > 0$$

The second derivative will be positive for any $r > 0$, hence a minimum.

What change of r would there be if the volume was increased to 2000 cm^3?

$$r = \sqrt[3]{\frac{2000}{2\pi}} = \frac{10}{\sqrt[3]{\pi}}$$

$r = 6{\cdot}828$ r is doubled

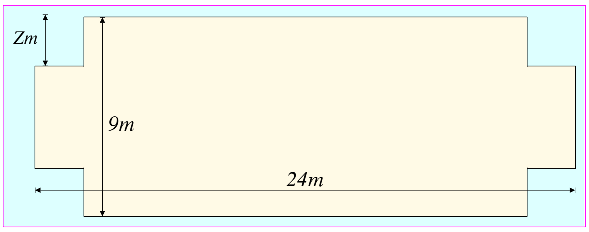
7 A piece of cardboard 9m × 24m is cut out to make a box. What is the value of z for the optimum volume.

Solution:

Write an equation for volume in terms of z:

Area of base of box =

$$(24 - 2z)(9 - 2z)$$



$$\therefore \text{ Volume of box: } \quad V = z(24 - 2z)(9 - 2z)$$
$$= z(216 - 48z - 18z + 4z^2)$$
$$= z(216 - 66z + 4z^2)$$
$$= 4z^3 - 66z^2 + 216z$$
$$\frac{dV}{dz} = 12z^2 - 132z + 216$$
$$12z^2 - 132z + 216 = 0 \qquad \text{for max/min}$$
$$z^2 - 11z + 18 = 0 \qquad \text{divide by 12}$$
$$(x - 2)(x - 9) = 0$$
$$\therefore \quad x = 2 \ \ or \ \ x = 9$$

When $x = 9$; $\quad (9 - 2z) \Rightarrow (9 - 18) \Rightarrow -9 \qquad$ Hence $x = 9$ is invalid.

$$\frac{d^2V}{dz^2} = 24z - 132$$

If $z = 2$; $\quad \dfrac{d^2V}{dz^2} = 48 - 132 = -84 \qquad$ Hence a maximum volume

EAFQLA

20.7 Differentiation Digest

If $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$ then the point is a maximum

If $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} > 0$ then the point is a minimum

If $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = 0$ then the point is either a maximum, minimum, a point of inflection or some other arrangement

Module C2

Core 2 Basic Info

Trigonometry; Sequences and series; Algebra; Integration.

The C2 exam is 1 hour 30 minutes long and is in two sections, and worth 72 marks (75 AQA).
Section A (36 marks) 5 – 7 short questions worth at most 8 marks each.
Section B (36 marks) 2 questions worth about 18 marks each.

OCR Grade Boundaries.

These vary from exam to exam, but in general, for C2, the approximate raw mark boundaries are:

Grade	100%	*A*	*B*	*C*
Raw marks	72	60 ± 2	53 ± 3	46 ± 2
UMS %	100%	80%	70%	60%

The raw marks are converted to a unified marking scheme and the UMS boundary figures are the same for all exams.

C2 Contents

C2 Assumed Basic Knowledge

Knowledge of C1 is assumed, and you may be asked to demonstrate this knowledge in C2.
You should know the following formulae, (many of which are NOT included in the Formulae Book).

1 Algebra

Remainder when a polynomial $f(x)$ is divided by $(x - a)$ is $f(a)$

2 Progressions

$$U_n = a + (n - 1)d \qquad AP$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] \qquad AP$$

$$U_n = ar^{n-1} \qquad GP$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \qquad GP$$

$$S_\infty = \frac{a}{(1 - r)} \quad \text{if } |r| < 1 \qquad GP$$

3 Trig

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

$$\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

Area of $\Delta = ½\, ab \sin C$

π radians $= 180^\circ$

Arc length of a circle, $L = r\theta$

Area of a sector of a circle, $A = ½\, r^2\theta$

4 Differentiation and Integration

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
ax^n	nax^{n-1}

Function $f(x)$	Integral $\int f(x)\,dx$	
ax^n	$\frac{a}{n + 1}x^{n+1} + c$	$n \neq -1$

Area between curve and x-axis $\quad A_x = \int_a^b y\,dx \quad (y \geqslant 0)$

Area between curve and y-axis $\quad A_y = \int_a^b x\,dy \quad (x \geqslant 0)$

5 Logs

$$sa^b = c \quad \Leftrightarrow \quad b = \log_a c$$

$$\log_a x + \log_a y \equiv \log_a(xy)$$

$$\log_a x - \log_a y \equiv \log_a\left(\frac{x}{y}\right)$$

$$k \log_a x \equiv \log_a\left(x^k\right)$$

C2 Brief Syllabus

1 Algebra and Functions

- use of the factor and remainder theorem
- carry out simple algebraic division (division of a cubic by a linear polynomial)
- sketch the graph of $y = a^x$, where $a > 0$, and understand how different values of a affect the shape of the graph
- understand the relationship between logarithms, indices, and the laws of logs (excluding change of base)
- use logarithms to solve equations of the form $a^x = b$, and similar inequalities.

2 Trigonometry

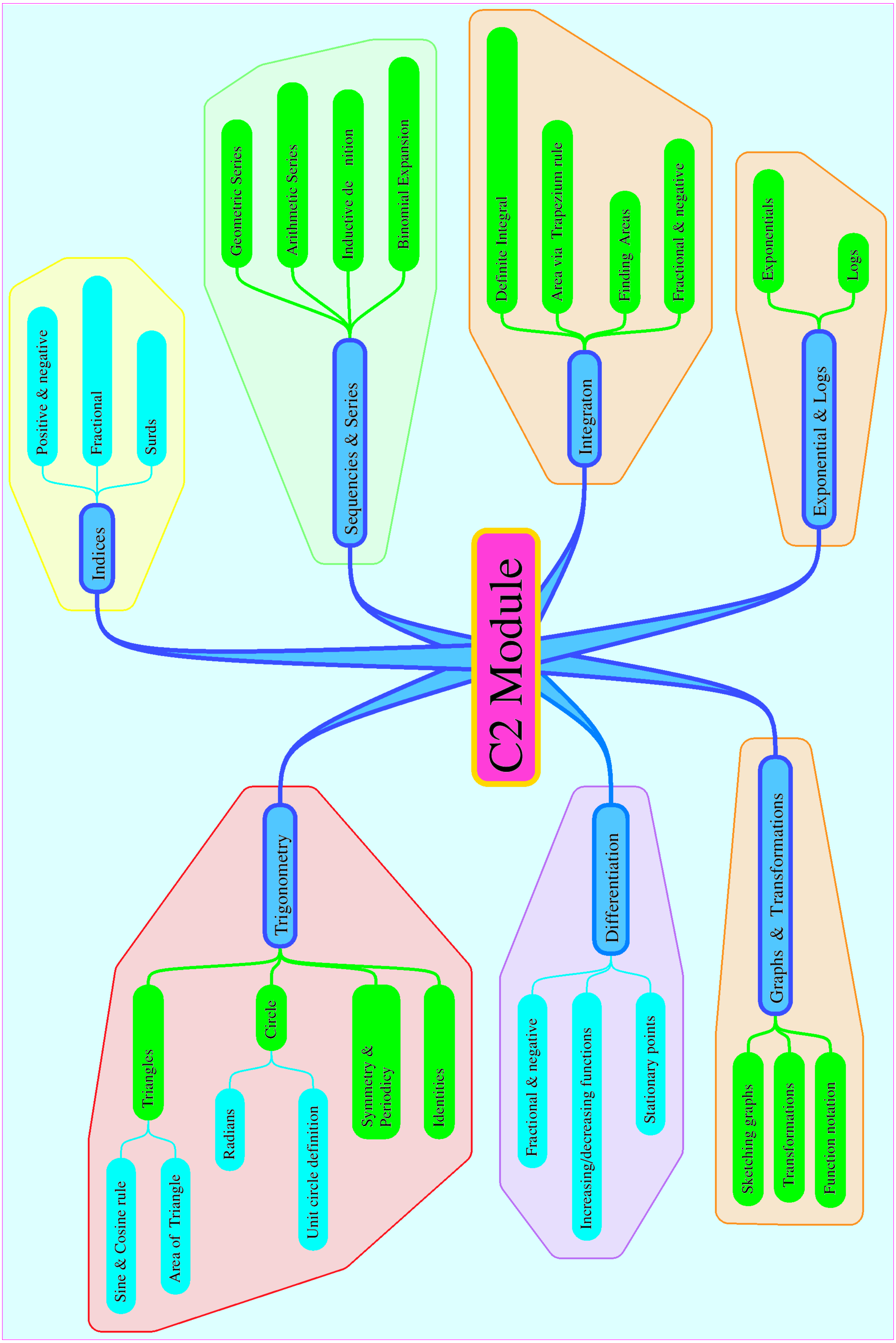
- use the sine and cosine rules in the solution of triangles (excluding the ambiguous case of the sine rule)
- use the area formula $\Delta = \frac{1}{2} ab \sin C$
- understand the definition of a radian, and use the relationship between degrees and radians
- use the formulae $s = r\theta$ and $A = \frac{1}{2} r^2\theta$ for the arc length and sector area of a circle
- relate the periodicity and symmetries of the sine, cosine and tangent functions to the form of their graphs
- use the identities $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\cos^2\theta + \sin^2\theta \equiv 1$
- use the **exact** values of the sine, cosine and tangent of 30°, 45°, 60° e.g $\cos 30 = \frac{1}{2}\sqrt{3}$
- find all the solutions, within a specified interval, of the equations $\sin(kx) = c$, $\cos(kx) = c$, $\tan(kx) = c$, and of equations (for example, a quadratic in $\cos x$).

3 Sequences & Series

- understand the idea of a sequence of terms, and use definitions such as $u_n = n^2$ and relations such as $u_{n+1} = 2u_n$ to calculate successive terms and deduce simple properties
- understand and use Σ notation
- recognise arithmetic and geometric progressions
- use the formulae for the n-th term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions (including the formula of the first n for the sum of natural numbers)
- use the condition $|r| < 1$ for convergence of a geometric series, and the formula for the sum to infinity of a convergent geometric series
- use the expansion of $(a + b)^n$ where n is a positive integer, including the recognition and use of the notations $\binom{n}{r}$ and $n!$ (finding a general term is not included).

4 Integration

- understand indefinite integration as the reverse process of differentiation, and integrate x^n (for any rational n except -1), together with constant multiples, sums and differences
- solve problems involving the evaluation of a constant of integration, (e.g. to find the equation of the curve through $(-2, 1)$ for which $\dfrac{dy}{dx} = 3x + 2$)
- evaluate definite integrals
- use integration to find the area of a region bounded by a curve and lines parallel to the coordinate axes, or between two curves or between a line and a curve
- use the trapezium rule to estimate the area under a curve, and use sketch graphs, in simple cases, to determine whether the trapezium rule gives an over-estimate or an under-estimate.


C2 Module
Indices
Positive & negative
Fractional
Surds
Sequencies & Series
Geometric Series
Arithmetic Series
Inductive de nition
Binomial Expansion
Integraton
Definite Integral
Area via Trapezium rule
Finding Areas
Fractional & negative
Exponential & Logs
Exponentials
Logs
Graphs & Transformations
Sketching graphs
Transformations
Function notation
Differentiation
Fractional & negative
Increasing/decreasing functions
Stationary points
Trigonometry
Triangles
Sine & Cosine rule
Area of Triangle
Circle
Radians
Unit circle definition
Symmetry & Periodicity
Identities

21 • C2 • Algebraic Division

21.1 Algebraic Division Intro

This is about dividing polynomials. Any polynomial expression with order m, which is divided by another polynomials, order n, will have an answer of order $(m - n)$. For example a cubic expression divided by a linear expression will have a quadratic solution.

Note the names of the parts of a division:

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

For a function $f(x)$ divided by $(ax - b)$ we can write:

$$\frac{f(x)}{ax - b} = g(x) + \frac{r}{ax - b}$$

$$f(x) = g(x)(ax - b) + r$$

A linear polynomial divided by a linear polynomial; result: *Quotient* is a constant, *Remainder* is a constant.

$$\frac{6x - 1}{2x + 1} \qquad 6x - 1 \equiv A(2x + 1) + R$$

A quadratic polynomial divided by a linear polynomial; result: *Quotient* is linear, *Remainder* is a constant.

$$\frac{x^2 + 6x - 1}{2x + 1} \qquad x^2 + 6x - 1 \equiv (Ax + B)(2x + 1) + R$$

A cubic polynomial divided by a linear polynomial; result: *Quotient* is a quadratic, *Remainder* is a constant.

$$\frac{x^3 + x^2 + 6x - 1}{2x + 1} \qquad x^3 + x^2 + 6x - 1 \equiv (Ax^2 + Bx + C)(2x + 1) + R$$

21.2 Long Division by $ax + b$

Long division is a useful technique to learn, although other methods can be used.

21.2.1 Example:

1 Divide $2x^3 - 3x^2 - 3x + 7$ by $x - 2$

	$2x^2 + x - 1$	
$x - 2$	$\overline{)\,2x^3 - 3x^2 - 3x + 7}$	Divide $2x^3$ *by* $x = 2x^2$
	$2x^3 - 4x^2$	Multiply $x - 2$ *by* $2x^2$
	$x^2 - 3x + 7$	Subtract & divide x^2 *by* $x = x$
	$x^2 - 2x$	Multiply $x - 2$ *by* x
	$-x + 7$	Subtract & divide $-x$ *by* $x = -1$
	$-x + 2$	Multiply $x - 2$ *by* -1
	5	Subtract to give the remainder

2 Divide $2x^3 + 5x^2 - 3$ by $x + 1$
Rewrite to add a place holder for the missing x term.
Divide $2x^3 + 5x^2 + 0x - 3$ by $x + 1$

$$
\begin{array}{r|l}
2x^2 + 3x - 3 & \\
x + 1 \,\big)\, 2x^3 + 5x^2 + 0x - 3 & \text{Divide } 2x^3 \text{ by } x = 2x^2 \\
2x^3 + 2x^2 & \text{Multiply } x + 1 \text{ by } 2x^2 \\
3x^2 + 0x - 3 & \text{Subtract \& divide } x^2 \text{ by } x = 3x \\
3x^2 + 3x & \text{Multiply } x + 1 \text{ by } 3x \\
-3x - 3 & \text{Subtract \& divide } -3x \text{ by } x = -3 \\
-3x - 3 & \text{Multiply } x + 1 \text{ by } -3 \\
0 & \text{Subtract to give the remainder}
\end{array}
$$

21.3 Comparing Coefficients

Dividing a cubic equation by a linear equation means that the Quotient will be a quadratic.
Using this, we can compare coefficients. Thus:

$$\frac{5x^3 + 18x^2 + 19x + 6}{5x + 3} = ax^2 + bx + c$$

$$
\begin{aligned}
5x^3 + 18x^2 + 19x + 6 &= (5x + 3)(ax^2 + bx + c) \\
&= 5ax^3 + 5bx^2 + 5cx + 3ax^2 + 3bx + 3c \\
&= 5ax^3 + 5bx^2 + 3ax^2 + 5cx + 3bx + 3c
\end{aligned}
$$

Comparing coefficients, starting with the constant term, which is usually the easiest to find:

constant term :→	$6 = 3c$	$\therefore c = 2$	
x term :→	$19 = 5c + 3b$	$\therefore 19 - 10 = 3b$	$b = 3$
x^2 term :→	$18 = 5b + 3a$	$\therefore 18 - 15 = 3a$	$a = 1$
x^3 term :→	$5 = 5a$	$\therefore a = 1$	confirms value of a

$$\therefore \quad \frac{5x^3 + 18x^2 + 19x + 6}{5x + 3} = x^2 + 3x + 2$$

Now read the next section on the Remainder & Factor Theorem.

22 • C2 • Remainder & Factor Theorem

AQA C1 / OCR C2

22.1 Remainder Theorem

This is about dividing polynomials, (with order > 2), by a linear term.

If a polynomial $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$

This can be restated like this:

$$f(x) = (x - a)\,g(x) + f(a) \qquad \text{where } f(a) = \text{a constant, i.e. the remainder}$$

Note that the order of $g(x)$ is one less than $f(x)$.
Similarly:

If a polynomial $f(x)$ is divided by $(ax - b)$ then the remainder is $f\left(\frac{b}{a}\right)$

22.1.1 Example:

1 Find the remainder when $3x^3 - 2x^2 - 5x + 2$ is divided by $x - 2$.

Solution:

Let $f(x) = 3x^3 - 2x^2 - 5x + 2$
and $a = 2$

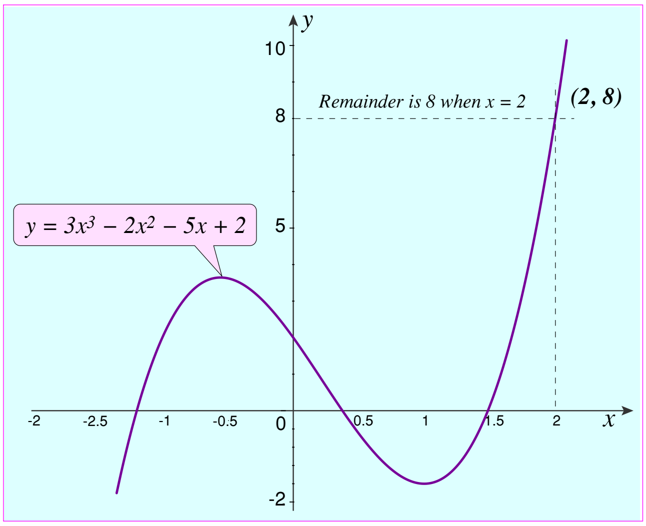
Hence, find $f(a)$:

$$f(2) = 3 \times 2^3 - 2 \times 2^2 - 10 + 2$$
$$= 24 - 8 - 8$$
$$= 8$$

Hence, we can say:

$$f(x) = (x - 2)\,g(x) + 8$$

This can be seen graphically here:



2 If $x^3 - 4x^2 + x + c$ is divided by $x - 2$, the remainder is 4, Find the value of c.

Solution:

Substitute $x = 2$ into $f(x)$

$$f(2) = 2^3 - 4 \times 2^2 + 2 + c = 4$$
$$8 - 16 + 2 + c = 4$$
$$c = 4 + 6$$
$$c = 10$$

22.2 Factor Theorem

This follows from the Remainder Theorem.

For a polynomial $f(x)$, if $f(a) = 0$ then $(x - a)$ is a factor.

i.e. the remainder is zero if $(x - a)$ is a factor.

This can be used to find the factors of any polynomial, usually after a bit of trial and error.

One immediate effect of this rule is that if $f(1) = 0$ then $(x - 1)$ is a factor. In other words, if all the coefficients of the expression add up to zero then $(x - 1)$ is a factor.

22.2.1 Example:

1 Find the factors for $x^3 + 6x^2 + 5x - 12$

Solution:

The coefficients of the expression add up to zero.

$$1 + 6 + 5 - 12 = 0$$

Therefore, $(x - 1)$ is a factor.

$$x^3 + 6x^2 + 5x - 12 \equiv (x - 1)(x^2 + bx + 12)$$

Compare x^2 coefficients: $6x^2 \equiv -x^2 + bx^2$

$$6 = -1 + b \qquad \therefore \quad b = 7$$

$$x^3 + 6x^2 + 5x - 12 \equiv (x - 1)(x^2 + 7x + 12)$$

$$x^3 + 6x^2 + 5x - 12 \equiv (x - 1)(x + 3)(x + 4)$$

2 If $f(x) = x^3 - 5x^2 - 2x + 24$, show that $(x - 4)$ is a factor and find the other two linear factors.

Solution:

If $(x - 4)$ is a factor then $f(4) = 0$

$$f(4) = 4^3 - 5 \times 4^2 - 8 + 24$$

$$= 64 - 80 - 8 + 24$$

$$= 0$$

The function $f(x)$ can now be written:

$$x^3 - 5x^2 - 2x + 24 \equiv (x - 4)(x^2 + bx + c)$$

Compare constants: $24 = -4c$

$$c = -6$$

Compare x terms: $-2 = c - 4b$

$$-2 = -6 - 4b$$

$$b = 1$$

$$\therefore \quad x^3 - 5x - 2x + 24 \equiv (x - 4)(x^2 + x - 6)$$

$$f(x) = (x - 4)(x - 3)(x + 2)$$

Show that $(x - 3)$ & $(x + 2)$ are factors of $f(x)$

$f(3) = 27 - 45 - 6 + 24 = 0 \qquad \therefore (x - 3)$ is a factor.

$f(-2) = -8 - 20 + 4 + 24 = 0 \qquad \therefore (x + 2)$ is a factor.

$$f(x) = (x - 4)(x - 3)(x + 2)$$

3 If $f(x) = 2x^3 + x^2 + bx - c$, and that $(x - 1)$ & $(x + 1)$ are factors, find the values of b & c, and the remaining factor.

Solution:

As $(x - 1)$ is a factor then $f(1) = 0$

$$f(1) = 2 + 1 + b - c = 0$$

$$b = c - 3 \qquad \dots (1)$$

As $(x + 1)$ is a factor then $f(-1) = 0$

$$f(-1) = -2 + 1 - b - c = 0$$

$$b = -c - 1 \qquad \dots (2)$$

$\therefore \quad c - 3 = -c - 1$ combine (1) & (2)

$$c = 1$$

$\therefore \quad b = -2$ substitute in (2)

$\therefore$ function is: $f(x) = 2x^3 + x^2 - 2x - 1 \qquad \dots (3)$

Since $(x - 1)$ & $(x + 1)$ are factors, let the 3rd factor be $(2x + t)$

$$\therefore \quad f(x) = (x - 1)(x + 1)(2x + t) \qquad \dots (4)$$

Compare constants from (3) and (4):

$$-1 = -1 \times 1 \times t$$

$$\therefore \quad t = 1$$

Hence: $f(x) = (x - 1)(x + 1)(2x + 1)$

4 If $f(x) = 2x^3 - ax^2 - bx + 4$, and that when $f(x)$ is divided by $(x - 2)$ the remainder is 2 & when $f(x)$ is divided by $(x + 1)$ the remainder is 5. Find the values of a & b.

Solution:

For $(x - 2)$ then the remainder is $f(2) = 2$

$$f(2) = 2 \times 8 - 4a - 2b + 4 = 2$$

$$= 16 - 4a - 2b + 4 = 2$$

$$\therefore \quad -4a - 2b = 2 - 4 - 16 = -18$$

$$\therefore \quad 2a + b = 9 \qquad \dots (1)$$

For $(x + 1)$ then the remainder is $f(-1) = 5$

$$f(-1) = -2 - a + b + 4 = 5$$

$$-a + b = 5 - 4 + 2$$

$$b = 3 + a \qquad \dots (2)$$

$\therefore \quad 2a + (3 + a) = 9$ combine (1) & (2)

$$3a + 6 = 9$$

$$a = 2$$

$b = 3 + 2 = 5$ substitute in (2)

$\therefore \quad f(x) = 2x^3 - 2x^2 - 5x + 4$ QED

5 Find the factors for $x^3 + 6x^2 + 11x + 6$

Solution:

To start the process, choose some values of x to try, but what?

The function has three linear factors, say $(x \pm s)(x \pm t)(x \pm u)$. Hence $stu = 6$.

Taking the factors of the constant, 6, will give us our starting point.

Factors of 6 are: 1, 2, 3, 6 and could be positive or negative. The likely factors to use are: 1, 2, 3.

Possible factors are $(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, $(x \pm 6)$.

Choose −6, −3, −2, −1 to start the process.

Try -6 $\qquad f(-6) = (-6)^3 + 6(-6)^2 - 66 + 6$

$= -216 + 216 - 60$

$= -60$

$\therefore \quad x + 6$ is NOT a factor. i.e. $f(-6) \neq 0$

Try -3 $\qquad f(-3) = (-3)^3 + 6(-3)^2 - 33 + 6$

$= -27 + 54 - 27$

$= 54 - 54$

$= 0$

$\therefore \quad x + 3$ is a factor

Try -2 $\qquad f(-2) = (-2)^3 + 6(-2)^2 - 22 + 6$

$= -8 + 24 - 22 + 6$

$= 0$

$\therefore \quad x + 2$ is a factor

Try -1 $\qquad f(-1) = (-1)^3 + 6(-1)^2 - 11 + 6$

$= -1 + 6 - 11 + 6$

$= 0$

$\therefore \quad x + 1$ is a factor

Hence: $\qquad f(x) = (x + 3)(x + 2)(x + 1)$

22.3 Topic Digest

- For a polynomial $f(x)$, if $f(a) = 0$ then $(x - a)$ is a factor of $f(x)$
- For a polynomial $f(x)$, if $f\left(\frac{b}{a}\right) = 0$ then $(ax - b)$ is a factor of $f(x)$
- A polynomial $f(x)$ divided by $(ax - b)$ has a factor of $f\left(\frac{b}{a}\right)$

23 • C2 • Sine & Cosine Rules

23.1 Introduction

The Sine & Cosine Rules covers the trig rules for any shaped triangles, not just right-angled triangles studied previously.

In order to solve these triangle problems, we need to know the value of one side plus two other bits of information, such as 2 sides, 2 angles, or one side and an angle.

There are 4 cases to consider with two rules:

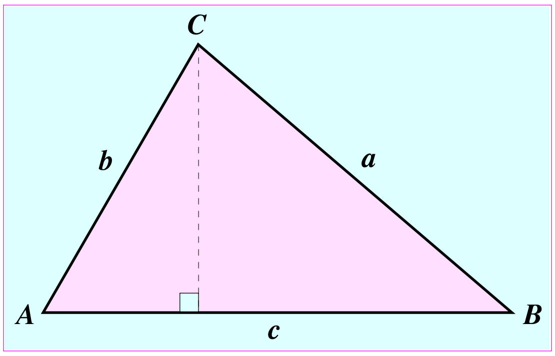
- Sine Rule
  - 2 sides + 1 opposite angle (SSA)
  - 2 angles + 1 side (AAS or ASA)
- Cosine Rule
  - 3 sides (SSS)
  - 2 sides + 1 included angle (SAS)

23.2 Labelling Conventions & Properties

By convention, the vertices are labelled with capital letters and the opposite sides by the corresponding lower case letter.

i.e. a is opposite $\angle A$
b is opposite $\angle B$
c is opposite $\angle C$

For sides a & b, C is called the **included** angle etc.



Recall that:

- Angles in a triangle add up to 180°
- The longest side of the triangle is opposite the largest angle, whilst the shortest side is opposite the smallest angle

23.3 Sine Rule

The Sine rule, for any triangle gives:

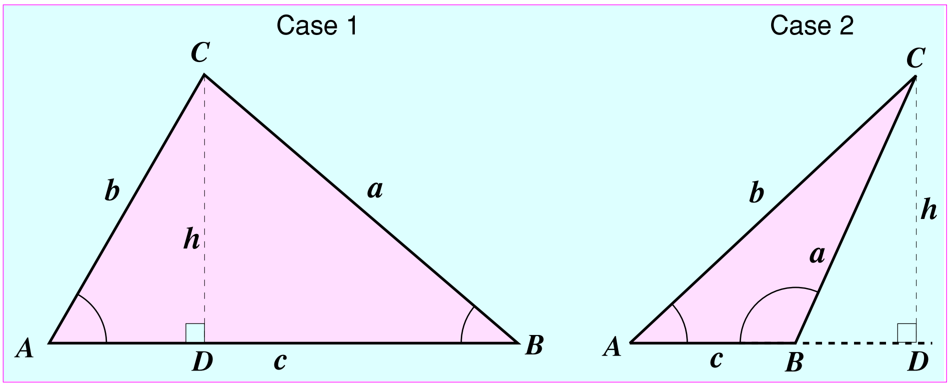
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ use this version with an unknown side - unknown on top

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ use this version with an unknown angle - unknown on top

i.e. put the unknown on top.

23.3.1 Sine Rule Proof



Case 1 (acute angled triangle)

From ΔACD $\quad h = b \sin A$

From ΔBCD $\quad h = a \sin B$

$\therefore \quad b \sin A = a \sin B$

$\therefore \quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

Similarly: $\quad \dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

Case 2 (obtuse angled triangle)

From ΔACD $\quad h = b \sin A$

From ΔBCD $\quad h = a \sin (180° - B)$

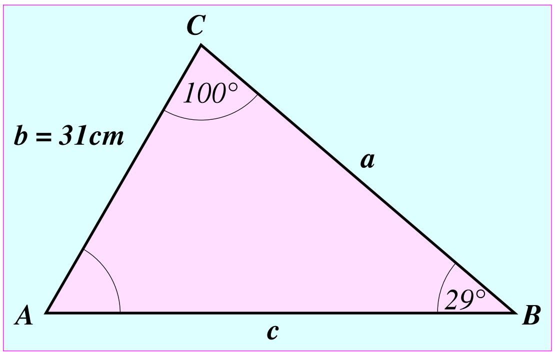
$\quad = a \sin B$

$\therefore \quad b \sin A = a \sin B$

Similarly: $\quad \dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

23.3.2 Example:

1 Find the remaining angle and missing sides of $\triangle ABC$:



Remaining angle, $\angle A = 180 - 100 - 29 = 51°$

$$\frac{a}{\sin 51} = \frac{31}{\sin 29} = \frac{c}{\sin 100}$$

$$\therefore \quad a = \frac{31 \sin 51}{\sin 29}$$

$$a = 49{\cdot}69\ cm\ (2dp)$$

$$\frac{31}{\sin 29} = \frac{c}{\sin 100}$$

$$c = \frac{31 \sin 100}{\sin 29}$$

$$= 62{\cdot}97\ cm\ (2dp)$$

2 A triangle ABC has sides of AB = 3, BC = 4, and angle A is 40°. Find angle C.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 40}{4} = \frac{\sin C}{3}$$

$$\sin C = \frac{3 \sin 40}{4}$$

$$\sin C = 0{\cdot}482$$

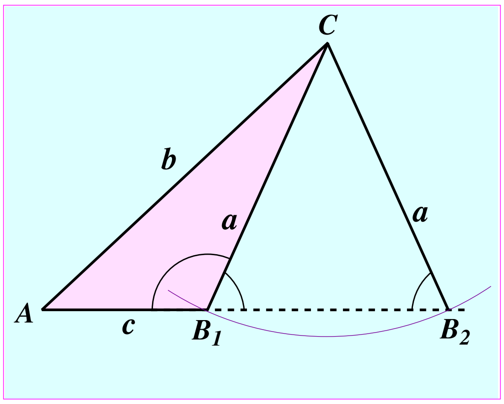
$$C = 28{\cdot}82$$

23.4 The Ambiguous Case (SSA)

This case occurs when given two sides and a non included angle (SSA) and we want to find the unknown angle.

In this example, we want to find $\angle B$. The line BC can take two positions that both satisfy the triangle when the sides a, b and $\angle A$ are known.

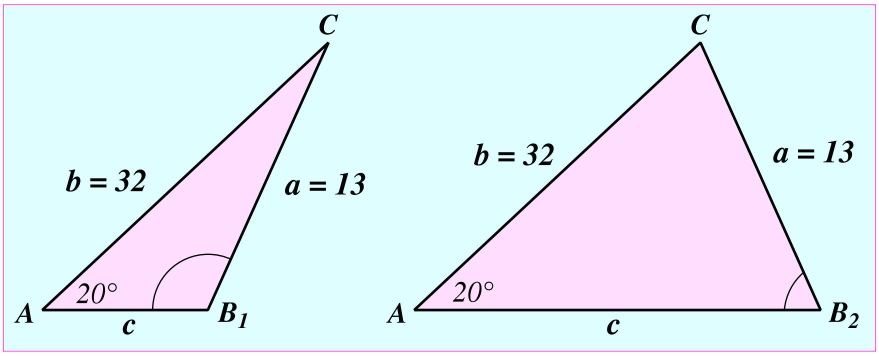
The triangle B_1CB_2 forms an isosceles triangle.



Another way to look at this problem, is to recognise that if the unknown angle is opposite the longest side then there will be two possible solutions, (except if the unknown angle is a right angle).

23.4.1 Example:

If $\angle A = 20°, \ a = 13\, cm, \ b = 32\, cm$ find the two triangles formed.



Solution:

To find the angles B_1 & B_2

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\sin B = \frac{b \sin A}{a} = \frac{32}{13} \sin 20$$

$$= 2{\cdot}4615 \times 0{\cdot}342 = 0{\cdot}8419$$

$$B = \sin^{-1} 0{\cdot}8419$$

$$\therefore \quad B_2 = 57{\cdot}34° \ (2\ dp)$$

$$B_1 = 180 - 57{\cdot}34 = 122{\cdot}66° \ (2\ dp)$$

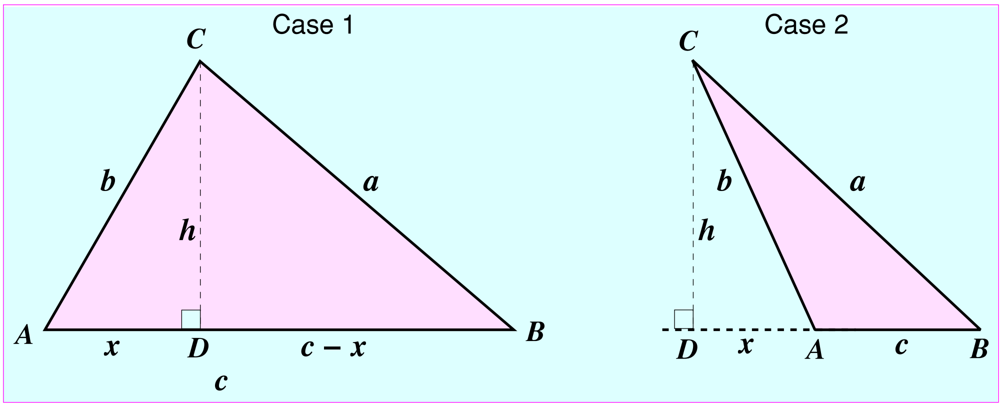
23.5 Cosine Rule

The Cosine rule, for any triangle gives:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note the cyclic nature of the equation: $a \to b \to c \to a$ and that the angle is the included angle.

23.5.1 Cosine Rule Proof



Case 1 (acute angled triangle)

From ΔACD $\quad h^2 = b^2 - x^2$

From ΔBCD $\quad h^2 = a^2 - (c - x)^2$

$$\therefore \quad a^2 - (c - x)^2 = b^2 - x^2$$
$$a^2 - c^2 + 2cx - x^2 = b^2 - x^2$$
$$\therefore \quad a^2 = b^2 + c^2 - 2cx$$

From ΔACD $\quad x = b\cos A$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$$

Similarly:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Case 2 (obtuse angled triangle)

From ΔBCD $\quad h^2 = a^2 - (c + x)^2$

From ΔACD $\quad h^2 = b^2 - x^2$

$$\therefore \quad a^2 - (c + x)^2 = b^2 - x^2$$
$$a^2 - c^2 - 2cx - x^2 = b^2 - x^2$$
$$\therefore \quad a^2 = b^2 + c^2 + 2cx$$

From ΔACD $\quad x = b\cos(180° - A)$

$$= -b\cos A$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$$

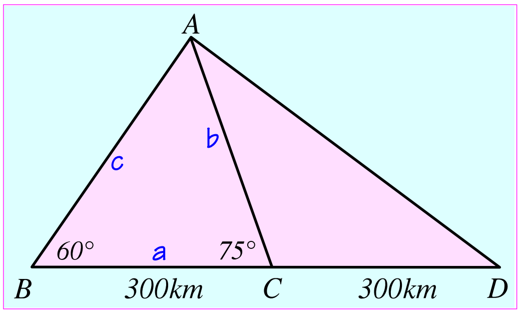
Similarly:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

23.5.2 Example:

1 Refer to the diagram and find the shortest distance between point A and the line BD.

Find the distance AD.



Solution:

To find the shortest distance between point A and the line BD, draw a line from A to BD, perpendicular to BD. Then find the length of AB, for which you need the angle $\angle BAC$. Then it is a matter of using the definition of a sine angle to work out the length of the perpendicular line.

$$\angle BAC = 180 - (60 + 75) = 45$$

To find AB, have AAS which needs the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{300}{\sin 45} = \frac{c}{\sin 75}$$

$$c = \frac{300 \sin 75}{\sin 45}$$

$$c = 288{\cdot}28\ km$$

Draw a line from A to BD at point R

$$\sin 60 = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AR}{c}$$

$$AR = c \sin 60 = 288{\cdot}28 \sin 60$$

$$= 249{\cdot}66\ km$$

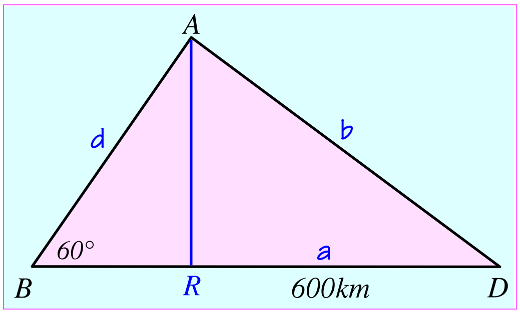
Look at the triangle ABD and calculate AD from the cosine rule (SAS)

$$b^2 = a^2 + d^2 - 2ad \cos B$$

$$= 288{\cdot}28^2 + 600^2 - 2 \times 288{\cdot}28 \times 600 \cos 60$$

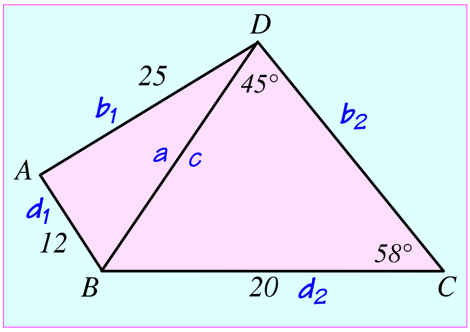
$$b^2 = 270137{\cdot}35$$

$$\therefore \quad b = 519{\cdot}74\ km$$



oecfrl

2 From the diagram, find length BD and $\angle BAD$



Solution:

To find the length BD, we have AAS therefore use the sine rule:

$$\frac{20}{\sin 45} = \frac{c}{\sin 58}$$

$$c = \frac{20 \sin 58}{\sin 45}$$

$$c = 23{\cdot}98$$

To find $\angle BAD$ use the cosine rule (SSS)

$$\cos A = \frac{b_1^2 + d_1^2 - a^2}{2 b_1 d_1}$$

$$= \frac{25^2 + 12^2 - 23{\cdot}98^2}{2 \times 25 \times 12}$$

$$= 0{\cdot}323$$

$$A = \cos^{-1}(0{\cdot}323)$$

$$= 71{\cdot}14°$$

oecfrl

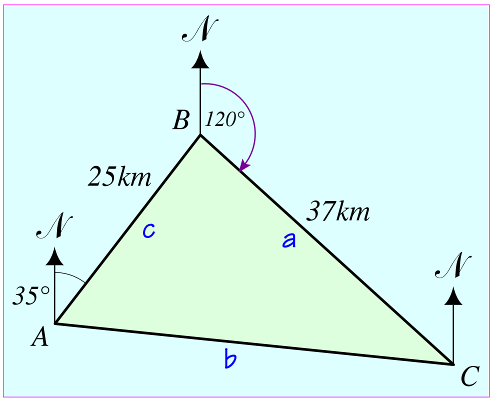
23.6 Bearings

Bearing problems are a favourite topic. You need to be familiar with the rules for angles and parallel lines. Also note that compass bearings are measured clockwise from North.

23.6.1 Example:

1 A microlight flies on a triangular cross county course, from A to B to C and back to A for tea.

Show that ∠ABC is 95°, and find the distance from C to A.



Solution:

∠ABC can be split into two parts, and from the rules for parallel lines & angles on a straight line:

$$\angle ABC = 35 + 90 = 95°$$

As we now have two sides and an included angle (SAS), we use the cosine rule.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$b^2 = 37^2 + 25^2 - 2 \times 37 \times 25 \times \cos 95$$

$$b^2 = 1994 - 1850 \times (-0{\cdot}0872)$$

$$b^2 = 1994 + 161{\cdot}238$$

$$b = \sqrt{2155{\cdot}238}$$

$$b = 46{\cdot}424$$

Now find the bearing required, by finding $\angle BCA$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 95}{46{\cdot}424} = \frac{\sin C}{25}$$

$$\sin C = \frac{\sin 95}{46{\cdot}424} \times 25$$

$$= 0{\cdot}5365$$

$$C = \sin^{-1}(0{\cdot}5365) = 32{\cdot}44°$$

Bearing from C to A $= 360 - (60 + 32{\cdot}44) = 267{\cdot}55°$

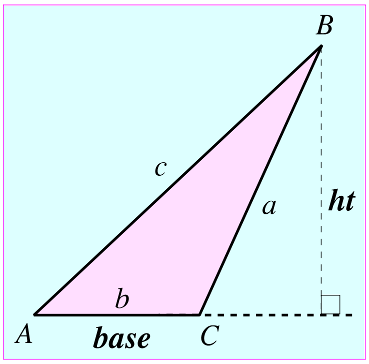
oecfrl

23.7 Area of a Triangle

From previous studies recall that the area of a triangle is given by:

$$Area = \frac{1}{2} base \times perpendicular\ height$$

$$A = \frac{1}{2} b\, h$$



From the trig rules, we know that the height, h, is given by

$$\sin A = \frac{opp}{hypotenuse} = \frac{h}{c}$$

$$h = c \sin A$$

Hence:

$$Area = \frac{1}{2} bc \sin A$$

Similarly for the other angles:

$$Area = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C$$

where the angle is always the included angle.

There are other formulae for the area of a triangle, such as:

From the sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$$a = \frac{b \sin A}{\sin B}$$

Substitute into the area formula: $Area = \frac{1}{2} ab \sin C$

$$Area = \frac{1}{2}\left(\frac{b \sin A}{\sin B}\right) b \sin C$$

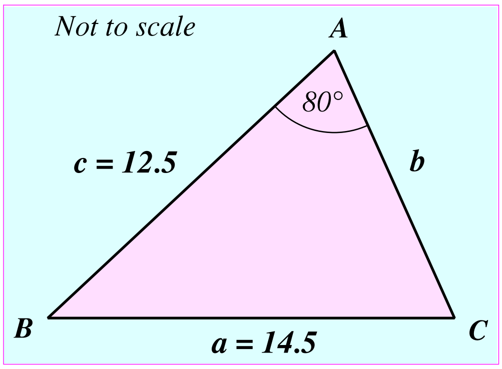
$$Area = \frac{b^2}{2} \times \frac{\sin A \sin C}{\sin B} = \frac{b^2 \sin A \sin C}{2 \sin B}$$

This is great if you have a given triangle with three angles and one side.

23.7.1 Example:

1

From the given sketch, find the area of the triangle. Dimensions in cm.



Solution:

$$Area = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

We have been given a, & c, so we need $\sin B$, to find the area.

$$Area = \frac{1}{2}ac \sin B$$

Using the sine rules to find angle C, then angle B.

$$\frac{\sin A \checkmark}{a \checkmark} = \frac{\sin B}{b} = \frac{\sin C}{c \checkmark}$$

$$\sin C = \frac{c \sin A}{a}$$

$$C = \sin^{-1}\left(\frac{c \sin A}{a}\right) = \sin^{-1}\left(\frac{12.5 \sin 80}{14.5}\right)$$

$$B = 180 - 80 - \sin^{-1}\left(\frac{12.5 \sin 80}{14.5}\right)$$

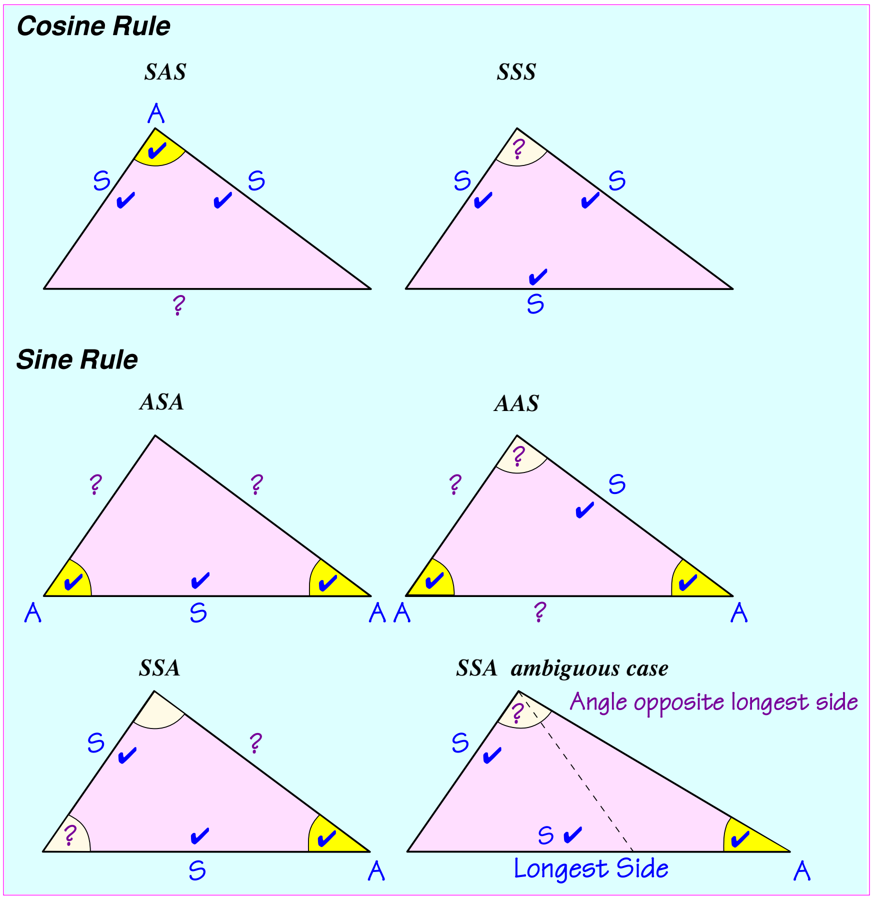
$$= 100 - 58{\cdot}1 = 41{\cdot}9°$$

$$Area = \frac{1}{2}ac \sin B = \frac{1}{2} \times 14.5 \times 12.5 \sin 41{\cdot}9$$

$$= 58{\cdot}10\ cm^2$$

23.8 Cosine & Sine Rules in Diagrams

Simply put: for SAS & SSS use the cosine rule, for anything else use the Sine Rule.



When to use the Sine & Cosine Rule

23.9 Heinous Howlers

Check your calculator is set to degrees or radians as appropriate.

Know the formulae for the sine rule and area of a triangle - they are not in the exam formulae book.

23.10 Digest

Use the cosine rule whenever you have:

- Cosine Rule
  - 2 sides and the included angle (SAS) – to find the unknown side
  - All 3 sides (SSS) – to find the unknown angle

For all other situations use the sine rule.

To find the length of a side:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

To find an angle:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Use the sine rule whenever you have:

- Sine Rule
  - 2 angles + 1 side (AAS or ASA) – to find the unknown side
  - 2 sides + 1 opposite angle (SSA) – to find the unknown angle
  - Note: if the unknown angle is opposite the longer of the two sides, then there are two possible angles (the ambiguous case), right angles excepted.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ side unknown - use this version

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ angle unknown - use this version

(i.e. put the unknown bit on top)

Recall that $\sin x = k$ always has two solutions for angles between 0 and 180°.

$$x = \sin^{-1}k$$

and $$x = 180° - \sin^{-1}k$$

Area of a triangle (SAS):

$$Area = \frac{1}{2}ab\sin C$$

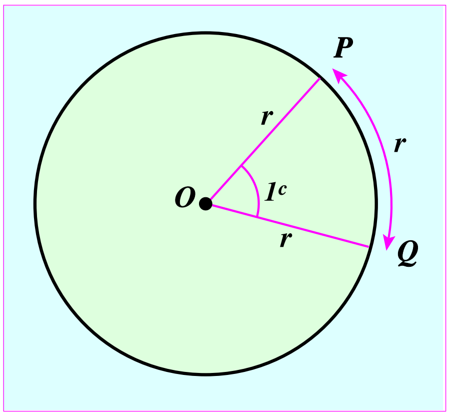
$$Area = \frac{1}{2}ac\sin B$$

$$Area = \frac{1}{2}bc\sin A$$

24 • C2 • Radians, Arcs, & Sectors

24.1 Definition of Radian

One radian is the angle subtended at the centre of a circle by an arc, whose length is equal to the radius of the circle.



Circumference $C = 2\pi r$ if $r = 1$ then

$$C = 2\pi$$

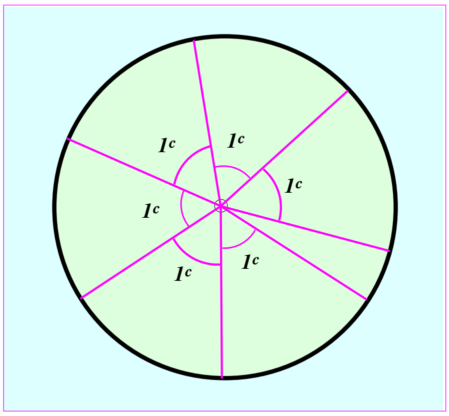
Hence 2π *radians* $= 360°$

$$180° = \pi \text{ rad}$$
$$90° = \frac{\pi}{2} \text{ rad}$$
$$1° = \frac{\pi}{180} \text{ rad}$$

One radian (1^c or 1 rad) = 57·296° (3dp)

Since a radian is defined by the ratio of two lengths, it has no units.

A circle can be divided up into 6.3 radians (approx).

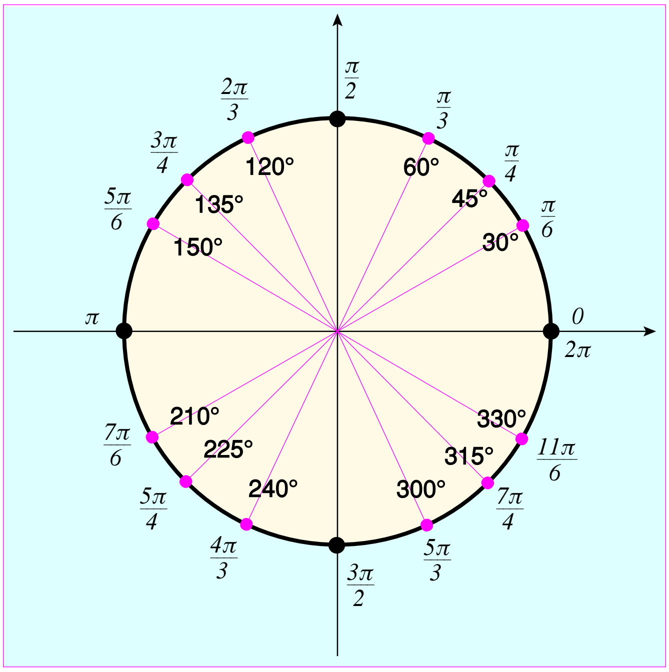


To convert from degrees to radians: $\times \dfrac{\pi}{180}$ rads

To convert from radians to degrees: $\times \left(\dfrac{180}{\pi}\right)^{\circ}$

24.2 Common Angles

Some angles have conversions that lead to exact conversions between degrees and radians.



Need to know — common angles in radians

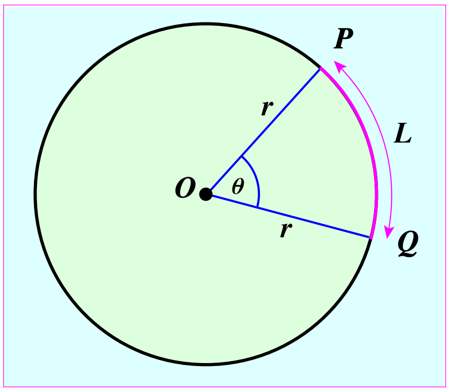
24.3 Length of an Arc

The length of the arc L with angle θ at the centre of a circle with radius r is:

$$L = \text{fraction of circle} \times \text{circumference}$$

$$L = \frac{\theta}{2\pi} \times 2\pi r = r\theta \quad (\theta \text{ in radians})$$

$$L = \frac{\theta}{360} \times 2\pi r = \frac{\pi r \theta}{180} \quad (\theta \text{ in degrees})$$



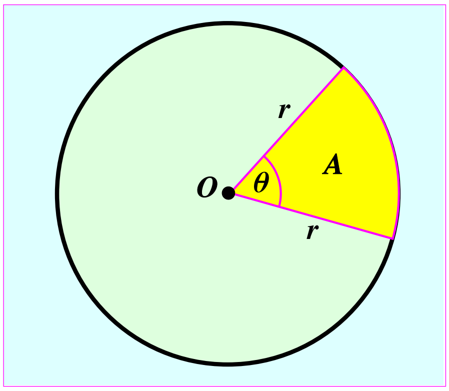
24.4 Area of Sector

The area of sector A is given by:

$$A = \text{fraction of circle} \times \text{area}$$

$$A = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta \quad (\theta \text{ in radians})$$

$$A = \frac{\theta}{360} \times \pi r^2 = \frac{\pi r^2 \theta}{360} \quad (\theta \text{ in degrees})$$



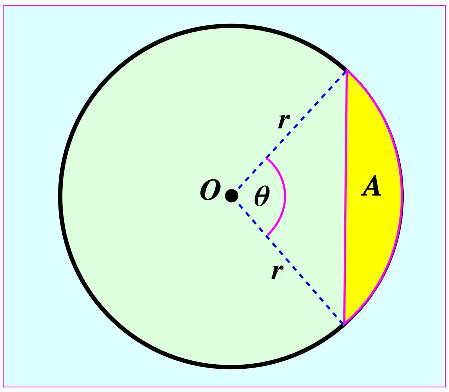
24.5 Area of Segment

The area of a segment of a circle with radius r is given by the area of the sector minus the area of the triangle:

$$A = \text{area of sector} - \text{area of triangle}$$

$$A = \tfrac{1}{2} r^2 \theta - \tfrac{1}{2} r^2 \sin\theta \quad (\theta \text{ in radians})$$

$$A = \tfrac{1}{2} r^2 (\theta - \sin\theta) \quad (\theta \text{ in radians})$$

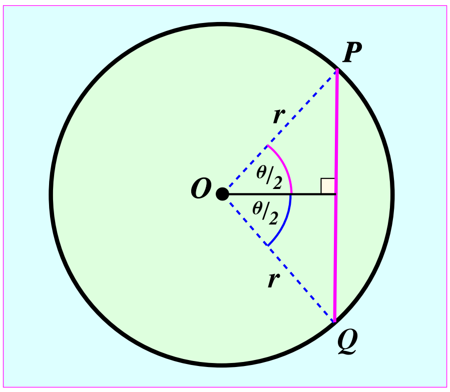


24.6 Length of a Chord

Recall:

Length of chord: $PQ = 2r \sin\frac{\theta}{2}$

θ in radians or degrees.

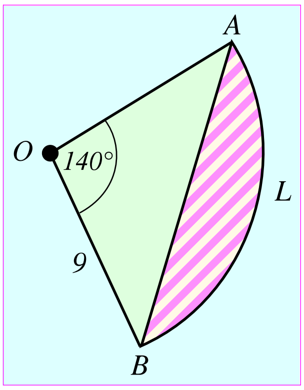


24.7 Radians, Arcs, & Sectors: Worked Examples

24.7.1 Example:

1 Find the perimeter and the area of the shaded region, giving the answers to 3 significant figures.

Convert the angle to radians.



Solution:

The perimeter of the shaded area is made up of the arc AB, plus the chord AB.
Convert angle to radians:

$$\theta = 140° \times \frac{\pi}{180} = \frac{7\pi}{9} \text{ radians}$$

Arc length is:

$$L = r\theta = 9 \times \frac{7\pi}{9}$$

$$= 7\pi$$

Chord length:

$$AB = 2r\,sin\,\frac{\theta}{2}$$

$$= 18\,sin\left(\frac{7\pi}{9} \times \frac{1}{2}\right)$$

$$= 18\,sin\,\frac{7\pi}{18} = 16{\cdot}91$$

$\therefore$ Perimeter $= 7\pi + 16{\cdot}91 = 38{\cdot}91$

$= 38{\cdot}9$ 3 *sf*

Area of shaded region:

$$A = \frac{1}{2}r^2(\theta - sin\,\theta) \qquad \theta \text{ in radians}$$

$$= \frac{81}{2}\left(\frac{7\pi}{9} - sin\,\frac{7\pi}{9}\right)$$

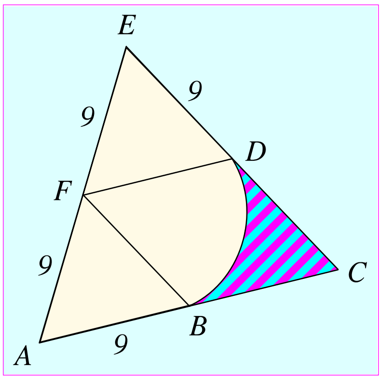
$$= \frac{81}{2} \times 1{\cdot}800$$

$$= 72{\cdot}93$$

$= 72{\cdot}9$ 3 *sf*

oecfrl h/t Fritz K

2 The triangle ACE is an equilateral triangle, with sides 18cm long. Find the area of the shaded region.



Solution:

The shaded area is found by finding the area of the rhombus $BCDF$ and subtracting the area of the sector BDF.

Area of the rhombus $BCDF$:

$$\angle BCD = 60° = \frac{\pi}{3} \; radians$$

Area of rhombus = 2 × Area of triangle BCD

$$A_{\diamond} = 2 \times \left(\frac{1}{2} BC \times CD \times \sin\frac{\pi}{3}\right)$$

$$= 9^2 \sin\frac{\pi}{3} = \frac{81\sqrt{3}}{2} \approx 70{\cdot}15$$

Area of the sector BDF: $$A_C = \frac{1}{2} FD^2 \times \frac{\pi}{3} = \frac{\pi}{6} FD^2$$

$$= \frac{\pi}{6} \times 9^2 = \frac{81\pi}{6} = \frac{27\pi}{2}$$

Area of the shaded region: $$A_S = \frac{81\sqrt{3}}{2} - \frac{27\pi}{2}$$

$$\approx 27{\cdot}74 \; cm^2$$

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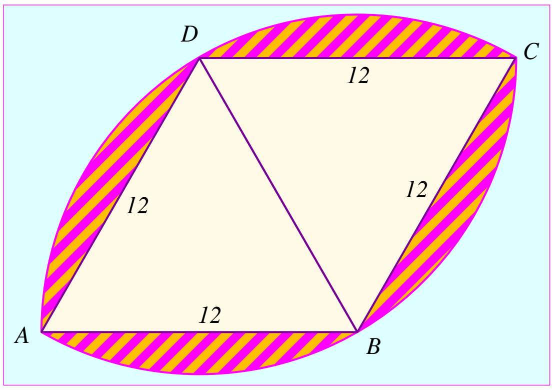
3 The triangles ABD and BCD are equilateral triangles, with sides 12 cm long. The triangles are set within two sectors centred on points B & D.

Find the area of the shaded region.



Solution:

The shaded area is found by finding the area of the two sectors ABC & ADC and subtracting the area of the rhombus ABCD.

Area of the rhombus $ABCD$:

Area of rhombus = 2 × Area of triangle ABD

$$A_{\diamond} = 2 \times \left(\frac{1}{2} AB \times AD \times \sin\frac{\pi}{3}\right) = AB \times AD \times \sin\frac{\pi}{3}$$

$$= 12^2 \sin\frac{\pi}{3} = \frac{144\sqrt{3}}{2} = 72\sqrt{3} \approx 124{\cdot}71$$

Area of the sector ABC:

$$A_C = \frac{1}{2}DB^2 \times \frac{2\pi}{3} = \frac{\pi}{3}DB^2$$

$$= \frac{\pi}{3} \times 144 = \frac{144\pi}{3} = 48\pi$$

Area of the shaded region:

$$A_S = 2 \times 48\pi - 72\sqrt{3}$$

$$A_S = 96\pi - 72\sqrt{3}$$

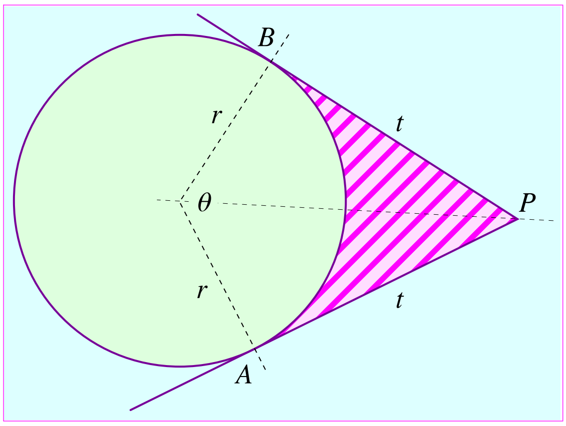
$$\approx 176{\cdot}89\ cm^2$$

4 Find an expression that gives the area of the shaded part of the diagram.

Radius: AC & $CB = r$

$\angle ACB = \theta$

Lines AP & BP are tangent to the circle.



Solution:

The shaded area is found by finding the area of the kite $ACBP$, and subtracting the area of the sector ABC.

Area of the kite $ACBP$:

Area of kite = 2 × Area of triangle ACP

$$A_\diamond = 2 \times \left(\frac{1}{2}AC \times AP\right) = AC \times AP = rt$$

but: $\tan\frac{\theta}{2} = \frac{t}{r} \quad \Rightarrow \quad t = r\tan\frac{\theta}{2}$

$$\therefore \quad A_\diamond = r^2 \tan\frac{\theta}{2}$$

Area of the sector ABC:

$$A_C = \frac{1}{2}r^2\theta$$

Area of the shaded area:

$$A_S = r^2 \tan\frac{\theta}{2} - \frac{1}{2}r^2\theta$$

$$= r^2\left(\tan\frac{\theta}{2} - \frac{\theta}{2}\right)$$

24.8 Topical Tips

- If the question asks for an exact answer, leave the answer in terms of a surd or π.
- Radians should be used at all times when dealing with the derivative and integral calculus

24.9 Common Trig Values in Radians

Exact equivalent of common trig values in radians.

Degrees	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*AT*	0	*AT*	0

24.10 Radians, Arcs, & Sectors Digest

$180° = \pi$ radians

Arc length $= r\theta$ $\qquad L = \dfrac{\pi r\theta}{180}$ (θ in degrees)

Length of chord $= 2r \sin\dfrac{\theta}{2}$ (θ in degreeas or radians)

Area of sector $= \frac{1}{2} r^2\theta$ (θ in radians) $\qquad A = \dfrac{\pi r^2\theta}{360}$ (θ in degrees)

Area of segment $= \frac{1}{2} r^2(\theta - \sin\theta)$ (θ in degreeas or radians)

25 • C2 • Logarithms

25.1 Basics Logs

Logs are exponents!

The logarithm of a number, N, is the exponent or power to which a base number must be raised to produce that number.

Thus: $8 = 2^3$ (*base* 2) $16 = 4^2$ (*base* 4)

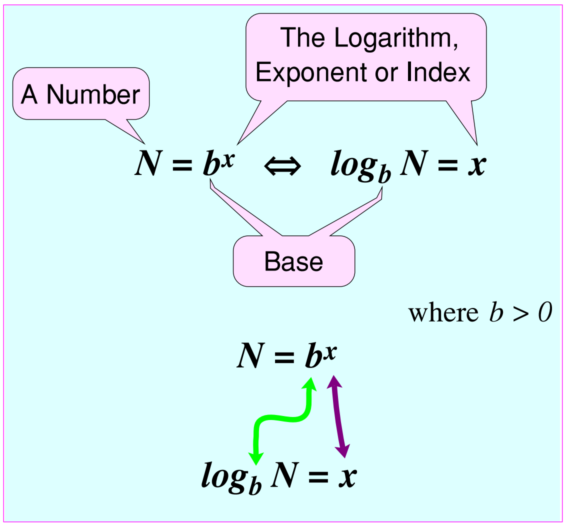
$\log_2 8 = 3$ $\log_4 16 = 2$

In simpler terms, what we are really asking is the question "How many times do we have to multiply the base number by itself to get our number N." The logarithm tells you what the exponent, power or index is.

In algebraic terms:

If: $N = b^x$ then $\log_b N = x$ true if $b > 0;\ b \neq 1$

The constant b is the "base" and the exponent, x, is the logarithm, 'index' or 'power'.



The Log Definition: remember this relationship.

Note that:

as: $b = b^1$ then $\log_b b = 1$

and: $b^0 = 1$ then $\log_b 1 = 0$

if $\log_b x = \log_b y$ then $x = y$

Note the restriction that the base, b, has to be a positive number and greater that zero. You can't evaluate an equation like $y = (-3)^x$ for all values of x.

If $b > 1$ then $N = b^x$ increases as x increases for all values of x.

If $0 < b < 1$ then $N = b^x$ decreases as x increases for all values of x.

N is always +ve, as the log does not exist for −ve values.

However, the exponent or logarithm can be negative, and this implies division rather that multiplication.

The log function can also be defined as the inverse of an exponential function (see later).

25.2 Uses for Logs

Logs are a great way to reduce a large number to a smaller number, which can then be used for comparison purposes.

Hence logs to base 10 are often used for:

- Earthquake intensity scale (Richter scale)
- Sound intensity scale (decibel scale dB)
- Musical scales
- Measurement of pH [$\log_{10}$ (1/concentration of H^+ ions)]
- Radioactive decay
- Financial investment calculations
- Population growth studies
- Log graph paper with the x and/or y axes with log scales (turns an exponential curve into a straight line)

25.3 Common Logs

Common Logs use the base 10, and are very common in Engineering.

Logs to base 10 were used before the days of calculators to handle long multiplication & division, powers, and roots. The old slide rules are based on log scales.

Normally written without the base e.g. $log\, 67{\cdot}89 \;\equiv\; log_{10}\, 67{\cdot}89$

Thus:

$$log_{10}\, 1000 = 3 \qquad \equiv 10^3$$
$$log_{10}\, 100 = 2 \qquad \equiv 10^2$$
$$log_{10}\, 10 = 1 \qquad \equiv 10^1$$
$$log_{10}\, 1 = 0 \qquad \equiv 10^0$$
$$log_{10}\, 0{\cdot}1 = 0 \qquad \equiv 10^{-1}$$
$$log_{10}\, 0{\cdot}01 = 0 \qquad \equiv 10^{-2}$$
$$log_{10}\, 0{\cdot}001 = 0 \qquad \equiv 10^{-3}$$

and:

$$log_{10}\, 67{\cdot}89 = 1{\cdot}8318 \qquad \equiv 10^{1.8318}$$
$$log_{10}\, 6{\cdot}789 = 0{\cdot}8318 \qquad \equiv 10^{0{\cdot}8318}$$
$$log_{10}\, 0{\cdot}6789 = -1{\cdot}8318 \qquad \equiv 10^{-1{\cdot}8318}$$
$$log_{10}\, 0{\cdot}06789 = -2{\cdot}8318 \qquad \equiv 10^{-2{\cdot}8318}$$

25.4 Natural Logs

Natural logs, sometimes called Naperian logs, have a base of e (Euler's Number) and are written ln to distinguish them from common logs. You **must** use natural logs in calculus, hence mathematicians tend to use natural logs.

The value e is found in many scientific & natural processes. It is an irrational number (you cannot turn it into a fraction!)

$$e = 2{\cdot}7182818\ldots$$

Note: $\quad ln\, 1 = 0 \quad \& \quad ln\, e = 1$

More in the next section...

25.5 Log Rules - OK

$$log_a(MN) = log_a M + log_a N \qquad (1)$$

Proof:

By defn: $a^x = M \Leftrightarrow log_a M = x$

By defn: $a^y = N \Leftrightarrow log_a N = y$

Power Law ①
$$MN = a^x \times a^y = a^{(x+y)}$$
$$log_a(MN) = (x + y)$$
$$= log_a M + log_a N \qquad \text{QED}$$

Similarly:

$$log_a\left(\frac{M}{N}\right) = log_a M - log_a N \qquad (2)$$

By defn: $a^x = M \Leftrightarrow log_a M = x$

By defn: $a^y = N \Leftrightarrow log_a N = y$

Power Law ②
$$\frac{M}{N} = \frac{a^x}{a^y} = a^{(x-y)}$$
$$log_a\left(\frac{M}{N}\right) = x - y$$
$$= log_a M - log_a N \qquad \text{QED}$$

From this rule we see:

$$log_a\left(\frac{1}{N}\right) = log_a 1 - log_a N = 0 - log_a N$$

$$\therefore \qquad log_a\left(\frac{1}{N}\right) = -log_a N$$

Also:

$$log_a(M)^r = rlog_a M \qquad (3)$$

By defn: $a^x = M \Leftrightarrow log_a M = x$

Power Law ③
$$M^r = (a^x)^r = a^{rx}$$
$$log_a(M)^r = rx$$
$$= r\,log_a M \qquad \text{QED}$$

From this last rule we see that:

$$log_a \sqrt[n]{M} = log_a (M)^{\frac{1}{n}} = \frac{1}{n}log_a M$$

$$log_a \sqrt[n]{M} = \frac{1}{n}log_a M$$

$$log_a\left(\frac{M}{N}\right) = -log_a\left(\frac{N}{M}\right)$$

25.6 Log Rules Revision

Log rules work for any base, provided the same base is used throughout the calculation.

Any base used must be > 1.

The second and third log rules can be derived from the first rule thus:

$$\log_a M + \log_a N = \log_a (MN) \qquad (1)$$

But $$\log_a \frac{1}{N} = -\log_a N$$

$$\log_a M + \log_a \frac{1}{N} = \log_a M - \log_a N = \log_a \left(\frac{M}{N}\right) \qquad (2)$$

Assume $M = N$

$$\log_a N + \log_a N = 2\log_a N = \log_a \left(N^2\right)$$

In general: $$r\log_a M = \log_a (M)^r \qquad (3)$$

Equating indices

Note that in the same way as $5^{8x} = 5^4 \Rightarrow 8x = 4$ then similarly if $\log 8n = \log 4 \Rightarrow 8n = 4$

25.7 Change of Base

Nearly all log calculations are either log to base 10 or log to base e. Some engineering calculations are to base 2.

In general, try not to mix the bases in any calculation, but if a change of base is required use:

$$\log_a N = \frac{\log_b N}{\log_b a} \qquad (4)$$

25.7.1 Example:

Find $\log_2 128$:

$$\log_2 128 = \frac{\log_{10} 128}{\log_{10} 2}$$

$$\log_2 128 = \frac{2{\cdot}1072}{0{\cdot}3010} = 7$$

(OK - you can do this directly on your calculator, but it illustrates the technique)

If you want to use a factor to change bases, choose a base and number that reduce to 1 for the denominator.

25.7.2 Example:

Find a factor to convert base 6 logs to base 10.

$$\frac{\log_6 10}{\log_{10} 10} = \frac{\log_6 10}{1} = 1{\cdot}285$$

hence $$\log_6 N = 1{\cdot}285 \log_{10} N$$

Hence:

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$$

25.8 Worked Examples in Logs of the form $a^x = b$

25.8.1 Example:

1 Find x if: $3^{2x+1} = 5^{100}$

Take logs (base 10) both sides: (Resist the temptation to take logs to base 3 on one side and logs to base 5 on the other, which will give you a change of base to do).

$$\log 3^{2x+1} = \log 5^{100}$$

$$(2x + 1)\log 3 = 100 \log 5$$

$$2x + 1 = \frac{100 \log 5}{\log 3}$$

$$2x = \frac{100 \log 5}{\log 3} - 1$$

$$2x = \frac{100 \log 5}{\log 3} - 1 = \frac{100 \times 0{\cdot}699}{0{\cdot}477} - 1$$

$$x = \frac{145{\cdot}53}{2} = 72{\cdot}77\ (2dp)$$

2 A curve has the equation $y = \left(\frac{1}{2}\right)^x$. A point Q, on the line has a value of $y = \frac{1}{6}$.

Show that the x-co-ordinate of Q has the form: $1 + \frac{\log 3}{\log 2}$

At point Q: $y = \dfrac{1}{6} = \left(\dfrac{1}{2}\right)^x$

$$\frac{1}{6} = \frac{1}{2^x}$$

$$6 = 2^x$$

Take logs $\log 6 = \log 2^x$

$$= x \log 2$$

$$\therefore \quad x = \frac{\log 6}{\log 2}$$

$$x = \frac{\log (2 \times 3)}{\log 2} = \frac{\log 2 + \log 3}{\log 2}$$

$$\therefore \quad x = 1 + \frac{\log 3}{\log 2}$$

3 Given that: $y = 5 \times 10^{3x}$, show that $x = p \log_{10}(qy)$, and state the values of p and q.

Solution:

$$y = 5 \times 10^{3x}$$

$$\frac{y}{5} = 10^{3x}$$

$$\log\left(\frac{y}{5}\right) = 3x$$

$$x = \frac{1}{3}\log\left(\frac{y}{5}\right)$$

$$\therefore \quad p = \frac{1}{3} \qquad q = \frac{1}{5}$$

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4 Given that: $y = 6^{3x}$ and $y = 84$, solve for x.

Method 1

$$6^{3x} = 84$$

$$\log 6^{3x} = \log 84$$

$$3x \log 6 = \log 84$$

$$3x = \frac{\log 84}{\log 6}$$

$$x = \frac{\log 84}{3 \log 6}$$

$$x = 0{\cdot}824 \qquad \text{(3sf)}$$

Method 2

$$6^{3x} = 84$$

$$3x = \log_6 84$$

$$x = \frac{\log_6 84}{3}$$

$$x = 0{\cdot}824 \qquad \text{(3sf)}$$

5 Solve: $3^{2x+1} - 14(3^x) - 5 = 0$

Solution:

Recognise that: $a^{2x} = (a^x)^2$

$$3^{2x}(3) - 14(3^x) - 5 = 0$$

$$(3^x)^2(3) - 14(3^x) - 5 = 0$$

$$3(3^x)^2 - 14(3^x) - 5 = 0$$

This is a quadratic in 3^x $\quad \therefore$ let $z = 3^x$

$$3(z)^2 - 14z - 5 = 0$$

$$(3z + 1)(z - 5) = 0$$

$$\therefore \quad z = -\frac{1}{3} \quad or \quad z = 5$$

But 3^x cannot be –ve, hence $3^x = 5$:

$$\log 3^x = \log 5$$

$$x \log 3 = \log 5$$

$$\therefore \quad x = \frac{\log 5}{\log 3} = 1{\cdot}46 \ 3sf$$

25.9 Inverse Log Operations

25.9.1 First Investigation

From our basic definition

$$N = b^x \quad \Leftrightarrow \quad log_b N = x$$

we note that the process is reversible, i.e. this is an inverse function.
If we substitute various numerical values for x, we derive the following:

$$\text{If } x = 0 \quad \text{then } N = b^0 \quad \Rightarrow \quad N = 1 \quad \therefore \quad log_b 1 = 0$$

$$\text{If } x = 1 \quad \text{then } N = b^1 \quad \Rightarrow \quad N = b \quad \therefore \quad log_b b = 1$$

$$\text{If } x = 2 \quad \text{then } N = b^2 \quad \Rightarrow \quad \therefore \quad log_b b^2 = 2$$

$$\text{If } x = n \quad \text{then } N = b^n \quad \Rightarrow \quad \therefore \quad log_b b^n = n$$

$$\therefore \quad log_b b^x = x$$

Reversing the definitions and substituting $log_b N$ for x in $N = b^x$

$$x = log_b N \quad \Leftrightarrow \quad b^x = N \quad \therefore \quad b^{log_b N} = N$$

As these are true for any value of x, then we have these identities:

$$log_b b^x \equiv x \quad \text{and} \quad b^{log_b N} \equiv N \qquad N > 0, \ x \in \mathbb{R}$$

(Hence we find: $log_{10} 10 = 1$ & $ln\, e = 1$

25.9.2 Second Investigation

from our basic definition

$$(1) \quad N = b^x \quad \Leftrightarrow \quad log_b N = x \quad (2)$$

Substitute (1) into (2) $\quad log_b b^x = x \quad$ (first proof)

≈≈≈≈≈≈≈≈≈≈≈≈≈≈≈≈≈

Now from (1) $\quad b^x = N$

Take logs both sides $\quad log_b \left(b^x\right) = log_b N$

But $\quad log_b N = x \quad \therefore \quad log_b b^x = x \quad$ (second proof)

≈≈≈≈≈≈≈≈≈≈≈≈≈≈≈≈≈

Now from (1) $\quad b^x = N$

But $\quad x = log_b N \quad \therefore \quad b^{log_b N} = N \quad$ (third proof)

These results are very useful in solving log problems. Some examples will help clarify things:

25.9.3 Example:

1 Take e^x and take logs to base e. From the log rules we have:

$$\ln e^x = x \ln e$$

but: $\ln e = 1 \qquad \therefore \ln e^x = x$

In general:

$$y = e^x \quad \Leftrightarrow \quad \ln y = x$$

$$\therefore \quad y = e^{\ln y}$$

$$\therefore \quad y^a = \left(e^{\ln y}\right)^a = e^{a \ln y}$$

2 Take the number 128 and take logs to base 2.

$$\log_2 128$$

Raise the base 2 to the log of 128: $2^{\log_2 128}$

But: $128 = 2^7 \qquad \therefore \qquad 2^{\log_2 128} = 2^{\log_2 2^7}$

From the log rules: $\qquad \therefore \qquad 2^{\log_2 128} = 2^{7 \times \log_2 2}$

But: $\log_2 2 = 1 \qquad\qquad 2^{\log_2 128} = 2^7 = 128$

Raising a base number to the log of another number, using the same base, results in the same number being generated. Hence this is called an inverse operation.

In general: $\qquad a^{\log_a m} = m$

3 Given that:

$$2 \log_{10}\left(\frac{x}{y}\right) = 1 + \log_{10}\left(10x^2y\right)$$

Find y to 3 dp.

$$2(\log_{10} x - \log_{10} y) = 1 + \log_{10} 10 + \log_{10} x^2 + \log_{10} y$$

$$2 \log_{10} x - 2 \log_{10} y = 1 + 1 + 2 \log_{10} x + \log_{10} y$$

$$-\log_{10} y - 2 \log_{10} y = 2$$

$$3 \log_{10} y = -2$$

$$\log_{10} y = -\frac{2}{3}$$

$$y = 10^{-\frac{2}{3}} = 0{\cdot}215$$

25.10 Further Worked Examples in Logs

25.10.1 Example:

1 Find the value of y, given that:

$$3\log\left(\frac{x}{y}\right) = 2 + \log(10x^3y)$$

Answer to 3 dp.

Solution:

$$3(\log x - \log y) = 2 + \log 10 + 3\log x + \log y$$

$$3\log x - 3\log y = 2 + \log 10 + 3\log x + \log y$$

$$-3\log y = 2 + 1 + \log y$$

$$-4\log y = 3$$

$$\log y = -\frac{3}{4}$$

$$y = 10^{-\frac{3}{4}}$$

$$y = 1{\cdot}778$$

oecfrl

2 Evaluate:

$$\log_5 10 + \log_5 75 + \log_5 2 - \log_5 12$$

Solution:

Let $y = \log_5 10 + \log_5 75 + \log_5 2 - \log_5 12$

$$y = \log_5 \frac{10 \times 75 \times 2}{12}$$

$$y = \log_5 125$$

$$5^y = 125$$

$$y = 3$$

oecfrl

3 Solve $10^p = 0{\cdot}1$

Solution:

$$10^p = 0{\cdot}1$$

$$p = \log_{10} 0{\cdot}1$$

$$p = -1$$

or:

$$10^p = 0{\cdot}1$$

$$\log 10^p = \log 0{\cdot}1$$

$$p\log 10 = \log 0{\cdot}1$$

but: $\log_{10} 10 = 1$

$\therefore \quad p \times 1 = \log 0{\cdot}1$

$$p = -1$$

4 Simplify:

$$\frac{\log 1 - \log 16}{\log 1 - \log 2}$$

Solution:

$$\frac{\log 1 - \log 16}{\log 1 - \log 2} = \frac{0 - \log 16}{0 - \log 2} \qquad \text{since } 1 = 10^0$$

$$= \frac{\log (2 \times 8)}{\log 2}$$

$$= \frac{\log 2 + \log 8}{\log 2}$$

$$= 1 + \frac{\log 8}{\log 2} = 1 + 3$$

$$= 4$$

5 Solve: $3^{2x+1} - 14(3^x) - 5 = 0$

Solution:

Recognise that: $a^{2x} = (a^x)^2$

$$3^{2x}(3) - 14(3^x) - 5 = 0$$

$$(3^x)^2(3) - 14(3^x) - 5 = 0$$

$$3(3^x)^2 - 14(3^x) - 5 = 0$$

This is a quadratic in 3^x $\quad\therefore$ let $z = 3^x$

$$3(z)^2 - 14z - 5 = 0$$

$$(3z + 1)(z - 5) = 0$$

$$\therefore \quad z = -\frac{1}{3} \quad or \quad z = 5$$

But 3^x cannot be –ve, hence $3^x = 5$:

$$\log 3^x = \log 5$$

$$x \log 3 = \log 5$$

$$\therefore \quad x = \frac{\log 5}{\log 3} = 1{\cdot}46 \text{ } 3sf$$

6 Given that $2 \log n - \log(8n - 24) = \log 2$, show that $n^2 - 16n + 48 = 0$

Solution:

$$2 \log n - \log(8n - 24) = \log 2$$

$$\log n^2 - \log(8n - 24) = \log 2$$

$$\log \frac{n^2}{(8n - 24)} = \log 2$$

$$\therefore \quad \frac{n^2}{(8n - 24)} = 2$$

$$\therefore \quad n^2 - 16n + 48 = 0$$

EAFQLA

7 Given that $log_2 q = h$ and that $p = ½$

express: $log_2 \frac{p^4}{\sqrt{q}}$ in terms of h

Solution:

$$log_2 \frac{p^4}{\sqrt{q}} = log_2 p^4 - log_2 \sqrt{q}$$

$$= 4\, log_2 p - \frac{1}{2} log_2 q$$

$$= 4\, log_2 \frac{1}{2} - \frac{1}{2} h$$

$$= 4\left(log_2 1 - log_2 2\right) - \frac{1}{2} h$$

$$= 4\left(0 - 1\right) - \frac{1}{2} h$$

$$= -4 - \frac{1}{2} h$$

eedfxl

8 Find the roots of the equation: $2\, log_2 (2x + 3) + log_2 (x) - 3\, log_2 (2x) = 1$

Solution:

Now recognise that $log_2 2 = 1$

Hence: $2\, log_2 (2x + 3) + log_2 (x) - 3\, log_2 (2x) = log_2 2$

Converting back to index form:

$$\frac{(2x + 3)^2 x}{(2x)^3} = 2$$

$$\frac{(2x + 3)^2}{8x^2} = 2$$

$$(2x + 3)^2 = 16x^2$$

$$4x^2 + 12x + 9 = 16x^2$$

$$-12x^2 + 12x + 9 = 0$$ Divide thro' by (−3)

$$4x^2 - 4x - 3 = 0$$

$$(2x + 1)(2x - 3) = 0$$

$$\therefore \quad x = -\frac{1}{2} \quad or \quad x = \frac{3}{2}$$

9 A curve has the equation $y = 3\, log_{10} x - log_{10} 8$. Point P lies on the curve. P has the co-ordinates: $P = \left(3,\ log_{10}\left[\frac{27}{8}\right]\right)$

The point Q (6, q) also lies on the curve, show that the gradient of PQ is $log_{10} 2$

Solution:

At $x = 6$ $\quad y = log \frac{6^3}{8} \Rightarrow log \frac{216}{8} = log\, 27$

$$\text{gradient} = \frac{y_1 - y_1}{x_1 - x_2} = \frac{log\, 27 - log\left(\frac{27}{8}\right)}{6 - 3} = \frac{log\, 27 - (log\, 27 - log\, 8)}{3}$$

$$= \frac{log\, 8}{3} = log\, 8^{\frac{1}{3}} = log\, 2$$

10 Solve the inequality:

$$0{\cdot}5776^x \leqslant 0{\cdot}76$$

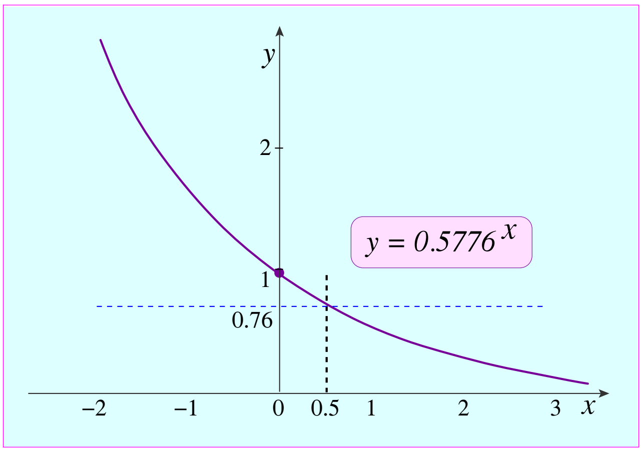
Solution:

$$0{\cdot}5776^x \leqslant 0{\cdot}76$$

$$x \log(0{\cdot}5776) \leqslant \log(0{\cdot}76)$$

But a log of a number < 1 is −ve

Hence: $x \geqslant \dfrac{\log(0{\cdot}76)}{\log(0{\cdot}5776)} = 0{\cdot}5$ (Note the change in the inequality sign)



25.11 Use of Logs in Practice

As was noted at the beginning of this section, logs are used in a number of different fields, such as:

The Richter Scale, M:

$$M = \frac{2}{3} \log \frac{E}{E_0} \qquad \text{where } E_0 = 10^{4.40}\ joules$$

where E is the energy released by an earthquake, and E_0 is the energy released by a standard reference earthquake.

Sound decibel scale (dB):

$$dB = 10 \log\left(p \times 10^{12}\right) \qquad \text{where } p = \text{ sound pressure}$$

pH scale:

$$pH = -\log\left[H^+\right] \qquad \text{where } \left[H^+\right] = \text{ concentration of H ions (moles/L)}$$

25.12 Heinous Howlers

Don't make up your own rules!

- $log(x + y)$ is ***not the same*** as $log\, x + log\, y$. Study the above table and you'll find that there's nothing you can do to split up $log(x + y)$ or $log(x - y)$.
- $\frac{log(x)}{log(y)}$ is ***not the same*** as $log\left(\frac{x}{y}\right)$. When you divide two logs to the same base, you are in fact using the change-of-base formula backwards. Note that $\frac{log(x)}{log(y)} = log_y(x)$, ***NOT*** $log\left(\frac{x}{y}\right)$!
- $(log\, x)(log\, y)$ is ***not the same*** as $log(xy)$. There's really not much you can do with the product of two logs when they have the same base.

Handling logs causes many problems, here are a few to avoid.

1 $ln(y + 2) = ln(4x - 5) + ln\, 3$

You cannot just remove all the *ln*'s so: $(y + 2) \neq (4x - 5) + 3$ ✘

To solve, put the RHS into the form of a single log first:

$$ln(y + 2) = ln\left[3(4x - 5)\right] \quad ✔$$

$$\therefore \quad (y + 2) = 3(4x - 5)$$

2 $ln(y + 2) = 2\, ln\, x$

You cannot just remove all the *ln*'s so: $(y + 2) \neq 2x$ ✘

To solve, put the RHS into the form of a single log first:

$$ln(y + 2) = ln\, x^2 \quad ✔$$

$$\therefore \quad (y + 2) = x^2$$

3 $ln(y + 2) = x^2 + 3x$

You cannot convert to exponential form this way: $(y + 2) \neq e^{x^2} + e^{3x}$ ✘

To solve, raise e to the whole of the RHS : $(y + 2) = e^{x^2 + 3x}$ ✔

25.13 Log Rules Digest

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$$\therefore \log_{10} 10 = 1 \quad \& \quad \ln e = 1$$

$$\log_b b^n = n$$

$$\therefore \log_{10} 10^n = n \quad \& \quad \ln e^n = n$$

Laws of Exponents	**Laws of Logarithms**	
$N = b^x$	$\log_b N = x$	$b > 0$
$b^0 = 1$	$\log_b 1 = 0$	
$b^1 = b$	$\log_b b = 1$	
$a^m a^n = a^{(m+n)}$	$\log_a (MN) = \log_a M + \log_a N$	
$\frac{a^m}{a^n} = a^{(m-n)}$	$\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$	
$\frac{1}{a^n} = a^{(-n)}$	$\log_a \left(\frac{1}{N}\right) = -\log_a N$	
$\sqrt[n]{m} = m^{\frac{1}{n}}$	$\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M$	
$(a^m)^n = a^{(mn)}$	$\log_a M^n = n \log_a M$	
$(a^m)^{\frac{1}{n}} = a^{\left(\frac{m}{n}\right)}$	$\log_a M^{\frac{1}{n}} = \frac{1}{n} \log_a M$	
Change of base $\Rightarrow$	$\log_a N = \frac{\log_b N}{\log_b a}$	
	$\log_a b = \frac{1}{\log_b a}$	
$\frac{a}{b} = \left(\frac{b}{a}\right)^{-1}$	$\ln \frac{a}{b} = -\ln \frac{b}{a}$	
	$a^{\log_a m} = m$	
$a^{\log_a x} = x$	$\log_a (a^x) = x$	
$10^{\log N} = N$	$\log (10^x) = x$	
$e^{\ln x} = x$	$\ln e^x = x$	
$e^{a \ln x} = x^a$	$a \ln e^x = ax$	*

Note:

$$\log x \equiv \log_{10} x \quad \& \quad \ln x \equiv \log_e x$$

26 • C2 • Exponential Functions

26.1 General Exponential Functions

An **exponential function** has the form:

$$f(x) = b^x \quad \text{or} \quad y = b^x \quad \text{where } b \text{ is the base and } b > 0,\ b \neq 1$$

Note that the power of the number is the variable x. The power is also called the exponent - hence the name exponential function.

E.g. 3^x, 4.5^x, 5^x are all exponentials.

Exponential functions have the following properties:

- The value of b is restricted to $b > 0$ and $b \neq 1$
  - Note that when $a = 0$, $b^x = 0$, and when $b = 1$, $b^x = 1$, hence the restrictions above
  - The function is not defined for negative values of b. (*e.g.* $-1^{0\cdot5} = \sqrt{-1}$)
- All exponential graphs have similar shapes
- All graphs of $y = b^x$ and $y = b^{-x}$ pass through co-ordinates (0, 1)
- Graphs pass through the point $(1, b)$, where b is the base
- The larger the value of b, the steeper the curve
- Graphs with a negative exponent are reflections of the positive ones, being reflected in the y-axis
- For $b > 1$ and +ve x, the gradient is always increasing and we have exponential growth
  For $b > 1$ and –ve x, the gradient is always decreasing and we have exponential decay
  For $0 < b < 1$ and +ve x, the gradient is always decreasing and we have exponential decay
- The x-*axis* of a exponential graph is an asymptote to the curve hence:
  - The value of y never reaches zero and is always positive
- For exponential graphs, the gradient divided by its y value is a constant
- Recall that $b^0 = 1$, for +ve values of b, and that $b^{-3} \equiv \dfrac{1}{b^3}$

26.2 The Exponential Function: e

Whereas a^x is an exponential function, there is one special case which we call THE exponential function.

By adjusting the value of the base b, we can make the gradient at the co-ordinate (0, 1) anything we want. If the gradient at (0, 1) is adjusted to 1 then our base, b, is found to be 2·71828…

The function is then written as:

$$y = e^x \quad \text{where } e = 2\cdot718281828 \ (9\ dp)$$

Like the number for π, e is an irrational number and never repeats, even though the first few digits may look as though they make a recurring pattern.

THE exponential function can also be found from the exponential series:

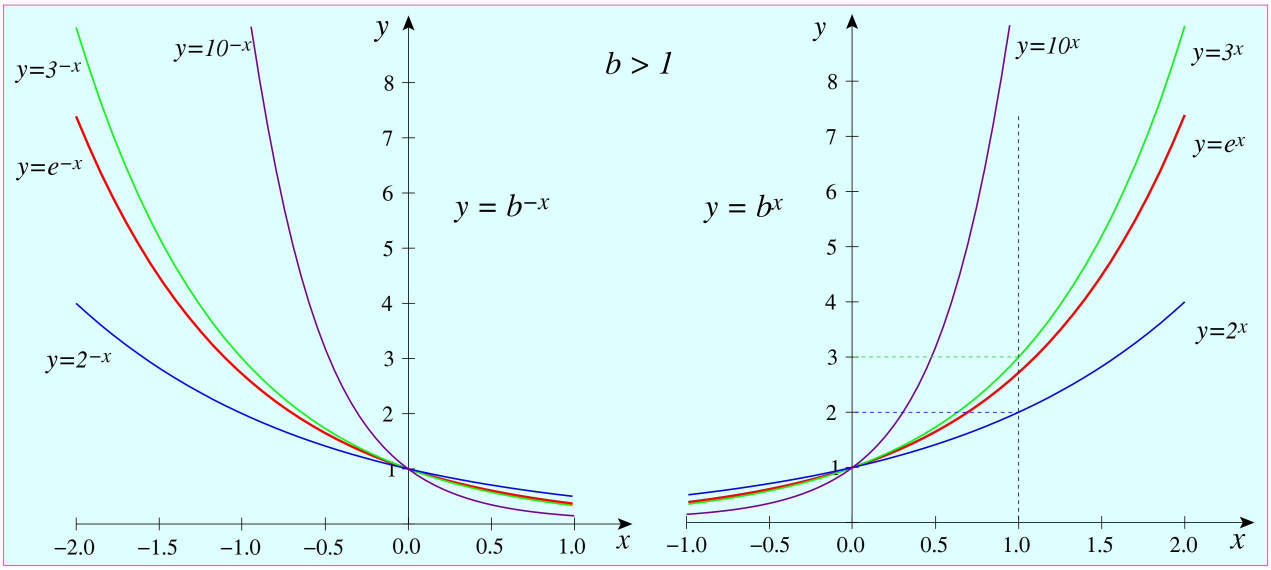
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \ldots$$

To find the value of e, set $x = 1$:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n!} + \ldots$$

In exponential graphs, the gradient divided by the y value ($^{dy}/_{dx} \div y$) is a constant. For e^x this value is 1, and we find that the gradient at any point is equal to y. Hence $\frac{dy}{dx} = e^x$.

26.3 Exponential Graphs



Properties of Exponential graphs:

- Graphs shown are for $y = b^x$ and $y = b^{-x}$, all with $b > 1, \ b \neq 1$
- Continuous for all real numbers
- No sharp corners
- All graphs pass through point (0, 1) and have similar shapes $(y = b^0 = 1)$
- For +ve values of x, graphs pass through point $(1, b)$, where b is the base
- For −ve values of x, graphs pass through point $(-1, b)$
- The negative exponential graphs are reflections of the positive ones, being reflected in the y-axis
- The larger the value of b, the steeper the curve
- For $b > 1$: Gradient increases as x increases,   (positive values of x)
  i.e. the rate of change increases (exponential growth)
- For $0 < b < 1$: Gradient decreases as x increases, $0 < b < 1$ (positive values of x)
  i.e. the rate of change decreases (exponential decay)
- The x-axis of a exponential graph is an asymptote to the curve hence:
  The value of y never reaches zero and is always positive so the curve lies above the x-axis
- Graph intersects any horizontal line only once, hence it is a one-to-one function. This means it has an inverse, the log function.
- For exponential graphs, the gradient divided by its y value is a constant

For $y = b^x$

$$x \rightarrow +\infty \Rightarrow y \rightarrow +\infty$$

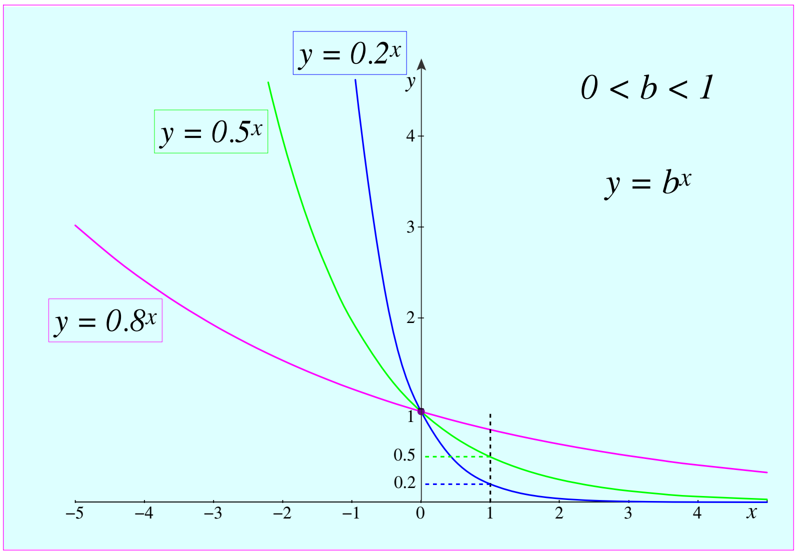
$$x \rightarrow -\infty \Rightarrow y \rightarrow 0$$

For $y = b^{-x}$

$$x \rightarrow +\infty \Rightarrow y \rightarrow 0$$

$$x \rightarrow -\infty \Rightarrow y \rightarrow +\infty$$

For +ve values of $b < 1$, similar graphs are drawn, but these represent decay curves. Note how the curves get steeper as b gets smaller. A negative value of x will produce reflected images in the y-axis (not shown).



These graphs follow from the law of indices:

e.g. $$y = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x = 0{\cdot}5^x$$

$$y = 0{\cdot}2^x = \left(\frac{2}{10}\right)^x = \left(\frac{1}{5}\right)^x = \frac{1}{5^x} = 5^{-x}$$

Note that the scales on the these exponential graphs are different.

26.4 Translating the Exponential Function

In mapping an exponential function remember that for $y = a^x$ the x-axis is an asymptote for the function. So when translating $y = a^x$ to $y = a^x + c$, then the asymptote is also translated by the same amount, to $y = c$.

26.4.1 Example:

Map $y = e^x$ to $y = e^x + 2$.

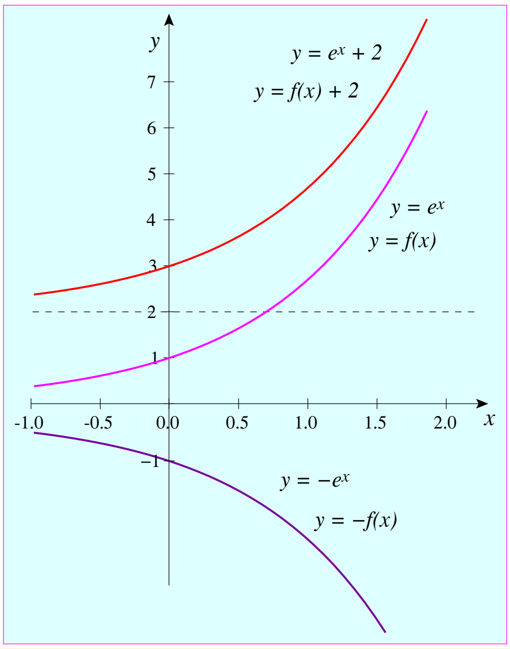
The translation vector is $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Note the asymptote drawn has moved to $y = 2$

~~~——~~~

Map $y = e^x$ to $y = -e^x$

This curve is a reflected image in the x-axis.
Note the asymptote in this case does not move.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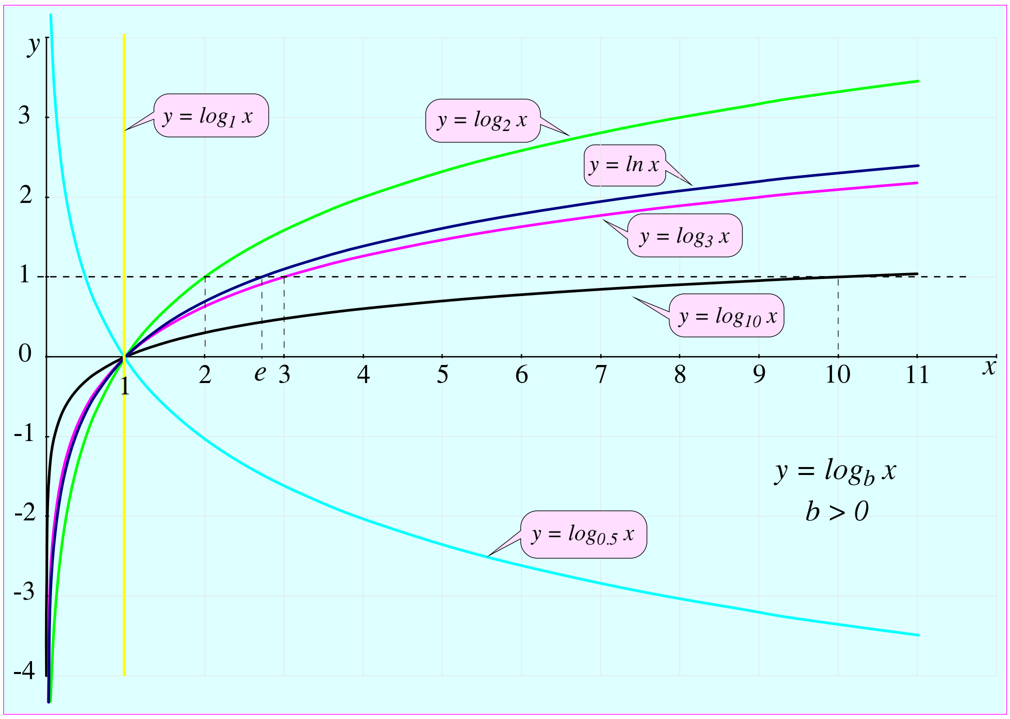
26.5 The Log Function Graphs



Graphs of the Log Family

Properties of Log graphs:

- Graphs for $f(x) = \log_b x \qquad b > 1,\ b \neq 1$
- Continuous in its domain of $(0, \infty)$ Range is $(-\infty, \infty)$
- No sharp corners
- Crosses the x-axis at the point $(1, 0)$
- Passes through point $(b, 1)$ where b is the base
- All have similar shapes
- Valid only for $x > 0$
- For $b > 1$: Graph increases as x increases, $b > 1$
  The smaller the value of b, the steeper the curve
- For $0 < b < 1$: Graph decreases as x increases, $0 < b < 1$
- As x increases, the gradient decreases.
- The y-axis of a log graph is an asymptote to the curve hence:
  The value of x never reaches zero and is always positive, so the curve lies to the right of the y-axis
- Graph intersects any horizontal line only once, hence it is a one-to-one function.
  This means it has an inverse, the exponential function.

26.6 Exponentials and Logs

The functions $y = b^x$ and $y = log_b x$ are inverse functions, i.e. the processes are reversible — one undoes the other.

$$y = 3^x \Leftrightarrow x = log_3 y$$

As with other inverse functions, these two functions, when plotted, are reflection of each other in the line $y = x$.

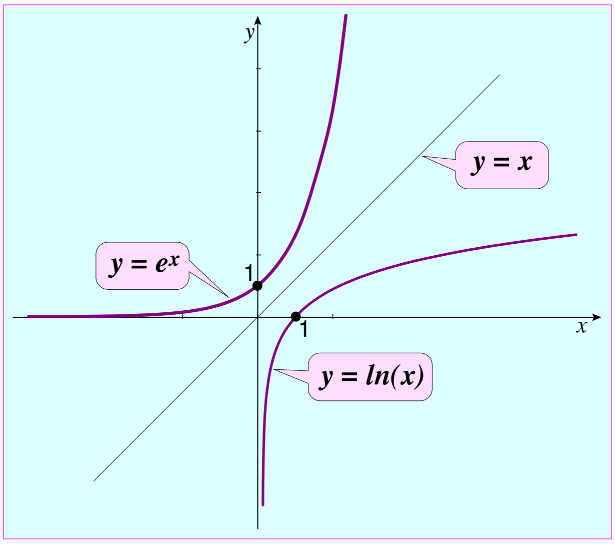
The exponential function, $y = e^x$ is the basis for natural logs, written log_e or ln.

The domain of $y = e^x$ is $x \in \mathbb{R}$ and the range is $y > 0$.

Hence the domain of $y = ln\, x$ is $x > 0$ with a range of $y \in \mathbb{R}$.

We also find that:

$$b^x = e^{x\, ln\, b}$$



26.7 Exponential and Log Worked Examples

26.7.1 Example:

1 Two curves intersect at a point P. Curve A is given by $y = a^x$, $a > 1$, and curve B is $y = 5b^x$, $0 < b < 1$. Show that the equation for the x-coordinate at point P is $x = \dfrac{1}{log_5 a - log_5 b}$

Solution:

$$5b^x = a^x$$

$$log_5 5 + log_5 b^x = log_5 a^x$$

$$1 + x log_5 b = x log_5 a$$

$$x log_5 a - x log_5 b = 1$$

$$x(log_5 a - log_5 b) = 1$$

$$x = \frac{1}{log_5 a - log_5 b}$$

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2 The curve $y = \left(\frac{1}{a}\right)^x$ has a coordinate of $y = \left(\frac{1}{b}\right)$. Find the x-coordinate.

Solution:

$$\left(\frac{1}{a}\right)^x = \frac{1}{b}$$

$$log\left(\frac{1}{a}\right)^x = log\frac{1}{b}$$

$$x\,(log\,1 - log\,a) = log\,1 - log\,b$$

but $\quad log\,1 = 0$

$$-\,x\,log\,a = -log\,b$$

$$x = \frac{log\,b}{log\,a}$$

If $a = 4$ & $b = 8$ show that $x = 1 + \dfrac{log\,2}{log\,4}$

$$x = \frac{log\,8}{log\,4}$$

$$x = \frac{log\,(4 \times 2)}{log\,4} = \frac{log\,4 + log\,2}{log\,4}$$

$$x = 1 + \frac{log\,2}{log\,4}$$

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3 Solve $log_5(5x + 10) - log_5 x = 2$

Solution:

$$log_5(5x + 10) - log_5 x = 2$$

$$log_5\frac{(5x + 10)}{x} = 2$$

$$\frac{(5x + 10)}{x} = 5^2$$

$$5x + 10 = 25x$$

$$5x - 25x = -10$$

$$20x = 10$$

$$x = \frac{1}{2}$$

oecfrl

27 • C2 • Sequences & Series

27.1 What is a Sequence?

A **sequence** or **number pattern** is a set of numbers, in a particular order, which follow a certain rule and creates a pattern.

E.g. 2, 4, 6, 8, 10, …

- Each number in the sequence is called a **term**, and is usually separated by a comma.
- Terms next to each other are referred to as **adjacent** terms or **consecutive** terms.
- Each term is related to the previous term either by a 'term–to–term' rule or a 'position–to–term' rule, or sometimes both.
- Every term in the sequence has a term or pattern number to show its position in the sequence. The n-th term is a general expression which means the value of a term at any position in the sequence.
- Note that the symbol '…' means that the sequence continues on and on and … :-)
- Sequences can be infinite, e.g. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10…
  or finite, e.g. 2, 5, 8, 10, 16… , 25 (where, in this example, 25 is the last term in the sequence).
- A sequence can be defined in two ways:
  - as a recurrence relationship, that depends on the preceding term or
  - as an algebraic relationship that gives the n-th term directly.
- Sequences can be either:
  - Divergent
  - Convergent
  - Periodic

27.1.2 Example:

Term position	1st	2nd	3rd	4th…	n th …
Sequence	2,	4,	6,	8, …	x, …

27.2 Recurrence Relationship

A **recurrence relationship** (also called an iterative formula or recursive definition) defines each term in the sequence by reference to the previous term. At least one term, usually the first term, should be specified.

E.g. The triangle numbers are:

1, 3, 6, 10, 15

The recursive definition of this sequence is given by:

$$U_n = U_{n-1} + n \qquad (\text{where } U_1 = 1)$$

Recurrence relations can be used to represent mathematical functions or sequences that cannot be easily represented non-recursively. An example is the Fibanocci sequence.

E.g. Fibanocci sequence:

1, 1, 2, 3, 5, 8, 13, 21…

The recursive definition is:

$$U_n = U_{n-1} + U_{n-2}$$

27.3 Algebraic Definition

An algebraic relationship defines the n-th term directly. There is no need to know the first term to find the n-th term.

E.g. The triangle numbers are:

$$1,\ 3,\ 6,\ 10,\ 15$$

The algebraic definition of this sequence is given by:

$$U_n = \frac{n(n+1)}{2}$$

27.4 Sequence Behaviour

27.4.1 Convergent Sequences

A sequence whose terms converge on some finite value, which we call the limit, L.
The sequence never quit reaches the limit, but gets exceedingly close to it.
We say that the sequence is convergent when:

$$U_n \to L \quad \text{as} \quad n \to \infty$$

E.g.

$$U_n = 3 + \frac{1}{n}$$

$$4,\ 3\,{}^1/_2,\ 3\,{}^1/_3,\ 3\,{}^1/_4,\ \ldots \to 3$$

$$\lim_{n \to \infty}\left(3 + \frac{1}{n}\right) = 3$$

As n becomes very large. $\frac{1}{n}$ becomes vanishing small and the sequence tends to 3.

A sequence may also oscillate and converge to a limit.

E.g.

$$U_n = \left(-\frac{1}{3}\right)^n$$

$$-\frac{1}{3},\ \frac{1}{9},\ -\frac{1}{27},\ \frac{1}{81},\ \ldots \to 0$$

$$\lim_{n \to \infty}\left(-\frac{1}{3}\right)^n = 0$$

27.4.2 Divergent Sequences

A sequence with terms that progressively become larger (more positive or more negative) without limit, and tend towards infinity.
We say that the sequence is divergent when:

$$U_n \to \pm\infty \quad \text{as} \quad n \to \infty$$

E.g.

$$U_n = 3n + 1$$

$$4,\ 7,\ 10,\ 13,\ \ldots \to \infty$$

A sequence may oscillate and diverge without limit.

E.g.

$$U_n = 2(-2)^n$$

$$-4,\ 8,\ -16,\ 32,\ -64,\ \ldots \to \infty$$

27.4.3 Periodic Sequences

A periodic sequence regularly repeat themselves and as such neither converge or diverge.

$U_{n+p} = U_n$ (where the period p is the smallest value to be true)

E.g.

$$U_n = 3 + (-1)^n$$

$$2,\ 4,\ 2,\ 4,\ 2,\ 4,\ \ldots$$

Period = 2 (repeats every 2 terms)

E.g.

$$U_n = \sin\left(\frac{n\pi}{2}\right)$$

$$1,\ 0,\ -1,\ 0,\ 1,\ -1,\ 0,\ 1\ldots$$

Period = 4 (repeats every 4 terms)

27.5 Worked Example

27.5.1 Example:

1 A sequence is defined by the following recurrence relation:

$$U_{n-1} = qU_n + r$$

$$U_1 = 700, \qquad U_2 = 300, \qquad U_3 = 140$$

Find q & r.

The limit of the sequence is L. Find an equation to express L in terms of q & r.

Solution:

To find q & r, make a simultaneous equation from the values of U_1 *to* U_3

$$300 = 700q + r \qquad (1)$$

$$140 = 300q + r \qquad (2)$$

$$160 = 400q \qquad (1) - (2)$$

$$\therefore \qquad q = \frac{160}{400} = 0{\cdot}4$$

Find r $\quad \therefore \quad 140 = 300 \times 0{\cdot}4 + r \qquad (2)$

$$r = 140 - 120 = 20$$

Limit $\quad L = U_\infty = U_{\infty + 1}$

$$\therefore \qquad L = qL + r$$

$$L - qL = r$$

$$L = \frac{r}{1 - q}$$

$$L = \frac{20}{0{\cdot}6} = 333\,{}^1\!/_3$$

27.6 Series

A **series** is created when all the terms of a sequence are added together. A series can be finite or infinite. It can also converge towards a particular value or diverge for ever.

E.g. $\quad 2 + 4 + 6 + 8 + 10\ldots$

27.7 Sigma Notation Σ

Sigma notation is used to write down a series in a simpler form. Mathematicians don't like having to constantly write out the same phrase, such as 'the sum of', so they use a symbol. To prove how educated they are, they use the Greek alphabet, where Σ corresponds to the English letter 'S'.

The simplest example is the sum of the counting numbers:

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + 4 + 5 + \ldots + n$$

where r is the term, n is the last term, and $r = 1$ gives the first term. This translates to "the sum of all the numbers from 1 to n"

The sigma notation also allows us to specify the range of values over which the series should be added.

E.g.

$$\sum_{r=4}^{6} 2^r = 2^4 + 2^5 + 2^6$$

27.7.2 Rules of Sigma Notation

The sigma notation can be handled according to these rules:

$$\sum_{r=1}^{n} (a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$$

$$\sum_{r=1}^{k} a_r + \sum_{r=k+1}^{n} a_r = \sum_{r=1}^{n} a_r \qquad r < k < n$$

$$\sum_{r=1}^{n} k a_r = k \sum_{r=1}^{n} a_r$$

$$\sum_{1}^{n} c = nc \qquad \text{where } c \text{ is a constant}$$

$$\sum_{1}^{n} 1 = n$$

27.7.3 Converting a Sequence to Sigma Form

To use the Sigma form for any sequence, you just need to find an expression for the n-th term in the sequence.

27.7.3.1 Example:

Convert the sequence 6, 10, 14, 18, … to sigma form:

Solution:

This is an arithmetic sequence with a common difference of 4. The n-th term is:

$$U_n = 4n + 2$$

The sum is expressed as:

$$\sum_{n=1}^{n} 4n + 2$$

27.7.4 Number of Terms in a Summation

The number of terms in a summation is given by:

Upper limit − Lower limit + 1

No. of terms in sum $\displaystyle\sum_{r=m}^{n} \Rightarrow n - m + 1$

27.7.4.1 Example:

$$\sum_{r=4}^{10} 2^r = 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$$

Number of terms $= (10 - 4) + 1 = 7$

27.7.5 Standard Sigma Results

Certain standard sums exist such as:

$$\sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2 = \left[\frac{1}{2} n(n+1)\right]^2 = \left[\sum_{r=1}^{n} r\right]^2$$

These standard results can be used to derive more complicated series.

27.7.5.1 Example:

From the standard results, find the sum of the sequence $(3n - 1)$.

Solution:

Using the rules above:

$$\begin{aligned}
\sum_{r=1}^{n} (3r - 1) &= \sum_{r=1}^{n} 3r - \sum_{r=1}^{n} 1 \\
&= 3 \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\
&= 3\left[\frac{n(n+1)}{2}\right] - n \\
&= \frac{3n(n+1)}{2} - n \\
&= \frac{3n(n+1)}{2} - \frac{2n}{2} \\
&= \frac{n}{2}[3(n+1) - 2] \\
&= \frac{n}{2}[3n + 3 - 2]
\end{aligned}$$

$$\sum_{r=1}^{n} (3r - 1) = \frac{n}{2}[3n + 1]$$

27.8 Sigma Notation: Worked Examples

1 Solve: $\sum_{r=1}^{20} k^2 - \sum_{r=2}^{19} k^2$

Solution:

$$\sum_{r=1}^{20} k^2 - \sum_{r=2}^{19} k^2 = \sum_{r=1}^{1} k^2 + \sum_{r=20}^{20} k^2$$

$$= 1 + 400 = 401$$

2 Show that: $\sum_{r=1}^{n} r = \frac{n}{2}(n + 1)$

Solution:

The sum of terms is given by: $S_n = \frac{n}{2}[2a + (n - 1)d]$

Now: $\sum_{r=1}^{n} r = 1 + 2 + 3 + 4 + \ldots + n$

Hence: $a = 1; \quad d = 1$

$\therefore$ $\sum_{r=1}^{n} r = \frac{n}{2}[2a + (n - 1)d]$

$$= \frac{n}{2}[2 + (n - 1)]$$

$$= \frac{n}{2}[n + 1]$$

3 Given that: $\sum_{r=n+3}^{2n} r = 312$ Find the value of n

Solution:

Now: $\sum_{r=1}^{2n} r = \sum_{r=1}^{n+2} r + \sum_{r=n+3}^{2n} r$

Hence: $\sum_{r=n+3}^{2n} r = \sum_{r=1}^{2n} r - \sum_{r=1}^{n+2} r$

$\sum_{r=1}^{2n} r = n[2n + 1]$ (From Q2 above)

$$\sum_{r=1}^{n+2} r = \frac{n+2}{2}[2 + (n + 2 - 1)] = \frac{(n + 2)(n + 3)}{2}$$

$\therefore$ $\sum_{r=n+3}^{2n} r = n(2n + 1) - \frac{(n + 2)(n + 3)}{2} = 312$

$$2(2n^2 + n) - (n^2 + 5n + 6) = 624$$

$$3n^2 - 3n - 630 = 0$$

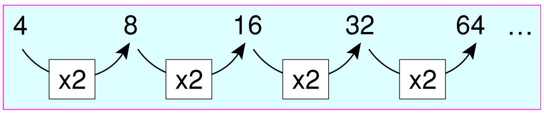
$$n^2 - n - 210 = (n - 15)(n + 14) = 0$$

$\therefore$ $n = 15$ (Positive integers only)

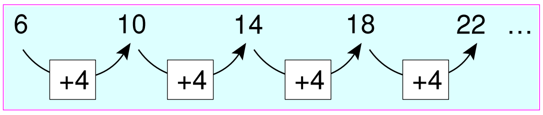
27.9 Finding a likely rule

To find the likely rule, try some of the following ideas:

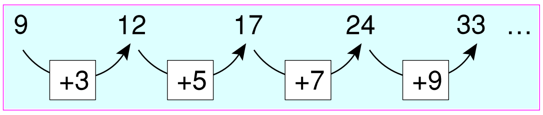
Is it a simple rule you know, like the times table?



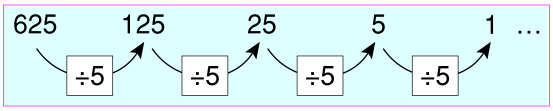
Is the difference between each adjacent term the same? i.e. a common difference.



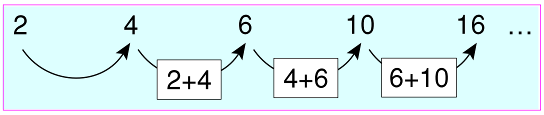
Is the difference between terms a changing pattern (e.g. odd numbers)?



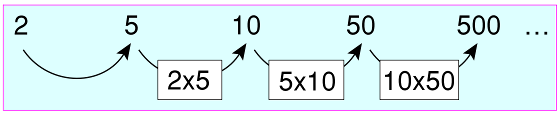
Is it dividing (or multiplying) each term by the same number?



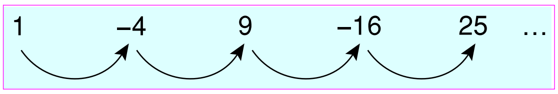
Is it adding the previous two terms together?



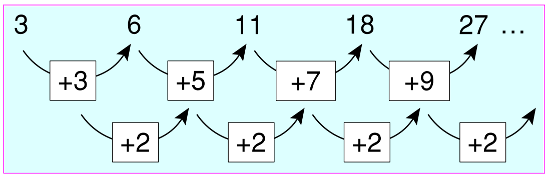
Is it multiplying the previous two terms together?



Is it a pattern with alternating signs? When k is odd: $(-1)^k = -1$. When k is even: $(-1)^k = 1$.



If any of the above do not work, try finding a pattern in the first set of differences, (quadratic sequence: $n^2 + 2$).



27.10 Some Familiar Sequences

Sequence Name	Sequence	Algebraic Defn	Recurrence Relation
Natural or counting numbers	1, 2, 3, 4, 5, 6, 7, …	$U_n = n$	$U_1 = 1, U_{n+1} = U_n + 1$
Even Numbers:	2, 4, 6, 8, 10, …	$U_n = 2n$	$U_1 = 2, U_{n+1} = U_n + 2$
Odd Numbers:	1, 3, 5, 7, 9, 11, …	$U_n = 2n - 1$	$U_1 = 1, U_{n+1} = U_n + 2$
Multiples of 3	3, 6, 9, 12, 15, …	$U_n = 3n$	$U_1 = 3, U_{n+1} = U_n + 3$
Multiples of 4	4, 8, 12, 16, 20, …	$U_n = 4n$	$U_1 = 4, U_{n+1} = U_n + 4$
Prime Numbers:	2, 3, 5, 7, 11, 13, 17, 19, …		
Square Numbers:	$1^2, 2^2, 3^2, 4^2, 5^2, \ldots$ ↓ ↓ ↓ ↓ ↓ 1, 4, 9, 16, 25, …	$U_n = n^2$	$U_1 = 1, U_{n+1} = \left(\sqrt{U_n} + 1\right)^2$
Difference between square numbers:	3, 5, 7, 9, 11, …	$U_n = 2n + 1$	$U_1 = 3, U_{n+1} = U_n + 2$
Triangle numbers:	1, 3, 6, 10, 15, …	$U_n = \frac{n(n+1)}{2}$	$U_1 = 1, U_{n+1} = U_n + n + 1$
Cube numbers:	$1^3, 2^3, 3^3, 4^3, 5^3, \ldots$ ↓ ↓ ↓ ↓ ↓ 1, 8, 27, 64, 125, …	$U_n = n^3$	$U_1 = 1, U_{n+1} = \left(\sqrt[3]{U_n} + 1\right)^3$
Powers of 2:	$2^1, 2^2, 2^3, 2^4, 2^5, \ldots$ ↓ ↓ ↓ ↓ ↓ 2, 4, 8, 16, 32, …	$U_n = 2^n$	$U_1 = 2, U_{n+1} = 2U_n$
Doubling (start with 2)	2, 4, 8, 16, 32, …	$U_n = 2^n$	$U_1 = 2, U_{n+1} = 2U_n$
Trebling (start with 3)	3, 9, 27, 81, 243, …	$U_n = 3^n$	$U_1 = 3, U_{n+1} = 3U_n$
Powers of 10:	10, 100, 1000, …	$U_n = 10^n$	$U_1 = 10, U_{n+1} = 10U_n$
Fibonacci numbers:	1, 1, 2, 3, 5, 8, 13, 21, …		$U_1 = 1, U_n = U_{n-1} + U_{n-2}$
Fraction Sequence	$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots$	$U_n = \frac{1}{n+2}$	$U_1 = \frac{1}{3}, U_{n+1} = \frac{U_n}{U_n + 1}$
Alternating Sequence	1, −3, 9, −27, 81, …	$U_n = (-3)^{n-1}$	$U_1 = 1, U_{n+1} = -3U_n$
Reducing Sequence	92, 78, 64, 50, 36, …	$U_n = 106 - 14n$	$U_1 = 92, U_{n+1} = U_n - 14$

Notes:

Triangle numbers are found by adding the natual numbers in order, thus: 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, …

Adding consecutive triangle numbers makes a square number, thus:

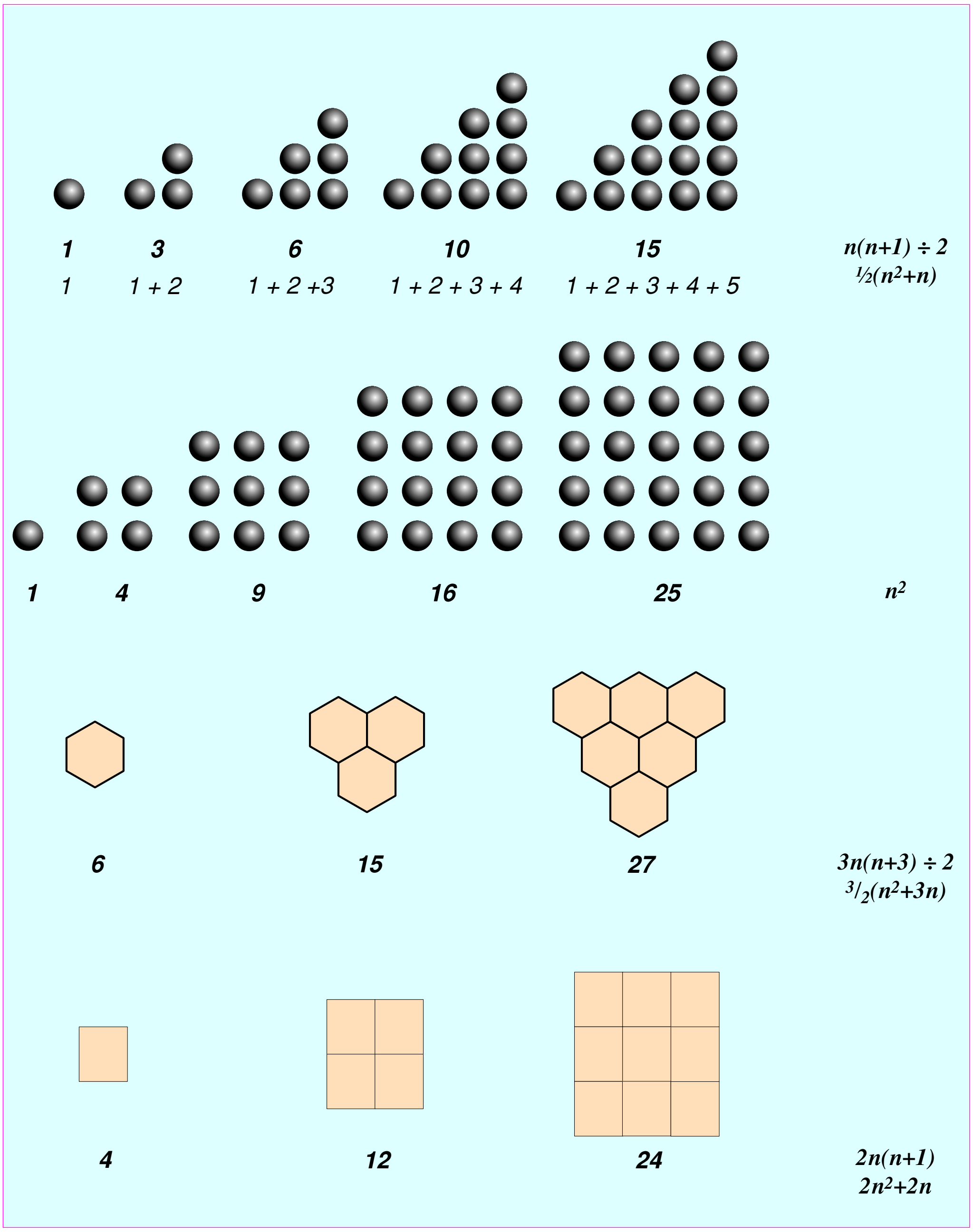
$3 + 6 = 9, \; 6 + 10 = 16, \; 10 + 15 = 25, \; \ldots$

Fibonacci numbers are formed by adding the last two number in the series together, thus:

$0 + 1 = 1, \; 1 + 1 = 2, \; 1 + 2 = 3, \; 2 + 3 = 5, \; \ldots$

27.11 Sequences in Patterns

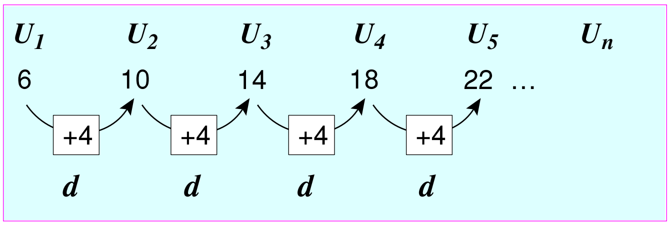
Here are some typical patterns that lead to problems on sequences:



28 • C2 • Arithmetic Progression (AP)

28.1 Intro to Arithmetic Progression

An **Arithmetic Progression** or sequence is based on a common difference between terms. Each term differs from its adjacent terms by a fixed amount. Arithmetic progression is sometimes abbreviated to AP.



Where U_1 is the first term, etc. and the n-th term is denoted by U_n. The common difference between terms is d.

The general definition of an AP can be given by the recurrence relation:

$$U_{n+1} = U_n + d \qquad \text{(where the integer } n \geqslant 1\text{)}$$

Also $$U_n = U_m + (n - m)d$$

Many series have the same recurrence relationship, so it is vitally important to state the first term.

The algebraic definition of an AP is:

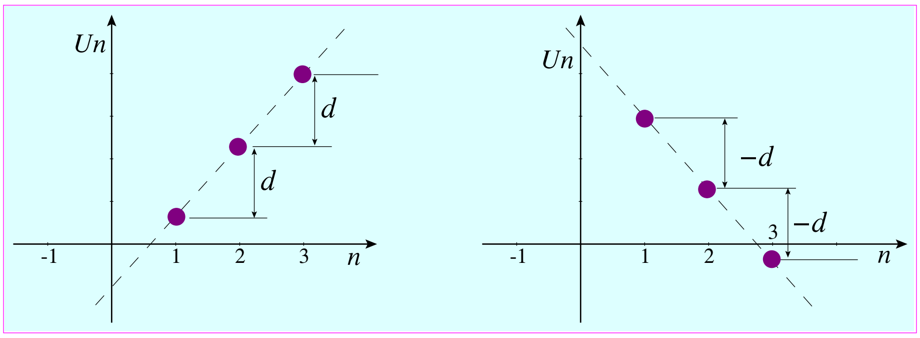
$$U_n = a + (n - 1)d$$

where a is the first term.

In general, an AP can be expressed as:

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ \dots\ a + (n - 1)d,\ \dots$$

The AP can be shown graphically thus:



Arithmetic Progression: a linear progression

All Arithmetic Progressions are linear.

In some exam questions, you may see an AP defined as:

$$U_n = an + b$$

$$U_n = an + U_0$$

where b represents the zeroth term U_0

28.1.1 Example:

A sequence is given by the equation $U_n = an + b$. Find a and b if $U_3 = 5$ & $U_8 = 20$.

$$U_3 = 3a + b = 5 \qquad (1)$$

$$U_8 = 8a + b = 20 \qquad (2)$$

From (1) $\qquad b = 5 - 3a \qquad (3)$

From (2) $\qquad b = 20 - 8a \qquad (4)$

Equate (3) & (4) $\qquad 5 - 3a = 20 - 8a$

$$8a - 3a = 20 - 5$$

$$5a = 15$$

$$a = 3$$

Sub a into (3) $\qquad b = 5 - 9$

$$b = -4$$

28.2 n-th Term of an Arithmetic Progression

Listing each term of an arithmetic progression:

$$U_1 = a$$

$$U_2 = a + d$$

$$U_3 = a + 2d$$

$$U_4 = a + 3d$$

$$\downarrow$$

$$U_n = a + (n - 1)d$$

This is the same as saying that we take $n - 1$ steps to get from U_1 to U_n.

Note that the expression for the n-th term is a linear expression in n. These sequences are usually derived from linear models.

28.3 The Sum of n Terms of an Arithmetic Progression

The sum of a **finite** arithmetic progression is called an **arithmetic series**.

The sum of n terms in an AP is simply n times the average of the first and last term.

Thus:

$$S_n = n\left[\frac{a+l}{2}\right] \quad or \quad S_n = \frac{n}{2}[a+l]$$

where $l = a + (n-1)d$

An alternative method is to make a series and then reverse the terms and add the two series together to give $2S_n$.

	U_1		U_2		U_{n-1}		U_n
S_n	a	$+$	$(a+d)$	$+ \ldots +$	$a+(n-2)d$	$+$	$a+(n-1)d$
S_n	$a+(n-1)d$	$+$	$a+(n-2)d$	$+ \ldots +$	$(a+d)$	$+$	a
$2S_n$	$2a+(n-1)d$	$+$	$2a+(n-1)d$	$+ \ldots +$	$2a+(n-1)d$	$+$	$2a+(n-1)d$

Therefore: $2S_n = n[2a+(n-1)d]$

Hence: $$S_n = \frac{n}{2}[2a+(n-1)d]$$

Note that:

$$S_1 = a$$

28.3.1 Example:

The sum of the first n natural numbers is:

$$1 + 2 + 3 + 4 + \ldots + n$$

where $a = 1,\ d = 1$

$$S_n = \frac{n}{2}(n+1) \quad or \quad \frac{1}{2}n(n+1)$$

and $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

In an AP, the sum of the terms that are equidistant from the beginning and end is always the same as the sum of the first and last terms.

Since a number of questions are based on manipulating the equation for S_n it is worth practising rewriting the equation in terms of n.

$$2S_n = n[2a+(n-1)d]$$

$$2S_n = 2an + dn(n-1)$$

$$2S_n = 2an + dn^2 - dn$$

$$2S_n = dn^2 + n(2a-d)$$

$$\therefore \quad dn^2 + n(2a-d) - 2S_n = 0$$

28.4 Sum to Infinity of an Arithmetic Progression

The sum to infinity of any progression depends on whether it is a convergent or a divergent series.
For an AP with a common difference d, if:

- d is positive, the sum will grow to $+\infty$
- d is negative, the sum will grow to $-\infty$

If the sum for an AP is multiplied out to remove the brackets, we have:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_n = \frac{n}{2}[2a + dn - d]$$

$$S_n = an + \frac{dn^2}{2} - \frac{dn}{2}$$

$$S_n = \frac{dn^2}{2} + n\left(a - \frac{d}{2}\right)$$

This is a quadratic equation and so as $n \rightarrow \infty$ then the sum $S_n \rightarrow \infty$.
Therefore, any AP is divergent, (except for the trivial case of $a = 0, \ \& \ d = 0$)

28.5 Sum of n Terms of an Arithmetic Progression: Proof

The proof goes like this:

$$S_n = a + (a+d) + (a+2d) + \ldots \ldots + (l-2d) + (l-d) + l \qquad (1)$$

Rewrite (1) $$S_n = l + (l-d) + (l-2d) + \ldots \ldots + (a+2d) + (a+d) + a \qquad (2)$$

Add (1) & (2) $$2S_n = (a+l) + (a+l) + (a+l) + \ldots \ldots + (a+l) + (a+l) + (a+l)$$

$$2S_n = n(a+l)$$

Since: $$l = a + (n-1)d$$

$$\therefore \quad S_n = \frac{n}{2}[a + a + (n-1)d]$$

$$= \frac{n}{2}[2a + (n-1)d]$$

28.6 Arithmetic Progression: Worked Examples

28.6.1 Example:

1 A 24m metal rod has been split into a number of different lengths forming an AP. The first piece is 0·4m long and the last piece is 3.6m long. Find the total number of pieces.

Solution:

$$S_n = n\left[\frac{a+l}{2}\right]$$

$$\therefore \quad n = \frac{2S_n}{a+l}$$

$$n = \frac{2 \times 24}{0{\cdot}4 + 3{\cdot}6} = \frac{48}{4}$$

$$n = 12$$

2 An arithmetic progression has a first term of 1, and a common difference of 4. The sum of the first n terms is 3160.
Show that $2n^2 - n - 3160 = 0$ and find the value of n.

Solution:

$$2S_n = n[2a + (n - 1)d]$$
$$2 \times 3160 = n[2 + (n - 1)4]$$
$$2 \times 3160 = 2n + 4n(n - 1)$$
$$3160 = n + 2n(n - 1)$$
$$3160 = 2n^2 - n$$
$$2n^2 - n - 3160 = 0$$

Find n by factorising:

3160×2
6320
40×158
80×79

$$(n + 79/2)(n - 80/2) = 0$$
$$(2n + 79)(n - 40) = 0$$
$$n = 40 \text{ (ignore the –ve value)}$$

3 The sum of the first 31 terms of an AP is 1302. Show that $a + 15d = 42$.
The sum of the 2nd and 9th terms is 21. Find a and d.

Solution:

$$2S_n = n[2a + (n - 1)d]$$
$$2 \times 1302 = 31[2a + (31 - 1)d]$$
$$\frac{2 \times 1302}{31} = 2a + 30d$$
$$42 = a + 15d \qquad (1)$$

$$U_2 = a + (2 - 1)d = a + d$$
$$U_9 = a + (9 - 1)d = a + 8d$$
$$\therefore \quad 21 = a + d + a + 8d$$
$$21 = 2a + 9d \qquad (2)$$

$$42 = a + 15d$$
$$84 = 2a + 30d \qquad \text{From (1)}$$
$$21 = 2a + 9d \qquad \text{From (2)}$$
$$63 = 21d \qquad \text{Subtract}$$
$$\therefore \quad d = 3$$

$$21 = 2a + 9 \times 3 \qquad \text{From (2)}$$
$$21 - 27 = 2a$$
$$a = -3$$

4 An AP has 200 terms with the first 4 terms as follows:

$49 + 55 + 61 + 67\ldots$

What is the sum of the **last** 100 terms?

Solution:

There are three ways to tackle this problem, noting that there is considerable room for confusion over the terms required to do the sum. The last 100 terms run from the 101st term to the 200th term. (It's the fence post problem!)

So find the value of the 101st and 200th terms and use either of the two formulae for the sum of terms. Use the 101st term as a in the formulae.

Alternatively, (method 3) find the sum of terms, S_{200}, and subtract the sum of terms to 100, S_{100}.

$$U_{200} = 49 + (200 - 1)6 = 1243$$
$$U_{101} = 49 + (101 - 1)6 = 649$$

Method 1:

$$S_n = n\left[\frac{a + l}{2}\right]$$
$$S_{101 \to 200} = 100\left[\frac{649 + 1243}{2}\right]$$
$$= 94600$$

Method 2:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$
$$S_{101 \to 200} = 50[2 \times 649 + (99)6]$$
$$= 94600$$

Method 3:

$$S_{200} = \frac{200}{2}[2 \times 49 + (199)6]$$
$$= 129200$$
$$S_{100} = \frac{100}{2}[2 \times 49 + (99)6]$$
$$= 34600$$
$$S_{200} - S_{100} = 129200 - 34600$$
$$= 94600$$

5 A sequence is given as $2,\ 6,\ 10,\ 14\ldots$ How many terms are required for the sum to exceed 162.

Solution:

$$S_n = \frac{n}{2}[4 + 4(n - 1)] = 2n^2$$
$$2n^2 = 162$$
$$n^2 = 81$$
$$n = 9$$

6 The sum of the first n terms of a sequence is given by $S_n = 3n^2 + n$.
Prove that the sequence is an AP, and find a and d.

Solution:

The method used to prove this is an AP, is to compare the given equation to the standard form of an AP, by re-arranging the equation.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_n = 3n^2 + n$$

$$S_n = n(3n + 1)$$

$$S_n = \frac{n}{2}(6n + 2)$$

$$S_n = \frac{n}{2}(6n - 6 + 6 + 2)$$

$$S_n = \frac{n}{2}(6(n-1) + 8)$$

$$S_n = \frac{n}{2}[2 \times 4 + (n-1)6]$$

where $a = 4, \quad d = 6$

Alternatively, prove by comparing our sequence to the sum of the nth term:

$$U_n = S_n - S_{n-1}$$

$$= 3n^2 + n - [3(n-1)^2 + n - 1]$$

$$= 3n^2 + n - [3n^2 - 5n + 2]$$

$$= 3n^2 + n - 3n^2 + 5n - 2$$

$$= 6n - 2$$

$$= 6(n-1) + 4$$

$$\equiv a + (n-1)d$$

where $a = 4, \quad d = 6$

7 Determine if the number 33 is a term in the sequence defined by $U_n = 5n - 2$.

Solution:

$$33 = 5n - 2$$

$$5n = 35$$

$$n = 6$$

Therefore, 33 is the 6th term in the sequence (as it is an integer value)

8 Determine if the number 100 is a term in the sequence defined by $U_n = 5n - 2$.

Solution:

$$100 = 5n - 2$$

$$5n = 102$$

$$n = 20{\cdot}4$$

Therefore, 100 is not a term in the sequence, it is between the 20th and 21st terms.

9 An AP has the terms $U_1, U_2, U_3, \ldots$ where $S_1 = 6, S_2 = 17$.
State the value of U_1, and calculate the common difference, d, and the value of U_5.

Solution:

$(i) \qquad U_1 = S_1 = 6$

$(ii) \qquad U_2 = S_2 - U_1$

$U_2 = 17 - 6 = 11$

$d = U_2 - U_1$

$d = 11 - 6 = 5$

$(ii) \qquad U_5 = U_1 + (n-1)d$

$U_5 = 6 + (5-1)5$

$U_5 = 26$

10 The ratio of the sixth and sixteenth terms of an AP is 4:9. The product of the first and third terms is 135. Assuming that the AP is positive, find the sum of the first 100 terms.

Solution:

Step (i) $\qquad \dfrac{U_6}{U_{16}} = \dfrac{4}{9}$

$\therefore \qquad 9U_6 = 4U_{16}$

but $\qquad U_6 = a + (n-1)d = a + 5d$

$U_{16} = a + (n-1)d = a + 15d$

$\therefore \qquad 9(a + 5d) = 4(a + 15d)$

$9a + 45d = 4a + 60d$

$5a = 15d$

$a = 3d \qquad (1)$

step (ii) $\qquad U_1 U_3 = 135$

$a(a + 2d) = 135$

$a^2 + 2ad = 135$

but from (1) $\qquad (3d)^2 + 6d^2 = 135$

$9d^2 + 6d^2 = 135$

$15d^2 = 135$

$d^2 = 9$

$d = \pm 3 \qquad$ hence $a = 9$

Using the +ve value for d, (a positive AP given)

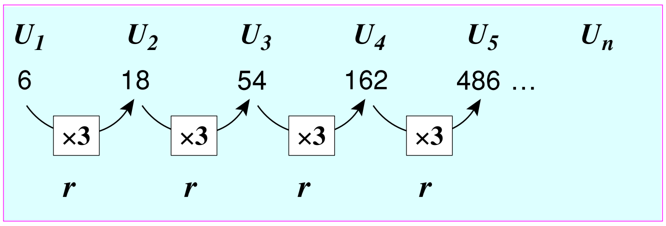
$$S_{100} = \frac{100}{2}\left[18 + (99)3\right] = 99225$$

EIFBL

29 • C2 • Geometric Progression (GP)

29.1 Geometric Progression (GP) Intro

An **Geometric Progression** or sequence is based on a common ratio between terms. Each term is found by multiplying the previous term by a constant, r. Sometimes abbreviated to GP.



Where U_1 is the first term, etc. and the n-th term is denoted by U_n. The common ratio between terms is r.

The general definition of an GP can be given by the recurrence relation:

$$U_{n+1} = U_n r \qquad \text{(where the integer } n \geqslant 1, \; r \neq 0)$$

Many series have the same recurrence relationship, so it is important to state the first term.

The algebraic definition is:

$$U_n = ar^{(n-1)}$$

where a is the first term.

In general, an GP can be expressed as:

$$a, \; ar, \; ar^2, \; ar^3, \; \dots, \; ar^{(n-1)}$$

29.2 The n-th Term of a Geometric Progression

Listing each term of an geometric sequence:

$$U_1 = a$$

$$U_2 = ar$$

$$U_3 = ar^2$$

$$U_4 = ar^3$$

$$\downarrow$$

$$U_n = ar^{(n-1)}$$

This is the same as saying that we take $n - 1$ steps to get from U_1 to U_n.

Note that the expression for the n-th term is an exponential expression in n. These sequences are usually derived from exponential models, such as population growth or compound interest models. It also means the use of logs on the exam paper.

The nth term can also be expressed as:

$$U_n = ar^n \times r^{-1} = \frac{a}{r} r^n$$

But $\qquad \dfrac{a}{r} = U_0$

$$U_n = U_0 r^n$$

$$U_n = ar^{(n-1)}$$

$$U_n = U_0 r^n$$

29.3 The Sum of a Geometric Progression

Adding the terms of a GP gives:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \qquad (1)$$

Multiply (1) by r, and note how pairs of terms match up.

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \qquad (2)$$

$$S_n - rS_n = a - ar^n \qquad \text{Subtracting (1) – (2)}$$

$$S_n(1 - r) = a(1 - r^n)$$

$$\therefore \quad S_n = \frac{a(1 - r^n)}{(1 - r)}$$

For $r > 1$ multiplying top and bottom by –1, gives a more convenient formula, (top & bottom are +ve)

$$S_n = \frac{a(r^n - 1)}{(r - 1)} \qquad |r| > 1 \Leftrightarrow r < -1 \text{ or } r > 1$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \qquad |r| < 1 \Leftrightarrow -1 < r < 1$$

$$S_n = \frac{a(r^n - 1)}{(r - 1)} \qquad |r| > 1 \Leftrightarrow r < -1 \text{ or } r > 1$$

Either of these formulae will work in finding the sum, but it is easier to use them as indicated above.

Note: The formulae above works well for large values of n. For small values of n (say 3 or less) then it is best to find the first few terms and just add them up!

29.3.1 Example:

1 Sum the first 25 terms of the series 5, –7·5, 11·25, –16·875…

Solution:

We find that $a = 5$, $r = -1{\cdot}5$, $n = 25$

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

$$S_n = \frac{5(1 - (-1{\cdot}5)^{25})}{1 - (-1{\cdot}5)}$$

$$= \frac{5(1 - (-25251{\cdot}168))}{2{\cdot}5}$$

$$= 2.5(1 + 25251{\cdot}168)$$

$$= 2.5 \times 25252{\cdot}168$$

$$= 63130{\cdot}42$$

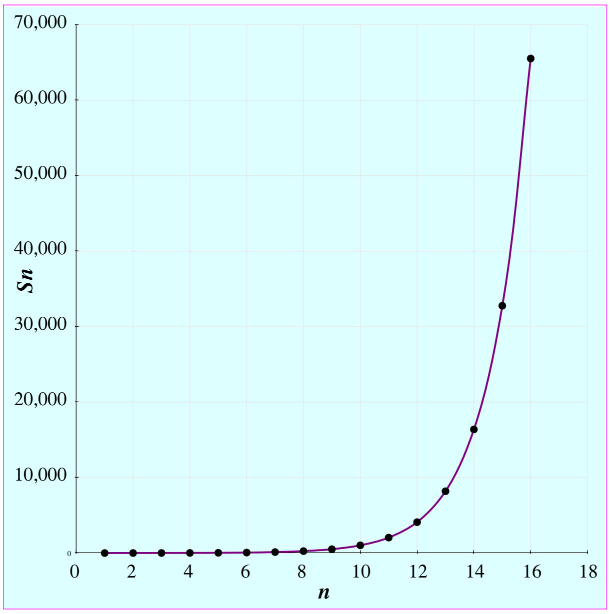
29.4 Divergent Geometric Progressions

A geometric progression can either be a divergent or a convergent series.

For any general GP, $a + ar + ar^2 + ar^3, \ldots$, if $r > 1$ or $r < -1$, the terms in the series become larger and larger and so the GP is divergent.

A commonly quoted example of a divergent geometric progression concerns a chess board, in which 1p is placed on the first square, 2p on the second, 4p on the third and so on. What is the total amount of money placed on the board?

This is a GP with a common ratio of 2 and a start value of 1. The progression is finite and ends at square 64. The graph below illustrates just the first 16 squares.



$$S_n = \frac{a(r^n - 1)}{(r - 1)} = \frac{1(2^{64} - 1)}{(2 - 1)} = 1{\cdot}85 \times 10^{19} pence$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)} = \frac{1(1 - 2^{64})}{(1 - 2)} = 1{\cdot}85 \times 10^{19} pence$$

29.5 Convergent Geometric Progressions

For any general GP, $a + ar + ar^2 + ar^3, \ldots$, if the common ratio is between -1 and $+1$, the terms in the series become smaller and smaller and so the GP is convergent.

A good example of a convergent series is to take a piece of string, length L, and cut it in half. Keep one half and cut it in half, keep one half and cut it in half, and so on…

The series, in theory, can go on for ever and will look like:

$$\frac{L}{2} + \frac{L}{4} + \frac{L}{8} + \frac{L}{16} + \ldots$$

This can be expressed in terms of n:

$$\frac{L}{2} + \frac{L}{2^2} + \frac{L}{2^3} + \frac{L}{2^4} + \ldots \frac{L}{2^n}$$

As $n \to \infty$ then the sum of all the cuts will get close to the original length L:

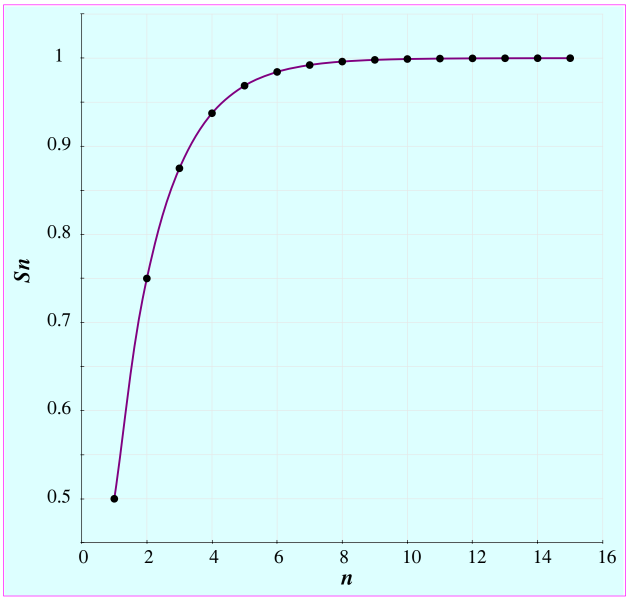
$$\frac{L}{2} + \frac{L}{2^2} + \frac{L}{2^3} + \frac{L}{2^4} + \ldots \frac{L}{2^n} \to L$$

$$\lim_{n \to \infty} \left[\frac{L}{2} + \frac{L}{2^2} + \frac{L}{2^3} + \frac{L}{2^4} + \ldots \frac{L}{2^n} \right] = L$$

This can be simplified by saying:

$$S_\infty = \lim_{n \to \infty} [S_n]$$

The graph below shows how the sum tends to 1, when $L = 1$.



29.6 Oscillating Geometric Progressions

For a GP, if the common ratio equals $+1$, the first term term is repeated again and again.

If the ratio equals -1, the GP oscillates between $+a$ and $-a$.

Clearly, neither of these GPs converge.

29.7 Sum to Infinity of a Geometric Progression

Any GP that has an infinite number of terms, but has a finite sum is said to be convergent.
So the sum to infinity only has a meaning if the GP is a convergent series.
The sum to infinity of a divergent series is undefined.
The general formula for the sum of a GP is:

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \qquad \text{which can be written as:} \qquad S_n = \left(\frac{a}{1 - r}\right) - \left(\frac{a}{1 - r}\right)r^n$$

However, if r is small and between $-1 < r < 1$, $(r \neq 0)$ then the term r^n tends to 0 as $n \to \infty$
Mathematically this is written:

$$\text{if } |r| < 1, \text{ then } \lim_{n \to \infty} r^n = 0$$

and the sum to infinity becomes:

$$S_\infty = \frac{a}{(1 - r)} \qquad |r| < 1$$

The GP is said to converge to the finite sum of S_∞

29.8 Geometric Progressions: Worked Examples

1 The first term of a GP is $8\sqrt{3}$ and has a second term of 12.
a) Show that the common ration is $\sqrt{3}/2$.
b) Find the 6-th term
c) Show that the sum to infinity is $16\left(2\sqrt{3} + 3\right)$

Solution:

a) Now

$$U_{n+1} = U_n r \qquad \therefore r = \frac{U_{n+1}}{U_n}$$

$$r = \frac{12}{8\sqrt{3}} = \frac{3}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{2 \times 3} = \frac{\sqrt{3}}{2}$$

b)

$$U_n = ar^{(n-1)} \qquad \Rightarrow \qquad 8\sqrt{3} \times \left(\frac{\sqrt{3}}{2}\right)^5$$

$$= \frac{2^3\sqrt{3}\left(\sqrt{3}\right)^5}{2^5} = \frac{\sqrt{3} \times 9\sqrt{3}}{2^2}$$

$$= \frac{27}{4}$$

c)

$$S_\infty = \frac{a}{(1 - r)} = \frac{8\sqrt{3}}{1 - \frac{\sqrt{3}}{2}} = \frac{8\sqrt{3}}{\frac{2 - \sqrt{3}}{2}}$$

$$= \frac{8\sqrt{3}}{2 - \sqrt{3}} \times 2 = \frac{16\sqrt{3}}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= \frac{16\sqrt{3}(2 + \sqrt{3})}{4 - 3} = 16\sqrt{3}(2 + \sqrt{3})$$

$$= 32\sqrt{3} + 16 \times 3 = 16\left(2\sqrt{3} + \sqrt{3}\right)$$

2 A sequence is defined by :

$$U_1 = 2 \quad \text{and} \quad U_{n+1} = 1 - U_n \quad \text{for } n \geqslant 0$$

Write down the values of U_2, U_3, U_4, U_5

Find:

$$\sum_{n=1}^{100} U_n$$

Solution:

$$U_2 = 1 - 2 = -1$$

$$U_3 = 1 - (-1) = 2$$

$$U_4 = 1 - 2 = -1$$

$$U_5 = 1 - (-1) = 2$$

Sequence is, therefore, an alternating series: 2, −1, 2, −1, 2, −1

The sum to $n = 100$ can be found by considering that there are 50 terms of '2' and 50 terms of '−1'

Hence:

$$\sum_{n=1}^{100} U_n = 100 - 50 = 50$$

Extension work:

This gives an opportunity to explore an alternating series. An alternating series is one in which the signs change after each term. The sequence also oscillates between two numbers of 2 and −1, with a mid-point of 0·5.

Now consider the alternate form of the sequence. This can be written as:

$$U_n = 0{\cdot}5 - 1{\cdot}5\,(-1)^n$$

Note the use of $(-1)^n$ in order the make the sign change. An important tool in mathematics.

The sum of the terms can be written in Sigma notation as:

$$\sum_{n=1}^{100} U_n = \sum_{n=1}^{100} 0{\cdot}5 - \sum_{n=1}^{100} 1{\cdot}5\,(-1)^n$$

$$= 50 - \sum_{n=1}^{100} 1{\cdot}5\,(-1)^n$$

$$= 50 - 1{\cdot}5 \sum_{n=1}^{100} (-1)^n$$

The second term can be considered as a GP with a common ratio of −1.

$$\sum_{n=1}^{100} U_n = 50 - 1{\cdot}5\, S_{100}$$

$$S_{100} = \frac{-100\,[1 - (-1)^{100}]}{(1 - (-1))} = 0$$

$$\sum_{n=1}^{100} U_n = 50 - 1{\cdot}5 \sum_{n=1}^{100} (-1)^{100}$$

$$= 50 - 0$$

$$= 50$$

3 As $n \to \infty$, $U_{n+1} \to L$, $U_n \to L$

$$\therefore \quad L = pL + q$$

$$L - pL = q$$

$$L(1 - p) = q$$

$$L = \frac{q}{1 - p}$$

4 Water is weekly pumped from a well, with 10,000 gallons being extracted in the first week. The common ratio is given as 0·85.
(*i*) Calculated the amount of water extracted at the end of week 4.
(*ii*) Find how long it takes for the amount of water to be extracted per week to fall to below 100 gallons, rounding up to the nearest week.
(*iii*) Find the total water extracted up to and including the week found in (*ii*) above, to 4sf.

Solution:

(*i*)

$$U_n = ar^{(n-1)}$$

$$U_4 = 10000(0{\cdot}85)^3$$

$$U_4 = 6141{\cdot}25 \textit{ gallons}$$

(*ii*)

$$ar^{(n-1)} < 100$$

$$10000\left[0{\cdot}85^{(n-1)}\right] < 100$$

$$0{\cdot}85^{(n-1)} < \frac{100}{10000}$$

$$0{\cdot}85^{(n-1)} < 0{\cdot}01$$

$$(n - 1)\,ln\,0{\cdot}85 < ln\,0{\cdot}01$$

$$(n - 1) > \frac{ln\,0{\cdot}01}{ln\,0{\cdot}85}$$

$$n > 1 + \frac{ln\,0{\cdot}01}{ln\,0{\cdot}85}$$

$$n > 29{\cdot}34$$

$$n = 30 \textit{ weeks}$$

(*iii*)

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

$$S_{30} = \frac{10000(1 - 0{\cdot}85^{30})}{(1 - 0{\cdot}85)}$$

$$S_{30} = 66157{\cdot}95$$

$$S_{30} = 66160 \textit{ gallons } (4sf)$$

OECFRL

5 A GP has the first term $a = 15$, and the second term of 14·1

(*i*) Show that $S_\infty = 250$

(*ii*) The sum of the first n terms is greater than 249. Show that $0{\cdot}94^n < 0{\cdot}004$

(*iii*) Find the smallest value of n to satisfy the inequality in (*ii*)

Solution:

(*i*)

$$r = \frac{14{\cdot}1}{15} = 0{\cdot}94$$

$$S_\infty = \frac{a}{(1 - r)} \qquad |r| < 1$$

$$S_\infty = \frac{15}{(1 - 0{\cdot}94)}$$

$$= \frac{15}{0{\cdot}06} = 250$$

(*ii*)

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

$$\frac{15(1 - 0{\cdot}94^n)}{(1 - 0{\cdot}94)} > 249$$

$$15(1 - 0{\cdot}94^n) > 249 \times 0{\cdot}06$$

$$1 - 0{\cdot}94^n > \frac{249 \times 0{\cdot}06}{15}$$

$$1 - 0{\cdot}94^n > 0{\cdot}996$$

$$-0{\cdot}94^n > -0{\cdot}004$$

$$\therefore \quad 0{\cdot}94^n < 0{\cdot}004 \qquad \text{(note change of inequality)}$$

(*ii*)

$$n \ln 0{\cdot}94 < \ln 0{\cdot}004$$

$$n > \frac{\ln 0{\cdot}004}{\ln 0{\cdot}94} \approx 89{\cdot}24 \ (2dp) \qquad \text{(note change of inequality)}$$

$\therefore$ Least value of $n = 90$

Note the trap and the reason for the change in inequality:

$$n \ln 0{\cdot}94 < \ln 0{\cdot}004$$

$$-0{\cdot}062n < -5{\cdot}52$$

$$0{\cdot}062n > 5{\cdot}52$$

$$n > \frac{5{\cdot}52}{0{\cdot}062}$$

As n increases, so does S_n until the limit is reached when $S_\infty = 250$. The value of $n \approx 89{\cdot}24$ represents the point at which the curve crosses a value of $S_n = 249$. Since n is an integer value, the smallest value to satisfy the inequality is 90.

Drawing a graph of $S_n : v : n$ will illustrate this.

OECFRL

6 The difference between the 4th and the 1st term of a GP is 3 times the difference between the 2nd and the 1st term. Find the possible values of the common ratio.

Solution:

$$U_n = ar^{(n-1)}$$

$$U_4 - U_1 = 3(U_2 - U_1)$$

$$ar^3 - a = 3(ar - a)$$

$$r^3 - 1 = 3(r - 1)$$

$$r^3 - 3r + 2 = 0$$

$$(r + 2)(r - 1)(r - 1) = 0$$

$$r = -2, \text{ or } 1$$

OECFRL

7 You decide to save some money for a rainy day, by joining a monthly savings scheme. The initial deposit is £100, and after 360 months the final payment will be £2110. What is the total paid into the scheme assuming the monthly payments increase by an inflation adjusted amount every month.

Solution:

From the question, $a = 100$, and $U_{360} = 2110$

$$U_n = ar^{(n-1)}$$

$$2110 = 100r^{(359)}$$

$$r^{(359)} = \frac{2110}{100} = 21{\cdot}1$$

$$r = \sqrt[359]{21{\cdot}1}$$

$$r = 1{\cdot}00853$$ (This represents an inflation of approx 0·8% per month)

$$S_n = \frac{a(r^n - 1)}{(r - 1)} \qquad r > 1$$

$$S_{360} = \frac{100(1{\cdot}00853^{360} - 1)}{(1{\cdot}00853 - 1)}$$

$$= \frac{100(21{\cdot}279 - 1)}{0{\cdot}00853} = \frac{2027{\cdot}99}{0{\cdot}00853}$$

$$= £237{,}749{\cdot}72$$ saved over 360 months (30 years)

OECFRL

8 This question is dressed up to hide the fact that it is a question on GPs.

A starship uses 2.5 tonnes of interstellar dust to make one standard hyperspace jump. A fault in the power crystals means that each subsequent jump consumes 3% more dust than the previous jump.

a) Calculate the amount of dust used in the 6th jump.

b) The engine fault has restricted the storage of dust to 206 tonnes. Show that $1.03^n \leqslant 3{\cdot}472$, where n represents the number of jumps.

c) Using logs, find the largest number of standard jumps that can be made with this restricted mass of dust.

Solution:

(a)

$$U_n = ar^{(n-1)}$$

$$a = 2.5 \qquad \& \qquad r = 1{\cdot}03 \qquad (3\%)$$

$$U_6 = 2.5 \times 1{\cdot}03^{(5)}$$

$$= 2{\cdot}898 \text{ tonnes (4sf)}$$

(b)

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

$$\therefore \quad \frac{2{\cdot}5(1{\cdot}03^n - 1)}{(1{\cdot}03 - 1)} \leqslant 206 \qquad \text{(number of jumps is an integer)}$$

$$2{\cdot}5(1{\cdot}03^n - 1) \leqslant 206 \times 0{\cdot}03$$

$$1{\cdot}03^n - 1 \leqslant \frac{206 \times 0{\cdot}03}{2{\cdot}5}$$

$$1{\cdot}03^n - 1 \leqslant 2{\cdot}472$$

$$\therefore \quad 1{\cdot}03^n \leqslant 3{\cdot}472$$

(b)

$$n \ln 1{\cdot}03 \leqslant \ln 3{\cdot}472$$

$$n \leqslant \frac{\ln 3{\cdot}472}{\ln 1{\cdot}03} \approx 42{\cdot}11\ (2dp)$$

Least value of $n = 42$

OECFRL

9 Given that $U_{n+1} = 0{\cdot}5\,U_n + 25$ and that the limit of U_n as $n \to \infty$ is U_L, form an equation for U_L and find its value.

Solution:

At the limit: $U_{n+1} = U_L \qquad \& \qquad U_n = U_L$

$$\therefore \quad U_L = 0{\cdot}5\,U_L + 25$$

$$0{\cdot}5\,U_L = 25$$

$$U_L = 50$$

10 A GP has a common ratio of 0·7, and a first term of 25.
Find the least value of n such that the n^{th} term is less than one.

Solution:

$$U_n = ar^{(n-1)}$$

$$U_n = 25 \times 0{\cdot}7^{(n-1)} < 1$$

$$0{\cdot}7^{(n-1)} < \frac{1}{25}$$

$$log_{0\cdot7}\frac{1}{25} < n - 1$$

$$9{\cdot}025 < n - 1$$

$$10{\cdot}025 < n$$

But n is an integer value, and the least value for n is 11
Test for correct solution:

$$25 \times 0{\cdot}7^{(11-1)} = 0{\cdot}706$$

$$25 \times 0{\cdot}7^{(10-1)} = 1{\cdot}0088$$

29.9 Heinous Howlers for AP & GP

- Don't mix up the AP & GP formulas, especially for the sum of terms.
- In quoting the AP formula $S_n = \frac{n}{2}[a + l]$ ensure you know what the l stands for. It is not 1!!!!
  l is the last term of an arithmetic sequence and $l = a + (n - 1)d$.

  Also make sure you know that the a stands for the first term.
- If the terms of an AP are decreasing, the common difference must be **negative**.
-

- In a GP, the nth term is given by $U_n = ar^{(n-1)}$. Do not use $(ar)^{n-1}$
-
-

29.10 AP & GP Topic Digest

Arithmetic Progression (AP)	Geometric Progression (GP)
First term: a	First term: a
Common difference: d	Common ratio: r
n-th term:	n-th term:
$U_n = a + (n-1)d$	$U_n = ar^{n-1}$
$U_n = nd + U_0$	$U_n = U_0 r^n$
$U_n - U_{n-1} = d$	$\frac{U_n}{U_{n-1}} = r$
Sum of first n terms:	Sum of first n terms:
$S_n = \frac{n}{2}[2a + (n-1)d]$	$S_n = \frac{a(r^n - 1)}{(r-1)} \quad \lvert r \rvert > 1$
$S_n = \frac{n}{2}[a + l]$ where l = last term	$S_n = \frac{a(1-r^n)}{(1-r)} \quad \lvert r \rvert < 1$
and $l = a + (n-1)d$	
Sum to infinity:	Sum to infinity:
N/A	$S_\infty = \frac{a}{(1-r)}$ if $\lvert r \rvert < 1$
Sum of next n terms:	
$U_{n+1} + U_{n+2} + \ldots \ldots + U_{2n} = S_{2n} - S_n$	
$S_n - S_{n-1} = U_n$	

$$\sum_{r=1}^{n} (a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$$

$$\sum_{r=1}^{k} a_r + \sum_{r=k+1}^{n} a_r = \sum_{r=1}^{n} a_r \qquad r < k < n$$

$$\sum_{r=1}^{n} k a_r = k \sum_{r=1}^{n} a_r$$

$$\sum_{1}^{n} c = nc \qquad \text{where } c \text{ is a constant}$$

$$\sum_{1}^{n} 1 = n$$

No. of terms in sum $\displaystyle\sum_{r=m}^{n} \Rightarrow n - m + 1$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Formula for the sum of the first n natural numbers

30 • C2 • Binomial Theorem

30.1 Binomials and their Powers

A binomial is simply a polynomial of two terms with the general form $(a + b)$, e.g. $(x + y)$, $(\sqrt{x} + 2y)$ or $(x^2 - \sqrt{2})$. A binomial **expansion** is about raising a binomial to a power and expanding out the expression.

Look at the following expansions for the general form $(a + b)^n$:

(Only +ve integers of n are considered in C2, other rational values are considered in C4).

$$(a + b)^0 = 1$$

$$(a + b)^1 = 1a + 1b$$

$$(a + b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a + b)^3 = 1a^3 + 3a^2b + 2ab^2 + 1b^3$$

$$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

From the expansions above, note the following properties:

- Number of terms in the expansion of $(a + b)^n$ is $n + 1$
- The first term is always a^n and the last term b^n
- The power or exponent of a starts at a^n and decreases by 1 in each term to a^0
- The power or exponent of b starts at b^0 and increases by 1 in each term to b^n
- In the k^{th} term: a will have a power of $(n - k + 1)$ and b a power of $(k - 1)$
- The sum of the powers of each term equals n
- The coefficients of each term follow a particular pattern that we know as Pascal's Triangle
- If the expansion is $(a - b)^n$ then the signs in the expansion alternate
- Each expansion is finite

30.2 Pascal's Triangle

Pascal's triangle is named after the mathematician Blaise Pascal who wrote about the unending triangle in 1653. The triangle was well know to mathematicians as early as 1100, but Pascal's name is associated with it because he published a study which summed up all that was known about it at the time.

Pascal's triangle gives us the binomial coefficient of each term in the expanded binomial. The sequence can be built up by adding the numbers in the row just above each given position. (Numbers outside the triangle are zero)

Some features of note:

- Row numbering starts at 0, (the power n) and row numbers then match the 2nd number in each row
- If each row is added up, a new sequence is created, (the powers of 2).

Row Number (n)		Row Sum
0	**1**	1
1	**1 1**	2
2	**1 2 1**	4
3	**1 3 3 1**	8
4	**1 4 6 4 1**	16
5	**1 5 10 10 5 1**	32
6	**1 6 15 20 15 6 1**	64
7	**1 7 21 35 35 21 7 1**	128
:	: : : : : : : :	:

30.2.1 Example:

1 Use Pascal's triangle to expand $(3 + 2x)^5$

Solution:

Use row 5 from Pascal's triangle to find the coefficients, which are 1, 5, 10, 10, 5, 1.

Set up a table to help calculate the terms:

Coefficient	*1st term*	*2nd term*	*Calculation*	*Total*
1	3^5	1	$1 \times 243 \times 1$	243
5	3^4	$(2x)$	$5 \times 81 \times 2x$	$810x$
10	3^3	$(2x)^2$	$10 \times 27 \times 4x^2$	$1080x^2$
10	3^2	$(2x)^3$	$10 \times 9 \times 8x^3$	$720x^3$
5	3^1	$(2x)^4$	$5 \times 3 \times 16x^4$	$240x^4$
1	1	$(2x)^5$	$1 \times 1 \times 32x^5$	$32x^5$

$$\therefore \qquad (3 + 2x)^5 = 243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32^5$$

2 Use Pascal's triangle to expand $\left(3x - \frac{1}{x}\right)^4$

Solution:

Use row 4 from Pascal's triangle to find the coefficients, which are 1, 4, 6, 4, 1.

Set up a table to help calculate the terms:

Coefficient	*1st term*	*2nd term*	*Calculation*	*Total*
1	$(3x)^4$	1	$1 \times 81x^4 \times 1$	$81x^4$
4	$(3x)^3$	$\left(-\frac{1}{x}\right)^1$	$4 \times 27x^3 \times \left(-\frac{1}{x}\right)$	$-108x^2$
6	$(3x)^2$	$\left(\frac{1}{x}\right)^2$	$6 \times 9x^2 \times \left(-\frac{1}{x}\right)^2$	54
4	$(3x)^1$	$\left(-\frac{1}{x}\right)^3$	$4 \times 3x \times \left(-\frac{1}{x}\right)^3$	$-12\frac{1}{x^2}$
1	1	$\left(\frac{1}{x}\right)^4$	$1 \times 1 \times \left(-\frac{1}{x}\right)^4$	$\frac{1}{x^4}$

$$\therefore \qquad \left(3x - \frac{1}{x}\right)^4 = 81x^4 - 108x^2 + 54 - \frac{12}{x^2} + \frac{1}{x^4}$$

Note the alternating signs in the expansion.

3 Use Pascal's triangle to find the coefficient of the x^4 term of the binomial $(5x + 2)^6$

Solution:

Use row 6 from Pascal's triangle to find the coefficients, which are 1, 6, 15, 20, 15, 6, 1.

Set up a table to help calculate the terms:

Coefficient	*1st term*	*2nd term*	*Calculation*	*Total*
1	$(5x)^6$	1		
6	$(5x)^5$	2		
15	$(5x)^4$	2^2	$15 \times 625x^4 \times 4$	$37500x^4$
20	$(5x)^3$			
15				

Coefficient of the x^4 term $= 37500$

30.3 Factorials & Combinations

Pascal's Triangle is fine for working out small powers of a binomial, but is very tedious for higher powers.

An alternative method of expanding a binomial is to use the binomial theorem, which involves the use of factorials, combinations and permutations.

30.3.1 Factorials

A factorial is a simple and short way to write down the product of all the positive integers from 1 to n thus:

$$n! = n(n-1)(n-2)\ldots(3)(2)(1) \qquad \text{(called } n \text{ factorial)}$$

e.g. $\quad 5! = 5 \times 4 \times 3 \times 2 \times 1 \qquad$ *or* $\qquad 5! = 5.4.3.2.1$

where, by definition:

$$0! = 1$$

Recursively this can be written as:

$$N! = N \times (N-1)!$$

30.3.2 Combinations & Permutations

Digressing into statistics for a moment, a permutation is an arrangement, whereas a combination is a selection.

A permutation is an arrangement of things, without repetition, and taking into account the order of things. It is always a whole number.

The number of permutations of n things, taken r at a time is given by:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

A combination is an selection of things, without repetition, but where the order is not important.

The number of combinations of n things, taken r at a time is given by:

$$\text{Now:} \qquad {^{n}C_{r}} \times r! = {^{n}P_{r}}$$

$$\therefore \quad ^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

This formula can be used to find the coefficient of each term in the binomial expansion.

Note that combinations are symmetric so that $^{12}C_{5} = {^{12}C_{7}}$. So choose the easiest one to calculate if doing it by hand (or use a calculator).

$$^{n}C_{r} = {^{n}C_{n-r}}$$

30.3.3 Alternative Symbology

Sometimes, alternative symbology is used for combinations:

$$^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

We say "n choose r", which is the number of ways of choosing r things from a pool of n items, where order is not important.

30.4 Binomial Coefficients

Calculating the binomial coefficient is a major part of using the binomial theorem. Using the combination format, Pascal's triangle can be redrawn thus:

Row Number (n)		Row Sum
0	1	1
1	${}^1C_0$ ${}^1C_1$	2
2	${}^2C_0$ ${}^2C_1$ ${}^2C_2$	4
3	${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$	8
4	${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$	16
5	${}^5C_0$ ${}^5C_1$ ${}^5C_2$ ${}^5C_3$ ${}^5C_4$ ${}^5C_5$	32
6	${}^6C_0$ ${}^6C_1$ ${}^6C_2$ ${}^6C_3$ ${}^6C_4$ ${}^6C_5$ ${}^6C_6$	64
7	${}^7C_0$ ${}^7C_1$ ${}^7C_2$ ${}^7C_3$ ${}^7C_4$ ${}^7C_5$ ${}^7C_6$ ${}^7C_7$	128
:	: : : : : : :	:

Pascals Triangle with Combinations

These are easily calculated on a calculator by using the ${}^nC_r$ or ${}_nC_r$ button on the calculator.

Note that ${}^nC_0 = 1$; ${}^nC_1 = n$; ${}^nC_{n-1} = n$; ${}^nC_n = 1$

Recall that the counter r starts at zero. It can become confusing if care is not taken over the difference between the term number and the counter r. If the term number is k, then $r = k - 1$.

Redrawing and simplifying Pascal's triangle we have:

Row Number (n)		Row Sum
0	1	1
1	1 1	2
2	1 n 1	4
3	1 n n 1	8
4	1 n ${}^4C_2$ n 1	16
5	1 n ${}^5C_2$ ${}^5C_3$ n 1	32
6	1 n ${}^6C_2$ ${}^6C_3$ ${}^6C_4$ n 1	64
7	1 n ${}^7C_2$ ${}^7C_3$ ${}^7C_4$ ${}^7C_5$ n 1	128
:	: : : : : : :	:

Pascals Triangle with Combinations Simplified

The properties of the binomial coefficients are:

- The binomial expansion is symmetrical, with ${}^nC_r = {}^nC_{n-r}$
- When $r = 0$ then ${}^nC_0 = 1$
- When $r = 1$ then ${}^nC_1 = n$
- When $r = n - 1$ then ${}^nC_{n-1} = n$
- When $r = n$ then ${}^nC_n = 1$
- From Pascals Triangle see that ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$ and ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$
- Binomial coefficients are all integers (theory of combinations)
- The sum of all the coefficients is 2^n
- To calculate: use either the ${}^nC_r$ button on the calculator, or use: ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$
- The expression ${}^nC_r$ means "n choose r" and is the number of ways to choose r things from a pool of n.
- Note that the ${}^nC_r$ format is only valid if n and r are positive integers.

Why use combinations?

For any given term in an expansion [say $(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$] then the number of combinations of a and b in that term will be the coefficient. For example, the third term includes a^2b^2.
That is 2 a's and 2 b's. How many ways are there of arranging them? The answer is 6, which are: *aabb, abab, abba, bbaa, baba, abba*. This is the same as the coefficient.

Calculating the binomial coefficient.

There are two ways of calculating the binomial coefficient. The first is the combination method and the second is a longer method, which will be required in C4.

Combination method: $${}^nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Long method: $$\textit{Coefficient} = \frac{n(n-1)(n-2)(n-3)(n-4)\dots(n-r+1)}{r!}$$

Note that in the longer method there are r terms on the top.

30.4.1 Example:

Calculate ${}^8C_5$

$${}^8C_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} = \frac{8\times7\times6\times5\times4}{5!} = \frac{8\times7\times\cancel{6}\times\cancel{5}\times\cancel{4}}{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1} = 56$$

or

$${}^8C_5 = \frac{n!}{r!\,(n-r)!} = \frac{8!}{5!\,(8-5)!} = \frac{8!}{5!\,3!} = \frac{8\times7\times\cancel{6}\times\cancel{5!}}{\cancel{5!}\times\cancel{3!}} = 56$$

Note how the digits of the factorials in the denominator add up to the same value as the factorial digit in the numerator. In this case $5 + 3 = 8$

30.5 Binomial Theorem

The **Binomial Theorem** codifies the expansion of $(a + b)^n$ where n is a positive integer.

When n is a positive integer the series is finite and gives an exact value for $(a + b)^n$ and is valid for all values of a & b. The expansion terminates after $n + 1$ terms.

The Binomial Theorem can be written in several forms, however learning the pattern of the first two versions are beneficial.

The long version is written thus:

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$$

Simplyfying a bit:

$$(a + b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}b^r + \ldots + nab^{n-1} + b^n$$

Replacing the binomial coefficients with the combination format gives:

$$(a + b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + {}^nC_3\,a^{n-3}b^3 \ldots + {}^nC_{n-1}\,ab^{n-1} + {}^nC_n\,b^n$$

Term no k: ① ② ③ ④ (n) $(n+1)$

Using the alternative symbology we have:

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

where ${}^nC_0 = \binom{n}{0} = 1$; ${}^nC_n = \binom{n}{n} = 1$; ${}^nC_1 = \binom{n}{1} = n$; ${}^nC_{n-1} = \binom{n}{n-1} = n$;

The general form of any term is given by the $(r + 1)^{th}$ term:

$${}^nC_r\,a^{n-r}b^r \quad \text{or} \quad \binom{n}{r}a^{n-r}b^r$$

Simplifying the first and last two coefficients we can write:

$$(a + b)^n = a^n + na^{n-1}b + {}^nC_2\,a^{n-2}b^2 + {}^nC_3\,a^{n-3}b^3 + \ldots + {}^nC_r\,a^{n-r}b^r + \ldots + nab^{n-1} + b^n$$

$$(a + b)^n = a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + nab^{n-1} + b^n$$

The compact method of defining the binomial theorem is:

$$(a + b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{or} \quad = \sum_{r=0}^{n}{}^nC_r\,a^{n-r}b^r$$

Note that the term counter, r, starts at zero.

30.6 Properties of the Binomial Theorem

A summary:

- Number of terms in the expansion of $(a + b)^n$ is $n + 1$
- The first term is always a^n and the last term b^n
- The power or exponent of a starts at a^n and decreases by 1 in each term to a^0
- The power or exponent of b starts at b^0 and increases by 1 in each term to b^n
- The general term of $(a + b)^n$ is the $(r + 1)^{\text{th}}$ term, which is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
- The k^{th} term will be: ${}^nC_{k-1}\, a^{n-(k-1)} b^{(k-1)}$

  where:

  a will have a power of $(n - k + 1)$ and b a power of $(k - 1)$

  $r = (k - 1)$
- The sum of the powers of each term equals n
- The coefficients of each term follow a well defined pattern
- The coefficient of the first and last term is always 1. ${}^nC_0 = 1$ & ${}^nC_n = 1$
- The coefficient of the second and last but one term is always n ${}^nC_1 = n$ & ${}^nC_{n-1} = n$
- If the expansion is $(a - b)^n$ then the signs in the expansion alternate
- Each expansion is finite provided that n is a positive integer

30.7 Binomial Theorem: Special Case

If 1 is substituted for a and x is substituted for b, then the expansion becomes:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + nx^{n-1} + x^n$$

$$(1 + x)^n = 1 + nx + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots + nx^{n-1} + x^n$$

This can be used to solve more complex problems and derive the full binomial expansion in the section above. Consider:

$$(a + x)^n = \left[a\left(1 + \frac{x}{a}\right)\right]^n = a^n\left(1 + \frac{x}{a}\right)^n$$

30.8 Finding a Given Term in a Binomial

Note the way that terms are counted—the binomial counter r starts at zero, but humans count the terms from one. Therefore, the k^{th} term is the $(r + 1)^{th}$ term and is given by:

$$^{n}C_{r}\, a^{n-r} b^{r} \qquad or \qquad \binom{n}{r} a^{n-r} b^{r}$$

To find the k^{th} term, $r = k - 1$, and is given by:

$$k^{th} \text{ term} = {}^{n}C_{k-1}\, a^{n-(k-1)} b^{k-1} \qquad or \qquad \binom{n}{k-1} a^{n-(k-1)} b^{k-1}$$

$$= {}^{n}C_{k-1}\, a^{n-k+1} b^{k-1} \qquad or \qquad \binom{n}{k-1} a^{n-k+1} b^{k-1}$$

30.8.1 Example:

Find the 9^{th} term of $(x - 2y)^{12}$. The coefficient is given by:

$$^{12}C_{9-1} = {}^{12}C_{8} = \frac{12!}{8!\,(12 - 8)!} = \frac{12!}{8!\,4!}$$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8!}{8! \times 4 \times 3 \times 2 \times 1} = \frac{11 \times 10 \times 9}{2}$$

$$= 495$$

Add in the x & y terms:

$$\text{9-th term} = 495x^{12-8}\, y^{8} = 495x^{4}y^{8}$$

To find a term with a given power, the general term in an expansion is give by:

$$^{n}C_{r}\, a^{n-r} b^{r}$$

30.8.2 Example:

Find the coefficient of the x^5 term in the expansion of $(2 - 2x)^7$

Solution:

The general term is given by:

$$^{n}C_{r}\, a^{n-r} b^{r}$$

In this case:

$$n = 7, \qquad a = 2, \qquad b = -2x$$

The x^5 term is when $r = 5$:

$$^{7}C_{5}\, 2^{7-5} (-2x)^{5} = {}^{7}C_{5}\, 2^{2} (-2x)^{5}$$

$$= {}^{7}C_{5}\, 4\left(-32x^{5}\right) = -{}^{7}C_{5}\, 128x^{5}$$

$$= -128 \times \frac{7!}{5!\,2!} x^{5} = -128 \times \frac{7 \times 6}{2} x^{5}$$

$$= -2688x^{5}$$

$\therefore$ The coefficient is: $\quad = -2688$

30.9 Binomial Theorem: Worked Examples

30.9.1 Example:

1 Expand $\left(x + \frac{2}{x}\right)^4$

Solution:

$$(a + b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^{4-1} b + {}^4C_2\, a^{4-2} b^2 + {}^4C_3\, a^{4-3} b^3 + {}^4C_4\, b^4$$

$$(a + b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 + {}^4C_3\, ab^3 + {}^4C_4\, b^4$$

But $\quad {}^4C_0 = {}^4C_4 = 1 \quad$ and $\quad {}^4C_1 = {}^4C_3 = 4$

$${}^4C_2 = \frac{4 \times 3}{2!} = 6$$

$$(a + b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$

Let $\quad a = x \quad$ and $\quad b = \frac{2}{x}$

$$\left(x + \frac{2}{x}\right)^4 = x^4 + 4x^3\left(\frac{2}{x}\right) + 6x^2\left(\frac{2}{x}\right)^2 + 4x\left(\frac{2}{x}\right)^3 + \left(\frac{2}{x}\right)^4$$

$$= x^4 + 8x^2 + 6x^2\left(\frac{4}{x^2}\right) + 4x\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= x^4 + 8x^2 + 24 + \frac{32}{x^2} + \frac{16}{x^4}$$

2 What is the coefficient of x^6 in $\left(x^2 + \frac{2}{x}\right)^{12}$?

Solution:

The general term in an expansion is give by:

$${}^nC_r\, a^{n-r} b^r$$

Require to find which value of r will give a term in x^6

Let $\quad a = x^2 \quad b = \frac{2}{x} \quad n = 12$

The term then becomes:

$${}^{12}C_r\, (x^2)^{12-r}\left(\frac{2}{x}\right)^r = {}^{12}C_r\, (x^{24-2r})\, 2^r x^{-r}$$

$$= 2^r \times {}^{12}C_r\, x^{24-3r} \qquad (1)$$

Now work out the value of r required to give x^6

$$24 - 3r = 6$$

$$3r = 24 - 6$$

$$r = 6$$

Substitute in (1) for the coefficient only

$$2^r \times {}^{12}C_r = 2^6 \times {}^{12}C_6$$

$$= 2^6 \times \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 64 \times \frac{11 \times 9 \times 8 \times 7}{6}$$

$$= 59136$$

3 Expand $(2 - x)^{10}$ up to the terms including x^3. Find an estimate for $(1.98)^{10}$ by using a suitable value for x.

Solution:

$$(2 - x)^{10} = 1a^{10} + 10a^9 b + {}^{10}C_2\, a^8 b^2 + {}^{10}C_3\, a^7 b^3 \ldots$$

Let $a = 2 \quad b = -x \quad n = 10$

$$(2 - x)^{10} = 2^{10} + 10 \times 2^9(-x) + {}^{10}C_2\, 2^8(-x)^2 + {}^{10}C_3\, 2^7(-x)^3 \ldots$$

$$= 2^{10} - 10 \times 2^9 x + {}^{10}C_2\, 2^8 x^2 - {}^{10}C_3\, 2^7 x^3 \ldots$$

$${}^{10}C_2 = \frac{10!}{2!\;8!} = \frac{10 \times 9 \times \cancel{8!}}{2 \times \cancel{8!}} = 5 \times 9 = 45$$

$${}^{10}C_3 = \frac{10!}{3!\;7!} = \frac{10 \times 9 \times 8 \times \cancel{7!}}{3 \times 2 \times \cancel{7!}} = 10 \times 3 \times 4 = 120$$

$$(2 - x)^{10} = 1024 - 5120x + 11520x^2 - 15360x^3 \ldots$$

Now $2 - x = 1.98 \quad \therefore\ x = 0{\cdot}02$

$$1{\cdot}98^{10} \cong 1024 - 102{\cdot}4 + 4{\cdot}608 - 0{\cdot}12288$$

$$\cong 926{\cdot}09 \text{ (2 } dp\text{)}$$

4 (a) Expand the binomial $(1 + 3x)^3$.
(b) Find the x coefficient in the expansion $(3 + x)^{10}$.
(c) Find the x coefficient in the expansion $(1 + 3x)^3(3 + x)^{10}$.

Solution:

(a)

$$(1 + 3x)^3 = 1 + 3(3x) + 3(3x)^2 + (3x)^3$$

$$(1 + 3x)^3 = 1 + 9x + 27x^2 + 27x^3$$

(b)

$$(3 + x)^{10} = 3^{10} + \binom{10}{1} 3^9(x) + \binom{10}{2} 3^8(x)^2 + \ldots$$

$$= 3^{10} + \frac{10!}{1!\,9!} 3^9(x) + \frac{10!}{2!\,8!} 3^8(x)^2 + \ldots$$

$$= 3^{10} + \frac{10 \times 9!}{9!} 3^9(x) + \frac{10 \times 9 \times 8!}{2 \times 8!} 3^8(x)^2 + \ldots$$

$$= 3^{10} + 19\,6830(x) + 45 \times 3^8(x)^2 + \ldots$$

Coefficient of $x = 196\,830$

(c)

$$(1 + 3x)^3(3 + x)^{10} = \left(1 + 9x + 27x^2 + 27x^3\right)\left(3^{10} + 196\,830(x) + \ldots\right)$$

x term: $\quad x = 196\,830x + 3^{10} \times 9x = 196\,830x + 531\,441x$

Coefficient of $x = 728{,}271$

5 The expression $(1 - 3x)^4$ expands to $1 - 12x + px^2 + qx^3 + 81x^4$.
(a) Find the values of p and q
(b) Find the coefficient for the x term of $(3 + x)^8$
(c) Find the coefficient for the x term of $(3 + x)^8(1 - 3x)^4$

Solution:

(a)

$(1 + b)^n = 1 + nb + {}^nC_2\, b^2 + {}^nC_3\, b^3 + \ldots + nb^{n-1} + b^n$ General expansion

$(1 + b)^4 = 1 + nb + {}^nC_2\, b^2 + nb^3 + b^4$ Has 5 terms

Substitute: $n = 4$; $b = -3x$; ${}^nC_2 = 6$

$(1 - 3x)^4 = 1 + 4(-3x) + 6(-3x)^2 + 4(-3x)^3 + (-3x)^4$

$(1 - 3x)^4 = 1 - 12x + 54x^2 - 108x^3 + 81x^4$

$\therefore$ $p = 54$; $q = -108$

(b)

$(a + b)^8 = {}^8C_0\, a^8 + {}^8C_1\, a^7b + {}^8C_2\, a^6b^2 + {}^8C_3\, a^5b^3 + \ldots$

Substitute: $n = 8$; $a = 3$; $b = x$; ${}^8C_2 = 28$

$(3 + x)^8 = 3^8 + 8(3)^7x + 28(3)^6x^2 + \ldots$

Coefficient of $x = 8(3)^7 = 17\,496$

(c)

$(1 - 3x)^4(3 + x)^8 = (1 - 12x + 54x^2 \ldots)(3^8 + 17\,496x \ldots)$ Only need up to x term

$= 17\,496x - 12x \times 3^8$

$= 17\,496x - 78\,732x$

$= -61\,236x$

Coefficient of $x = -61\,236$

EAFQLA

30.10 Alternative Method of Expanding a Binomial

This method relies on knowing some of the basic properties of the binomial discussed earlier. Whilst this method has its merits, it is much better to use the ${}^nC_r$ button on the calculator imho.

30.10.1 Example:

1 Expand the binomial $(x + y)^5$.

Alternative Method:

Step 1: Calculate the number of terms: $n + 1 = 6$

Step 2: Layout the binomial with the term numbers and just the x terms:

$$\begin{array}{lcccccc} & ① & ② & ③ & ④ & ⑤ & ⑥ \\ (x+y)^5 = & x^5 & + x^4 & + x^3 & + x^2 & + x^1 & + x^0 \end{array}$$

(note that the terms numbers start with 1)

Step 3: Add in the y terms:

$$\begin{array}{lcccccc} & ① & ② & ③ & ④ & ⑤ & ⑥ \\ (x+y)^5 = & x^5y^0 & + x^4y^1 & + x^3y^2 & + x^2y^3 & + x^1y^4 & + x^0y^5 \end{array}$$

Step 4: Add in the outer two coefficients for terms 1, 2, 5, & 6 and simplify x^0 & y^0:

$$\begin{array}{lcccccc} & ① & ② & ③ & ④ & ⑤ & ⑥ \\ (x+y)^5 = & x^5 & + 5x^4y & + x^3y^2 & + x^2y^3 & + 5xy^4 & + y^5 \end{array}$$

Step 5: Calculate the coefficients for the remaining terms 3 & 4. This is done by taking the coefficient and power of the previous x term and multiply them together and divide that by the term number of that previous term.

$$\begin{array}{lcccccc} & ① & ② & ③ & ④ & ⑤ & ⑥ \\ (x+y)^5 = & x^5 & + 5x^4y & + (10)x^3y^2 & + (10)x^2y^3 & + 5xy^4 & + y^5 \\ & & & \searrow\left(\frac{5 \times 4}{2}\right)\nearrow & \searrow\left(\frac{10 \times 3}{3}\right)\nearrow & & \end{array}$$

$$\therefore \qquad (x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

2 Find the coefficient of the x^3 term in the binomial $(2 - x)^{10}$.

Alternative Method:

Step 1: Calculate the number of terms: $n + 1 = 11$ (not really necessary for this example)

Step 2: Layout the binomial with the term numbers and just the constant (2) terms:

$$\overset{}{(2 - x)^{10}} = \overset{①}{2^{10}} + \overset{②}{2^{9}} + \overset{③}{2^{8}} + \overset{④}{2^{7}} + \overset{⑤}{2^{6}} + \ldots$$

Step 3: Add in the x terms:

$$(2 - x)^{10} = \overset{①}{2^{10}(-x)^0} + \overset{②}{2^9(-x)^1} + \overset{③}{2^8(-x)^2} + \overset{④}{2^7(-x)^3} + \overset{⑤}{}\ldots$$

$$(2 - x)^{10} = 2^{10} - 2^9x + 2^8x^2 - 2^7x^3 + \ldots$$

Step 4: Add in the first two coefficients for terms 1 & 2:

$$(2 - x)^{10} = \overset{①}{2^{10}} - \overset{②}{10 \times 2^9x} + \overset{③}{2^8x^2} - \overset{④}{2^7x^3} + \overset{⑤}{}\ldots$$

Step 5: Calculate the coefficients for the remaining terms 3 & 4. This is done by taking the coefficient and power of the previous x term and multiply them together and divide that by the term number of that previous term.

$$(2 - x)^{10} = \overset{①}{2^{10}} - \overset{②}{10 \times 2^9x} + \overset{③}{(45) \times 2^8x^2} - \overset{④}{(120) \times 2^7x^3} + \ldots$$

$$\searrow\left(\frac{10 \times 9}{2}\right)\nearrow \qquad \searrow\left(\frac{45 \times 8}{3}\right)\nearrow$$

Simplifying:

$$(2 - x)^{10} = \overset{①}{1024} - \overset{②}{5120x} + \overset{③}{11520x^2} - \overset{④}{15360x^3} + \ldots$$

The coefficient of the x^3 term = -15360

3 One source of confusion with this method can be if trying to expand something like $(1 + x)^4$. Care must be taken to include the 1 with its powers.

Solution:

Step 1, 2, 3 & 4: Combined

$$(1 + x)^4 = \overset{①}{1^4x^0} + \overset{②}{4 \times 1^3x^1} + \overset{③}{\square \times 1^2x^2} + \overset{④}{4 \times 1^1x^3} + \overset{⑤}{1^0x^4}$$

Step 5: Simplify and calculate the coefficient for the remaining term 3.

$$(1 + x)^4 = \overset{①}{1^4} + \overset{②}{4 \times 1^3x^1} + \overset{③}{6 \times 1^2x^2} + \overset{④}{4 \times 1^1x^3} + \overset{⑤}{x^4}$$

$$\searrow\left(\frac{4 \times 3}{2}\right)\nearrow$$

Simplifying:

$$(1 + x)^4 = \overset{①}{1} + \overset{②}{4x} + \overset{③}{6x^2} + \overset{④}{4x^3} + \overset{⑤}{x^4}$$

30.11 Heinous Howlers

- Binomial questions seem to cause no end of problems in the exams.
- Great care must be taken in getting the signs and powers correct. Lots of marks to loose here.
- In the formula book the expansion is quoted as:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r$$

Note that 1.2 in algebra means 1×2 not $1\frac{2}{10}$

- When substituting another term for the basic *a* or *b* in a binomial, a most common mistook is to forget to raise the substituted terms to the correct power. The liberal use of brackets will help avoid this particular howler.

30.11.1 Example:

Expand $(1 + 3x)^3$

$$(1 + b)^3 = 1 + 3b + 3b^2 + b^3$$

$$(1 + 3x)^3 = 1 + 9x + 3 \times 3x^2 + 3x^3 \quad ✘$$

The correct solution:

$$(1 + 3x)^3 = 1 + 3(3x) + 3(3x)^2 + (3x)^3$$

$$(1 + 3x)^3 = 1 + 9x + 27x^2 + 27x^3$$

- Evaluating simple fractions raised to a power also gives rise to a number of errors.

$$e.g. \quad \left(\frac{3}{x}\right)^2 = \frac{9}{x^2} \qquad NOT \quad \frac{9}{x} \quad or \quad \frac{3}{x^2}$$

30.12 Some Common Expansions in C2

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + nx^{n-1} + x^n$$

$$(1 + x)^n = 1 + nx + {}^nC_2\,x^2 + {}^nC_3\,x^3 + \ldots + nx^{n-1} + x^n$$

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$$

$$(1 - x)^3 = 1 - 3x + 3x^2 - x^3$$

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$(1 - x)^4 = 1 - 2x + 6x^2 - 4x^3 + x^4$$

Note how the signs change.

30.13 Binomial Theorem Topic Digest

The Binomial theorem, where n is a positive integer:

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$$

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots + {}^nC_r\, a^{n-r}b^r + \ldots + {}^nC_{n-1}\, ab^{n-1} + b^n$$

$$(a+b)^n = a^n + na^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots + {}^nC_r\, a^{n-r}b^r + \ldots + nab^{n-1} + b^n$$

where: $\quad {}^nC_1 = {}^nC_{n-1} = n \qquad (n \in \mathbb{N})$

Alternative symbology:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$$

$$(a+b)^n = a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + nab^{n-1} + b^n$$

where: $\quad \binom{n}{1} = \binom{n}{n-1} = n \qquad (n \in \mathbb{N})$

Special case for $(1+x)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + nx^{n-1} + x^n$$

$$(1+x)^n = 1 + nx + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots + nx^{n-1} + x^n$$

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \textit{or} \quad (a+b)^n = \sum_{r=0}^{n}{}^nC_r\, a^{n-r}b^r$$

Where:

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$${}^nC_r = {}^nC_{n-r}$$

$${}^nC_2 = \binom{n}{2} = \frac{n(n-1)}{2\times 1} \qquad {}^nC_3 = \binom{n}{3} = \frac{n(n-1)(n-2)}{3\times 2\times 1}$$

The k^{th} term:

$$= {}^nC_{k-1}\, a^{n-k+1}b^{k-1} \quad \textit{or} \quad \binom{n}{k-1}a^{n-k+1}b^{k-1}$$

For the term in a^{n-r} or b^r

$$= {}^nC_r\, a^{n-r}b^r \quad \textit{or} \quad \binom{n}{r}a^{n-r}b^r$$

Note: the combination format, ${}^nC_r$, is only valid if n & r are positive integers. For $n < 1$ then the full version of the Binomial theorem is required. More of this in C4.

When n is a positive integer the series is finite and gives an exact value of $(1+x)^n$ and is valid for all values of x. The expansion terminates after $n+1$ terms.

The use of the ${}^nC_r$ form for the combination symbol is simply because it is used on many calculators. Also shown as ${}_nC_r$ on some calculators.

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31 • C2 • Trig Ratios for all Angles

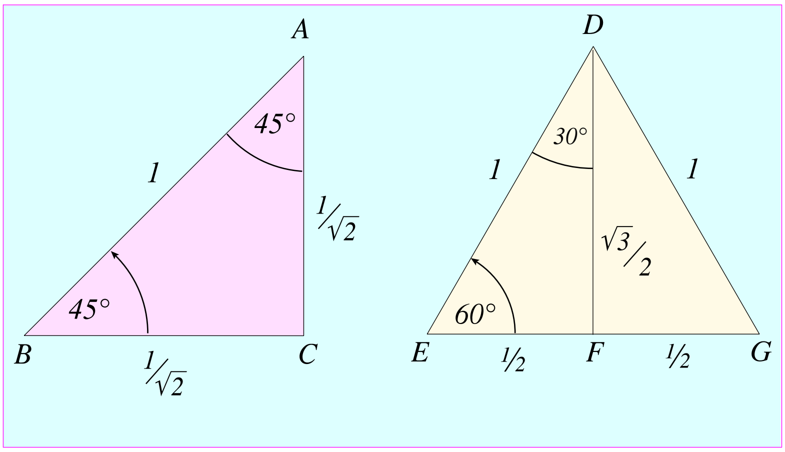
31.1 Trig Ratios for all Angles Intro

Prior to A level, the definitions of sine, cosine & tangent have been defined in terms of right angled triangles and acute angles. We now use Cartesian co-ordinates to define the trig ratios of any angle, even angles greater that 360°.

31.2 Standard Angles and their Exact Trig Ratios

However, you need to be very familiar with these standard angles and their exact ratios. You should be able to derive them in case you cannot remember them.

The trick is to use two regular triangles in which the hypotenuse is set to 1 unit. This simplifies the ratios and makes them easy to calculate. It is a simple matter to use pythag to calculate the lengths of the other sides and hence the trig ratios.



Unit Triangles

Recall SOH CAH TOA. Hence:

$$\sin 45 = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}$$

$$\cos 45 = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}$$

$$\tan 45 = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$$

$$\sin 60 = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\cos 60 = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\tan 60 = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$$

$$\sin 30 = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\cos 30 = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\tan 30 = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2}{1} \times \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

31.3 The Unit Circle

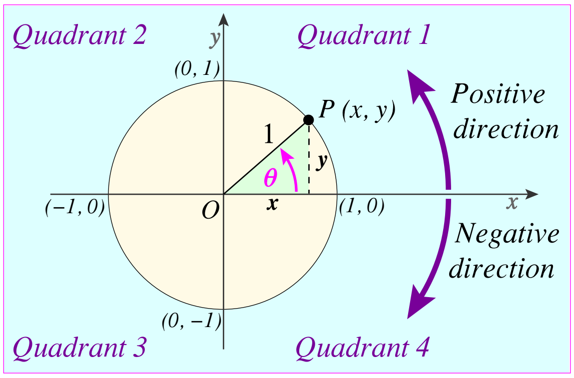
The unit circle is a standard way of representing angles over 90°. Cartesian co-ordinates are used to define the trig rations of any angle. The clever trick is to use a circle with a radius of 1 unit, hence the name. Once again this simplifies the definitions of the trig functions as shown below:

The Unit Circle

$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

$$\tan\theta = \frac{O}{A} = \frac{y}{x} \equiv \text{gradient of } OP$$



Properties of the Unit Circle:

- Radius $r = 1$ (always regarded as a positive value)
- Angles are measured from the positive x-axis in an anticlockwise direction
- Angles measured in a clockwise direction are said to be negative angles
- The circle is divided into 4 quadrants as seen
- Trig ratios in the first quadrant are equivalent to the definitions derived from a right angled triangle
- The x-axis represents $\cos\theta$ $\quad -1 \leqslant \cos\theta \leqslant 1$ for all θ
- The y-axis represents $\sin\theta$ $\quad -1 \leqslant \sin\theta \leqslant 1$ for all θ
- The coordinates of any point, P, on the unit circle are given by $(\cos\theta, \sin\theta)$
- $\tan\theta$ can be defined as the y-coordinate of the point $Q\ (1, \tan\theta)$
- $\tan\theta$ represents the gradient of the line OP and OQ
- The equation of a unit circle is $x^2 + y^2 = 1$

From the unit circle trig definitions we can see:

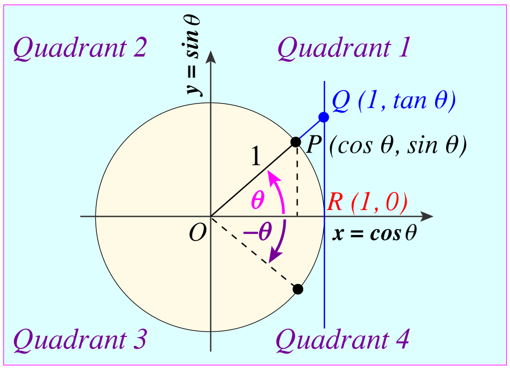
$x = \cos\theta \quad \& \quad y = \sin\theta$

Since $\tan\theta = \dfrac{y}{x} \quad \therefore \quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Note $\tan\theta = \dfrac{OQ}{OR} = \dfrac{OQ}{1} \quad \therefore \quad \tan\theta = y_Q$

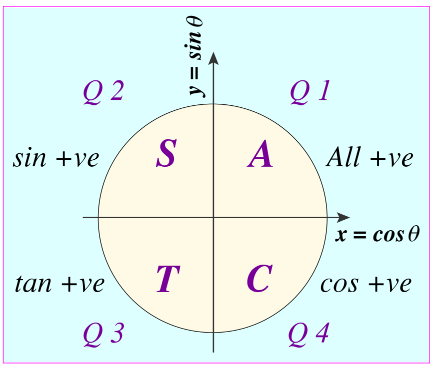
From the equation of a circle: $\quad x^2 + y^2 = 1$

hence: $\quad \sin^2\theta + \cos^2\theta = 1$



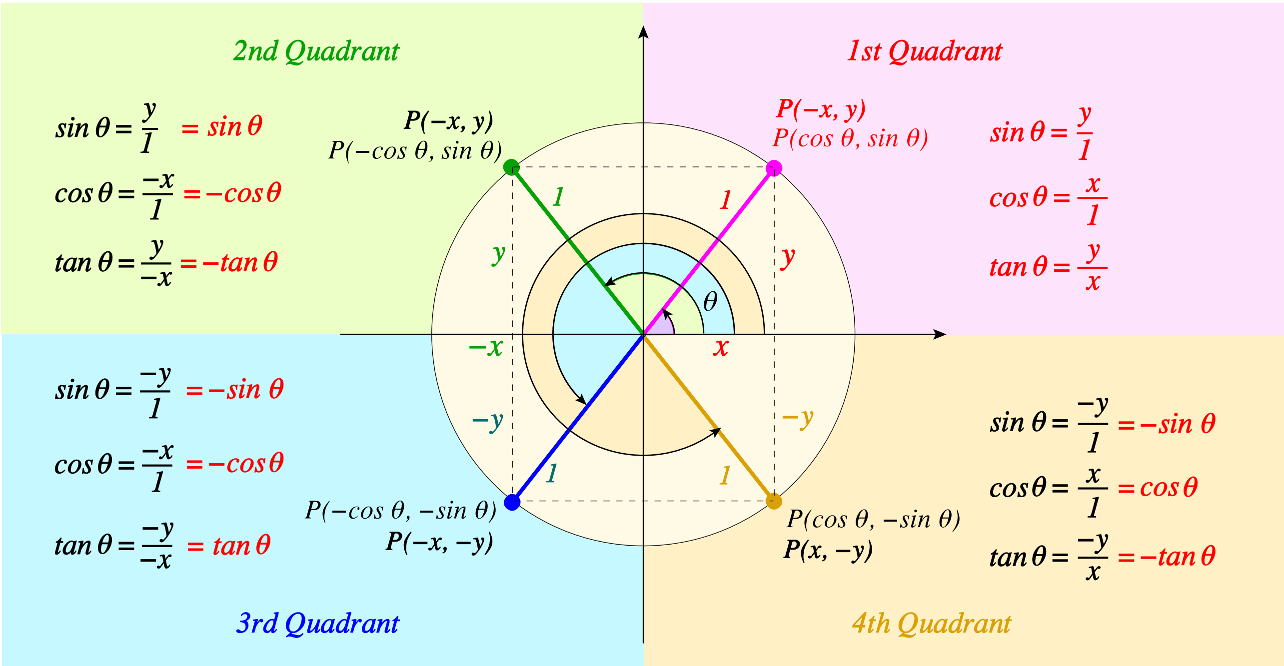
It also becomes easy to deduce the sign of each trig function in each quadrant. This gives us the standard CAST diagram:

The CAST Diagram showing the quadrants with positive trig functions.



Once you realise that the x-axis represents $\cos\theta$ and the y-axis $\sin\theta$ then it can be seen that trig functions for any angle have a close relationship with angles in the 1st quadrant.

The diagram below summarises this:

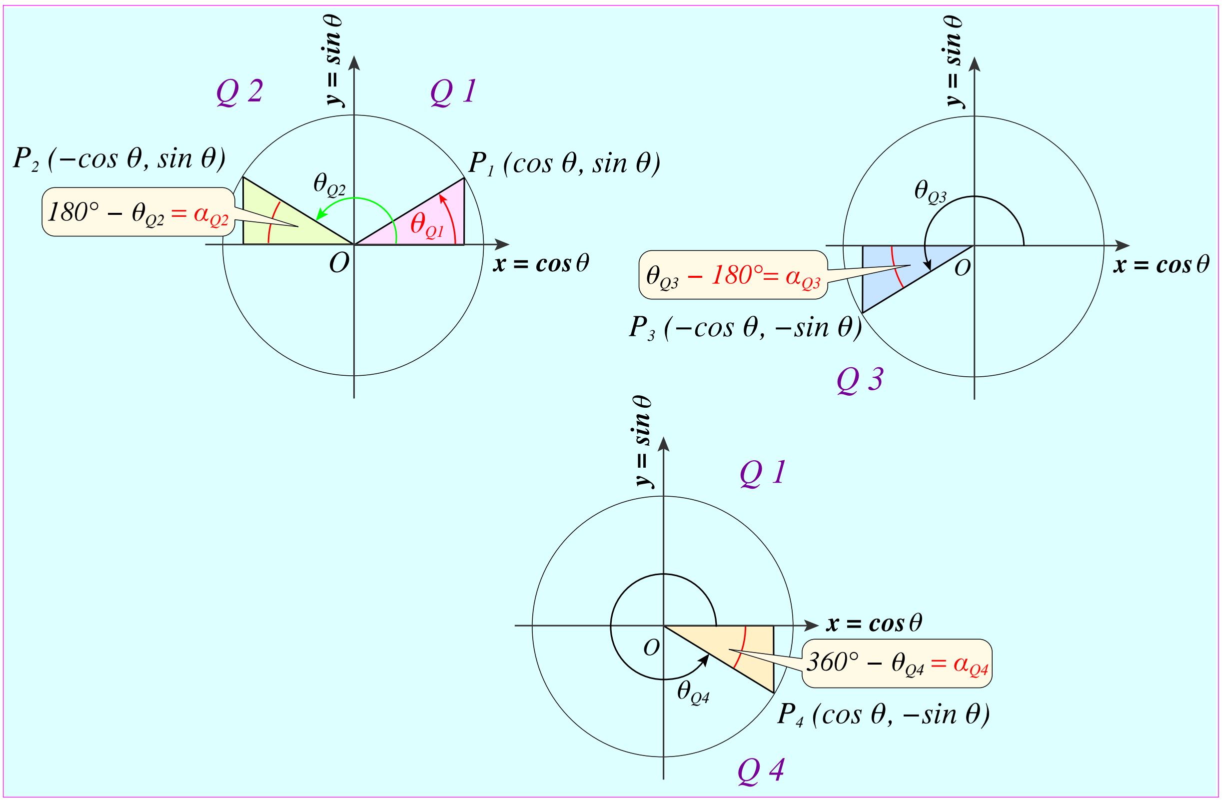


Unit Circle with Trig Definitions

31.4 Acute Related Angles

For any angle greater than 90°, the numerical value of a trig ratio can be found by finding the related acute angle, α, between the radius OP and the x-axis. The only difficulty is getting the sign right!

eg in Q2: $\sin\theta_{Q2} = \sin\alpha_{Q2}$ in Q3: $\tan\theta_{Q3} = \tan\alpha_{Q3}$ in Q4: $\cos\theta_{Q4} = \cos\alpha_{Q4}$



Acute Related Angles

31.5 The Principal & Secondary Value

An examination of the graphs of any trig function will tell you that for any given value of the function there are an infinite number of solutions for the angle θ.

In a typical exam question, you will be asked to solve a trig equation for all values of θ in a certain range or interval of values.

However, the calculator will only give one solution for θ, which is called the **Principal Value** (PV).

Try this on a calculator:

Recall that if: $\quad sin\,\theta = y \quad$ then $\quad \theta = sin^{-1} y$

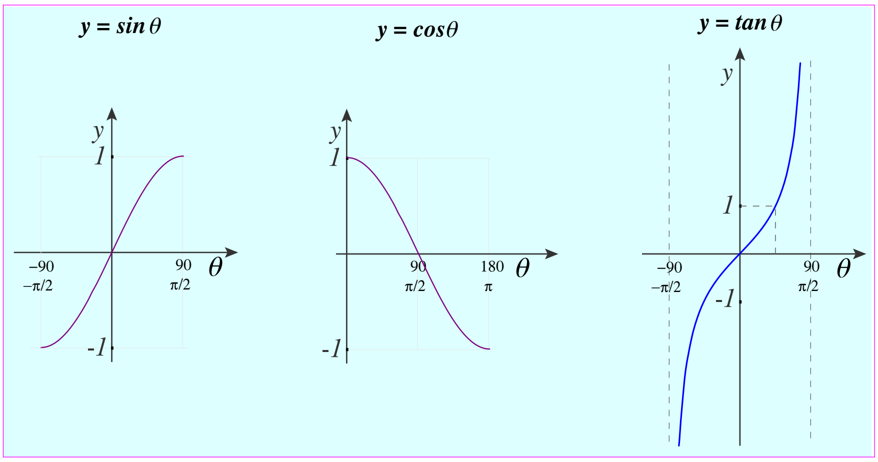
where $sin^{-1}\,\theta$ means the inverse, not the reciprocal!

On the calculator: $\quad sin\,210 = -\frac{1}{2} \quad$ but $\quad \theta = sin^{-1}\left(-\frac{1}{2}\right) \quad$ results in $\quad \theta = -30°$ (The PV)

So why does the 210° change to −30° when processed on the calculator?

The answer is that the calculator restricts its range of outputs to a certain range of values as shown below.

[This is because we are dealing with the inverse trig functions. Inverse trig functions are dealt with properly in C3].



Range of Principal Values

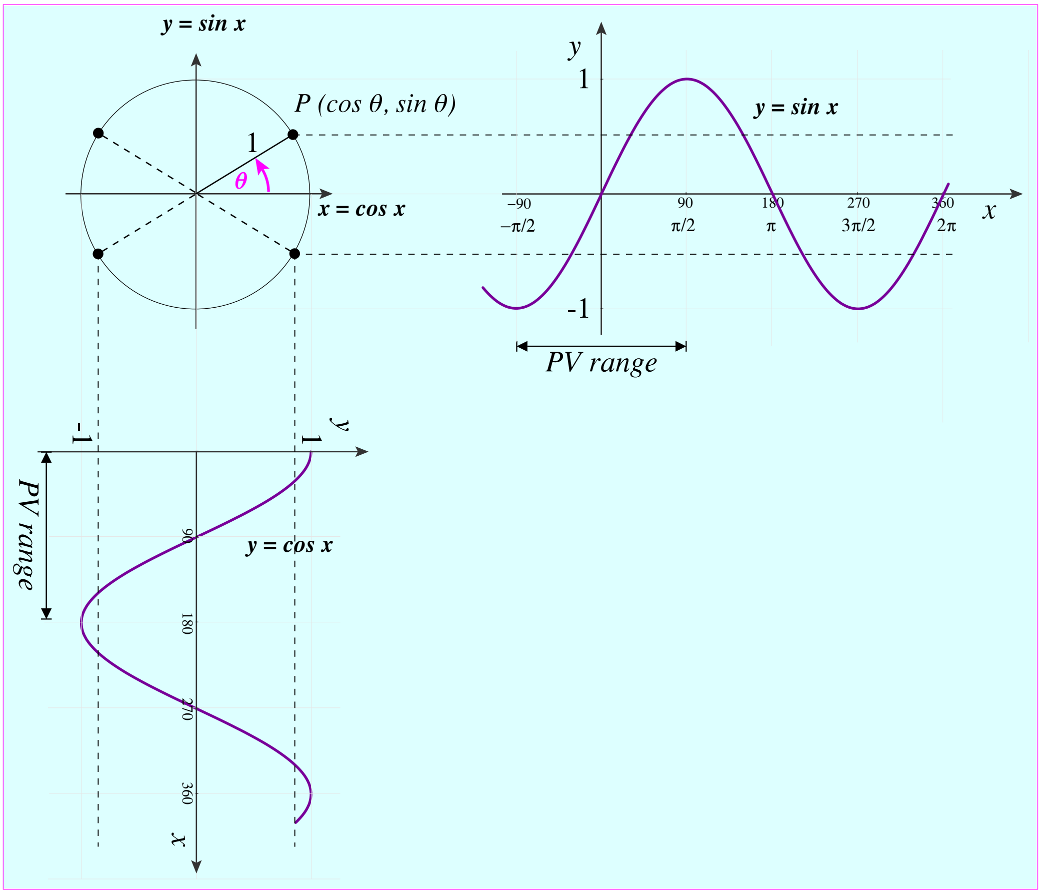
In solving trig equations, and depending on the trig function, the PV is restricted to these intervals:

PV for the sine function	$-\frac{\pi}{2} \leqslant sin^{-1} y \leqslant \frac{\pi}{2}$	$-90° \leqslant sin^{-1} y \leqslant 90°$	$Q1$ & $Q4$
PV for the cosine function	$0 \leqslant cos^{-1} y \leqslant \pi$	$0° \leqslant cos^{-1} y \leqslant 180°$	$Q1$ & $Q2$
PV for the tan function	$-\frac{\pi}{2} \leqslant tan^{-1} y \leqslant \frac{\pi}{2}$	$-90° \leqslant tan^{-1} y \leqslant 90°$	$Q1$ & $Q4$

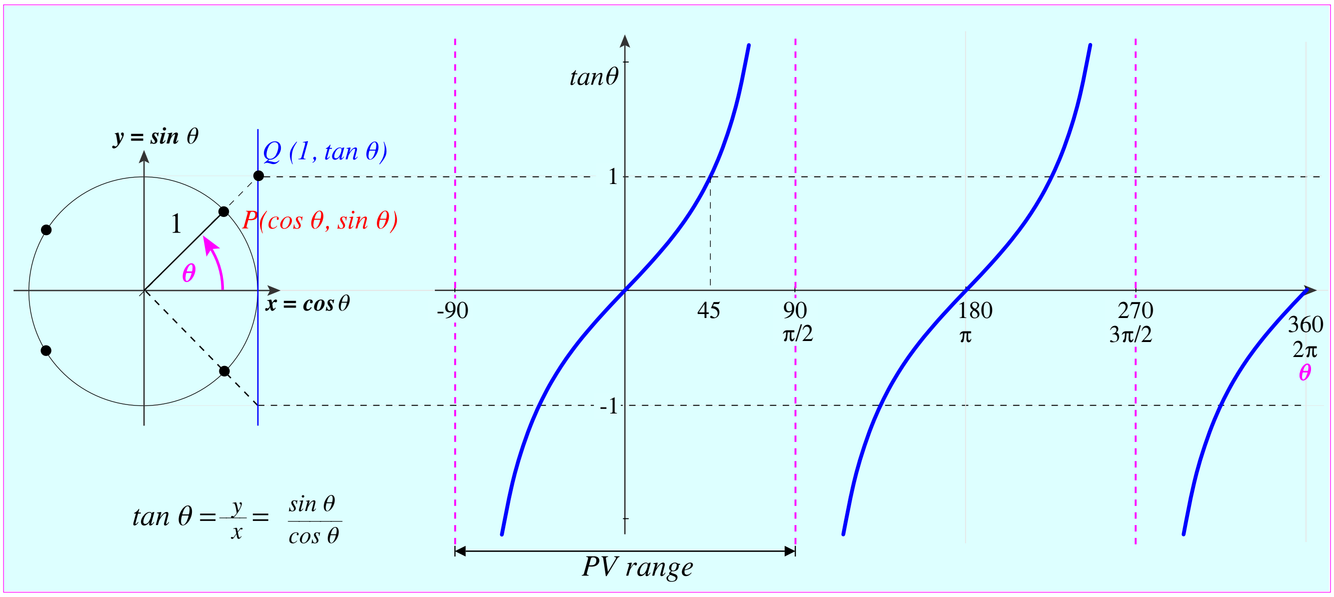
Where Q1 & Q4 refer to the quadrant numbers 1 & 4 etc.

So each trig function has two solutions in each 360° interval. This first solution is the PV, and is in the first quadrant and the second solution or secondary value (SV) is in another quadrant.

31.6 The Unit Circle and Trig Curves



Sine and Cosine Graphs and the Unit Circle



Tangent Graphs and the Unit Circle

31.7 General Solutions to Trig Equations

31.7.1 Solutions for Sin θ

From the unit circle and sine curve we can see that:

$$y = \sin(\theta) \qquad -1 \leqslant y \leqslant 1$$

$$\sin\theta = \sin(180° - \theta) \qquad Q2$$

$$\& \qquad -\sin\theta = \sin(180° + \theta) \qquad Q3$$

$$\& \qquad -\sin\theta = \sin(360° - \theta) \qquad Q4$$

The solutions for θ follow a pattern thus:

$$\theta = \ldots, -2\pi + PV, -\pi - PV, \mathbf{PV}, \pi - PV, 2\pi + PV, \ldots$$

$$\therefore \quad \theta = \sin^{-1}(y) + 2n\pi$$

$$\& \quad \theta = -\sin^{-1}(y) + (2n+1)\pi$$

where n is an integer value

31.7.2 Solutions for Cos θ

From the unit circle and cosine curve we can see that:

$$x = \cos(\theta)$$

$$\cos\theta = \cos(360° - \theta)$$

$$\& \qquad -\cos\theta = \cos(360° + \theta)$$

$$\& \qquad -\cos\theta = \cos(360° - \theta)$$

The solutions for θ follow a pattern thus:

$$\theta = \ldots, -2\pi + PV, -PV, \mathbf{PV}, 2\pi - PV, 2\pi + PV, \ldots$$

$$\therefore \quad \theta = \pm\cos^{-1}(y) + 2n\pi$$

where n is an integer value

31.7.3 Solutions for Tan θ

From the unit circle and tan curve we can see that:

$$\frac{y}{x} = \tan(\theta) = z$$

$$\& \qquad \tan\theta = \tan(180° - \theta)$$

$$\& \qquad -\tan\theta = \tan(180° + \theta)$$

The solutions for θ follow a pattern thus:

$$\theta = \ldots, -2\pi + PV, -\pi + PV, \mathbf{PV}, \pi + PV, 2\pi + PV, \ldots$$

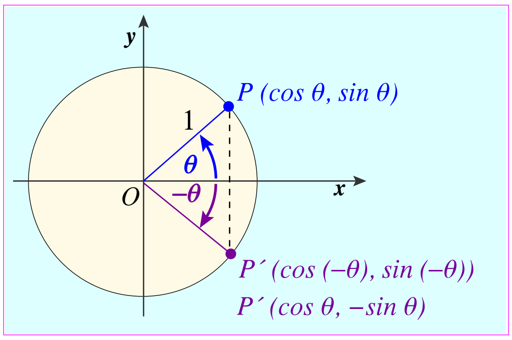
$$\therefore \quad \theta = \pm\tan^{-1}(z) + n\pi$$

where n is an integer value

31.8 Complementary and Negative Angles

31.8.1 Negative Angles

From the unit circle,
P & P' have the same x-coordinates but the y-coordinates have opposite signs.

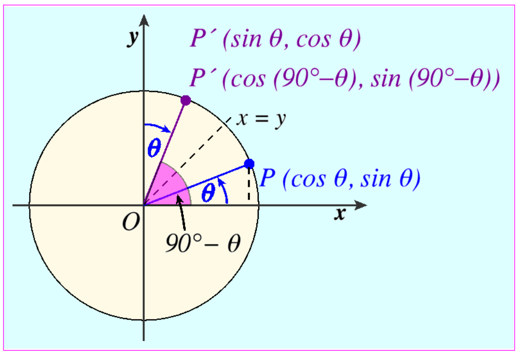


Thus we have:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$

31.8.2 Complementary Angles

Suppose point P' is set at an angle of $(90° - \theta)$, giving the coordinate of P' as $(\cos(90° - \theta), \sin(90° - \theta))$
Now P' is a reflection of P in the line $x = y$.
Since the coordinate of P is $(\cos\theta, \sin\theta)$ the reflected coordinate of P' is $(\sin\theta, \cos\theta)$.



Hence we can say:

$$\cos(90° - \theta) = \sin\theta \qquad or \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin(90° - \theta) = \cos\theta \qquad or \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

31.9 Coordinates for Angles 0°, 90°, 180° & 270°

For $\theta = 0°$, point P has the coordinates of $(1, 0)$

$$\therefore \quad \sin\theta = \frac{0}{1} = 0, \quad \cos\theta = \frac{1}{1} = 1, \quad \tan\theta = \frac{0}{1} = 0$$

For $\theta = 90°$, point P has the coordinates of $(0, 1)$

$$\therefore \quad \sin\theta = \frac{1}{1} = 1, \quad \cos\theta = \frac{0}{1} = 0, \quad \tan\theta = \frac{1}{0} = \infty \text{ (or not defined)}$$

For $\theta = 180°$, point P has the coordinates of $(-1, 0)$

$$\therefore \quad \sin\theta = \frac{0}{1} = 0, \quad \cos\theta = \frac{-1}{1} = -1, \quad \tan\theta = \frac{0}{-1} = 0$$

For $\theta = 270°$, point P has the coordinates of $(0, -1)$

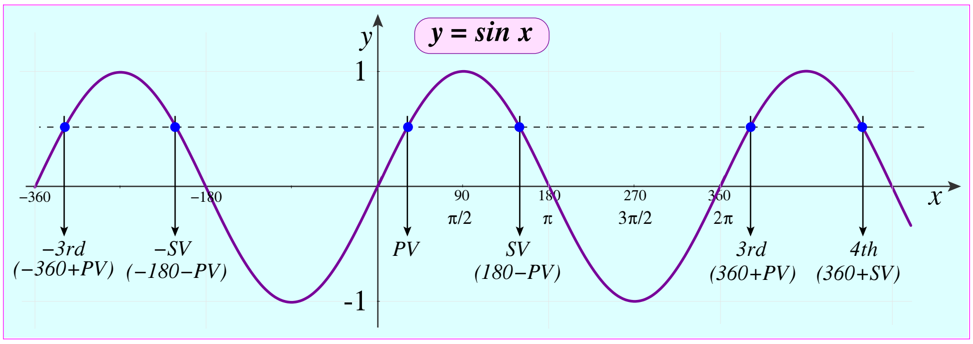
$$\therefore \quad \sin\theta = \frac{-1}{1} = -1, \quad \cos\theta = \frac{0}{1} = 0, \quad \tan\theta = \frac{-1}{0} = \infty \text{ (or not defined)}$$

31.10 Solving Trig Problems

These diagrams should help visualize the solutions for the three main trig ratios:

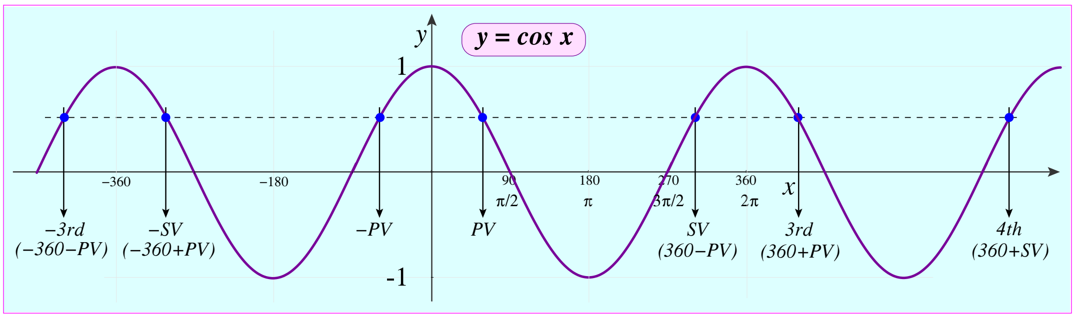
Solutions for Sine

$$PV = \sin^{-1} y$$
$$SV = 180 - PV$$
$$3rd = 360 + PV$$
$$4th = 360 + SV$$



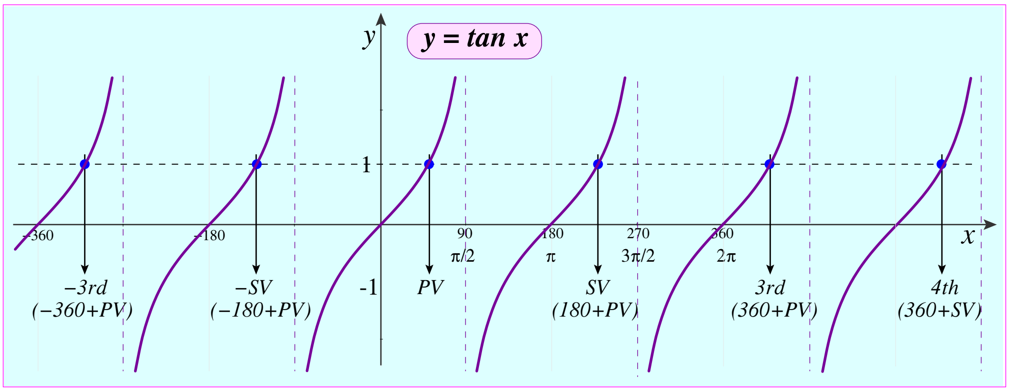
Solutions for Cos

$$PV = \cos^{-1} y$$
$$SV = 360 - PV$$
$$3rd = 360 + PV$$
$$4th = 360 + SV$$



Solutions for Tan

$$PV = \tan^{-1} y$$
$$SV = 180 + PV$$
$$3rd = 360 + PV$$
$$4th = 360 + SV$$



31.11 Trig Ratios Worked Examples

31.11.1 Example:

1 Solve $\sin 3x = 0{\cdot}5$ for: $0° \leqslant x \leqslant 180°$

Solution:

Draw a sketch!

$\sin 3x = 0.5$

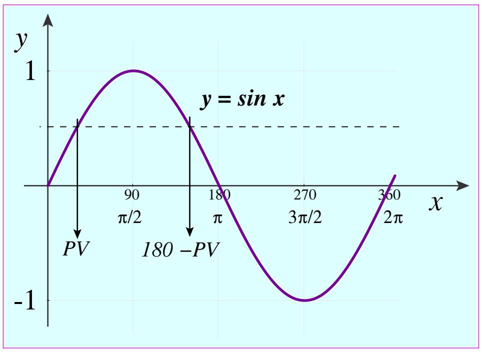
$3x = \sin^{-1}(0{\cdot}5)$

$3x = 30°$ (PV) $\therefore x = 10°$

$3x = 180 - 30° = 150°$ (SV) $\therefore x = 50°$

$3x = 360 + 30° = 390°$ $(3rd)$ $\therefore x = 130°$

$\therefore \quad x = 10°, 50°, 130°$



2 Solve $\sin^2 x = 1 - \frac{\sqrt{3}}{2}\cos x$ for: $0° \leqslant x \leqslant 180°$

Solution:

$$\sin^2 x = 1 - \frac{\sqrt{3}}{2}\cos x$$

$$\text{But } \sin^2 x + \cos^2 x = 1$$

$$\therefore 1 - \cos^2 x = 1 - \frac{\sqrt{3}}{2}\cos x$$

$$\cos^2 x = \frac{\sqrt{3}}{2}\cos x$$

$$\cos^2 x - \frac{\sqrt{3}}{2}\cos x = 0$$

$$2\cos^2 x - \sqrt{3}\cos x = 0$$

$$\cos x(2\cos x - \sqrt{3}) = 0$$

1st solution: $\cos x = 0$ $\therefore$ $x = 90°$, (other solutions out of range)

2nd solution: $(2\cos x - \sqrt{3}) = 0$

$\cos x = \frac{\sqrt{3}}{2}$ $\therefore$ $x = 30°$, (other solutions out of range)

3 Solve $\sin 2x = \sqrt{3}\cos 2x$ for: $0 \leqslant x \leqslant \pi$

Solution:

$$\sin 2x = \sqrt{3}\cos 2x$$

$$\frac{\sin 2x}{\cos 2x} = \sqrt{3}$$

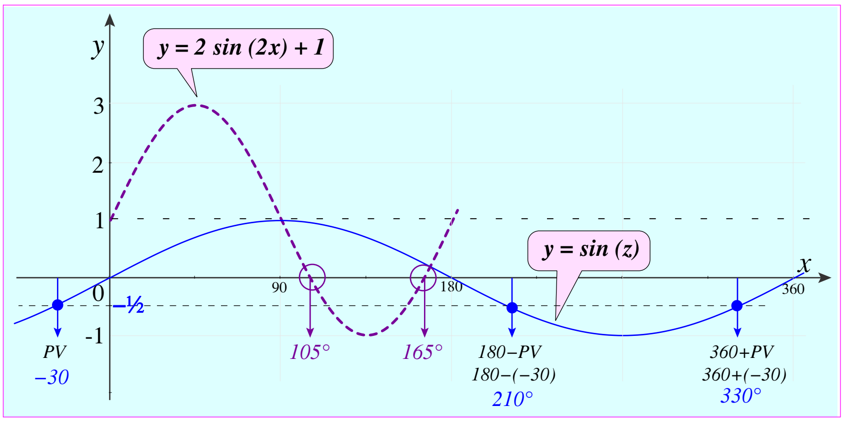
$$\tan 2x = \sqrt{3}$$

$$2x = \tan^{-1}\sqrt{3} \Rightarrow \frac{\pi}{3}\,(60°),\ \frac{\pi}{3} + \pi,\ \frac{\pi}{3} + 2\pi \Rightarrow \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{6}, \frac{2\pi}{3} \qquad (0 \leqslant x \leqslant \pi)$$

4 Find the values of x for which $2\,sin\,(2x) + 1 = 0$ for: $0° \leqslant x \leqslant 180°$

Solution:



Draw a sketch!

$$2\,sin\,(2x) + 1 = 0$$

Let $z = 2x$

$$sin\,(z) = -\frac{1}{2}$$

$$z = sin^{-1}\left(-\frac{1}{2}\right)$$

$z = -30°$ (not included as a solution)

Potential solutions are: $PV,\ 180 - PV,\ 360 + PV \quad (0° \leqslant x \leqslant 360°)$

$$z = 180 - (-30),\ 360 + (-30)$$

$$z = 210°,\ 330°$$

Hence: $2x = 210°,\ 330°$

$\therefore \quad x = 105°,\ 165°$

Note that the original equations to solve was $2\,sin\,(2x) + 1 = 0$ and the roots for this curve are shown in the diagram at $105°,\ 165°$.

5 Find the values of x for which $sin\,(2x - 25) = -0{\cdot}799$ for: $-90° \leqslant x \leqslant 180°$

Solution:

$$sin\,(2x - 25) = -0{\cdot}799$$

$$(2x - 25) = sin^{-1}\,(-0{\cdot}799)$$

$(2x - 25) = -53{\cdot}0 \quad (PV)$

Potential solutions are: $(-180 - PV),\ PV,\ 180 - PV,\ 360 + PV \quad (-90° \leqslant x \leqslant 180°)$

$$(2x - 25) = -180 + 53{\cdot}0,\ -53,\ 180 + 53,\ 360 - 53$$

$$(2x - 25) = -127,\ -53,\ 233,\ 307$$

$$2x = -102,\ -28,\ 258,\ 332$$

$x = -51,\ -14,\ 129,\ 166 \qquad -90° \leqslant x \leqslant 180°$

6 Solve $\sin(x - 30) + \cos(x - 30) = 0$ for: $0° \leqslant x \leqslant 360°$

Solution:

$$\sin(x - 30) + \cos(x - 30) = 0$$

$$\sin(x - 30) = -\cos(x - 30)$$

$$\frac{\sin(x - 30)}{\cos(x - 30)} = -1$$

$$\tan(x - 30) = -1$$

$$(x - 30) = \tan^{-1}(-1)$$

$$(x - 30) = -45 \qquad (PV)$$

Potential solutions are: $PV, \ 180 + PV, \ 360 + PV$

Solutions are: $(x - 30) = -45, \ 135, \ 405$

$$x = -15, \ 165, \ 435$$

$$x = 165, \ 435 \qquad 0° \leqslant x \leqslant 360°$$

7 Solve for x in the interval: $0° \leqslant x \leqslant 540°$

$$\frac{4 + 2\sin^2 x}{\cos(x) - 5} = 2\cos x$$

Solution:

$$4 + 2\sin^2 x = 2\cos x(\cos(x) - 5)$$

$$4 + 2\sin^2 x = 2\cos^2 x - 10\cos x$$

but $\cos^2 x + \sin^2 x = 1$

$$\therefore \quad 4 + 2\left(1 - \cos^2 x\right) = 2\cos^2 x - 10\cos x$$

$$6 - 2\cos^2 x = 2\cos^2 x - 10\cos x$$

$$\therefore \quad 4\cos^2 x - 10\cos x - 6 = 0$$

$$\therefore \quad 2\cos^2 x - 5\cos x - 3 = 0$$

$$(2\cos x + 1)(\cos x - 3) = 0$$

$\therefore \quad \cos x = 3$ (no solution since $\cos x > 1$)

$$\cos x = -\frac{1}{2}$$

$$x = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$$

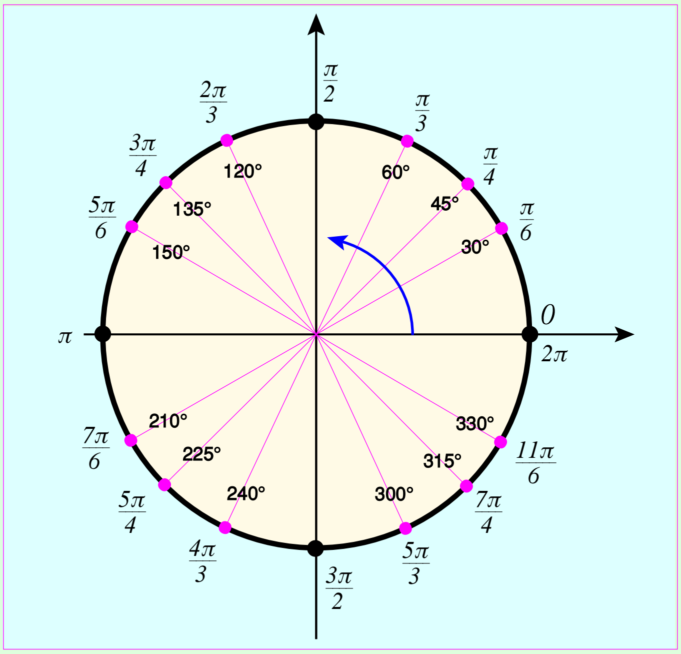
Potential solutions are: $PV, \ 360 - PV, \ 360 + PV \qquad (0° \leqslant x \leqslant 540°)$

Hence: $x = 120°, \ 240°, \ 480°$

EAFQLA

31.12 Trig Ratios for all Angles Digest

Degrees	Radians	sin	cos	tan
0	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	*AT*
180°	π	0	−1	0
270°	$\frac{3\pi}{2}$	−1	0	*AT*
360°	2π	0	1	0



Relationship Between Degrees and Radians

	Trig Ratio to solve		
Solution	***cos***	***sin***	***tan***
PV	cos^{-1}	sin^{-1}	tan^{-1}
SV	360 − *PV*	180 − *PV*	180 + *PV*
3*rd*	360 + *PV*	360 + *PV*	360 + *PV*
4*th*	360 + *SV*	360 + *SV*	360 + *SV*
5*th*	360 + 3*rd*	360 + 3*rd*	360 + 3*rd*
6*th*	360 + 4*th*	360 + 4*th*	360 + 4*th*

32 • C2 • Graphs of Trig Functions

32.1 Graphs of Trig Ratios

Sine Properties:

$$|\sin\theta| \leqslant 1$$

Periodic: every 360° or 2π radians

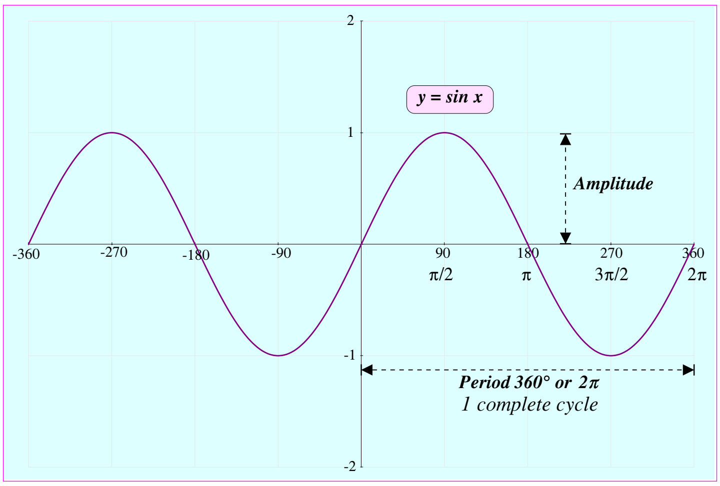
Hence: $\sin\theta° = \sin(\theta \pm 360)°$

& $\sin\theta = \sin(\theta \pm 2\pi)$

Symmetric about $\theta = \pm 90°, \pm 270°$ *etc*

$$\sin(90 - \theta)° = \sin(90 + \theta)°$$
$$\sin(90 - \theta)° = \cos\theta°$$
$$\sin(-\theta) = -\sin\theta$$
$$f(-\theta) = -\sin\theta = -f(\theta)$$

$\therefore$ Sine is classed as an odd function and the graph has rotational symmetry, order 2, about the origin.



Cosine Properties:

$$|\cos\theta| \leqslant 1$$

Periodic: every 360° or 2π radians

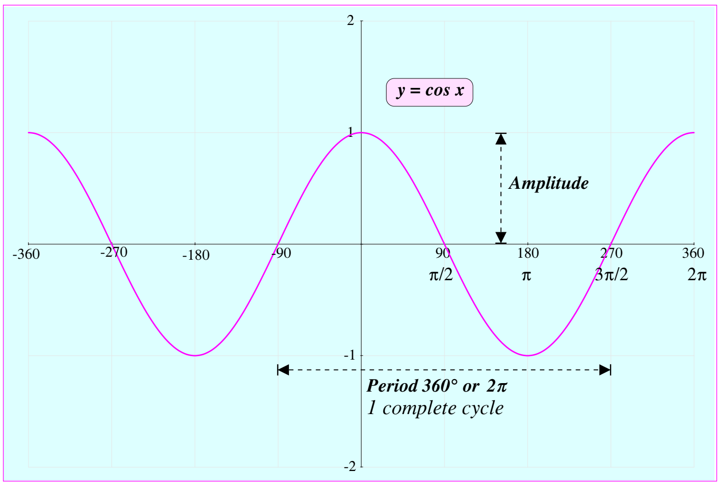
Hence: $\cos\theta° = \cos(\theta \pm 360)°$

& $\cos\theta = \cos(\theta \pm 2\pi)$

Symmetric about $\theta = 0°$

$$\cos(90 - \theta)° = \sin\theta°$$
$$\cos(-\theta) = \cos\theta$$
$$f(-\theta) = \cos\theta = f(\theta)$$

$\therefore$ Cosine is classed as an even function and the graph is symmetric about the *y*-axis.



Tangent Properties:

Periodic: every 180° or π radians

Hence: $\tan\theta° = \tan(\theta \pm 180)°$

& $\tan\theta = \tan(\theta \pm \pi)$

$$\tan(-\theta) = -\tan\theta$$

Asymptotes occur at odd multiples of 90°:

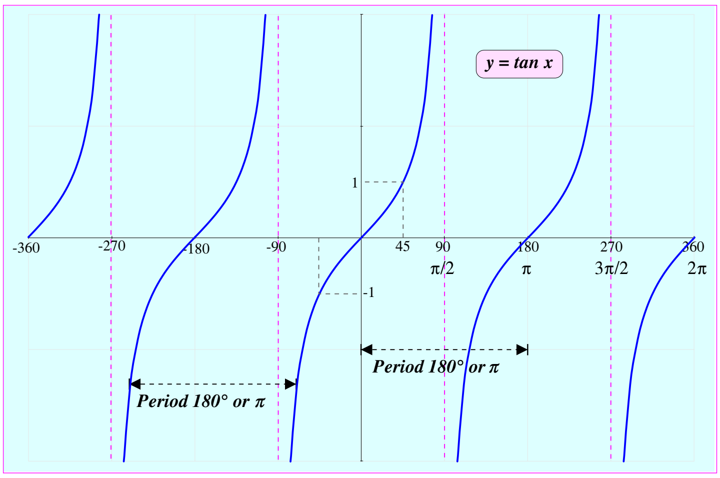
$$\theta = (2n + 1)90°$$

or: $\theta = \dfrac{\pi}{2} + n\pi$

i.e. $\pm 90°, \quad \pm 270°, \quad \ldots$

$$f(-\theta) = -\tan\theta = -f(\theta)$$

$\therefore$ tangent is classed as an odd function and the graph has rotational symmetry, order 2, about the origin.

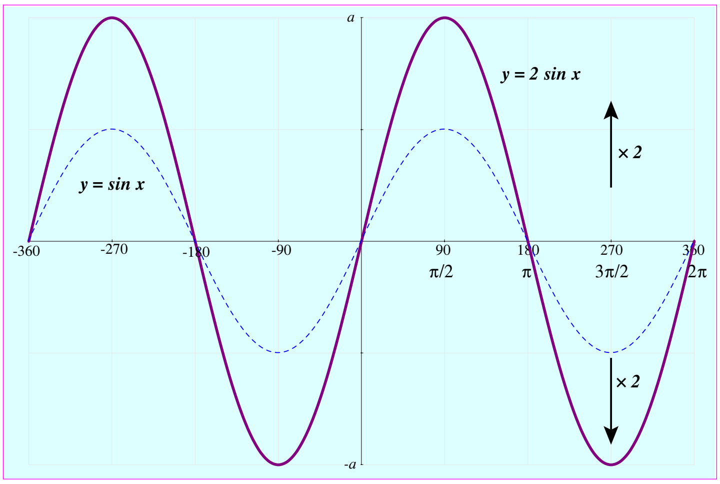


32.2 Transformation of Trig Graphs

Vertical Stretch

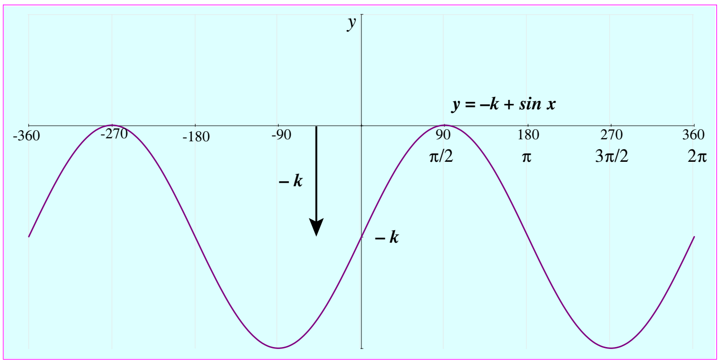
$$y = a \sin(x)$$

Scale factor = $\times\, a$



Vertical translation

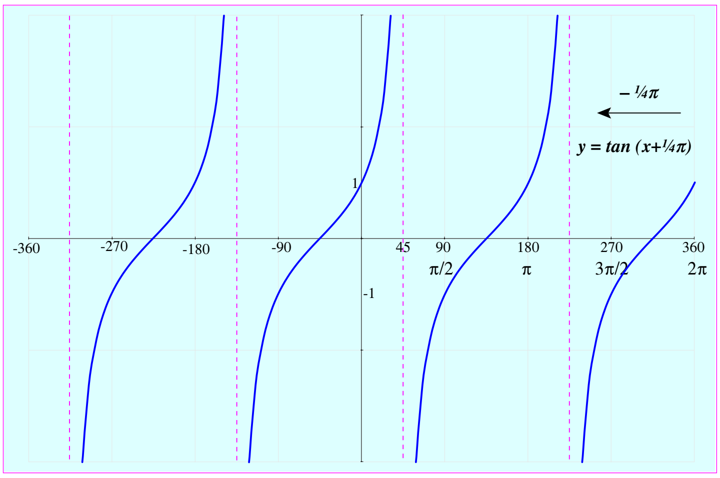
$$y = -k + \sin(x)$$



Horizontal translation

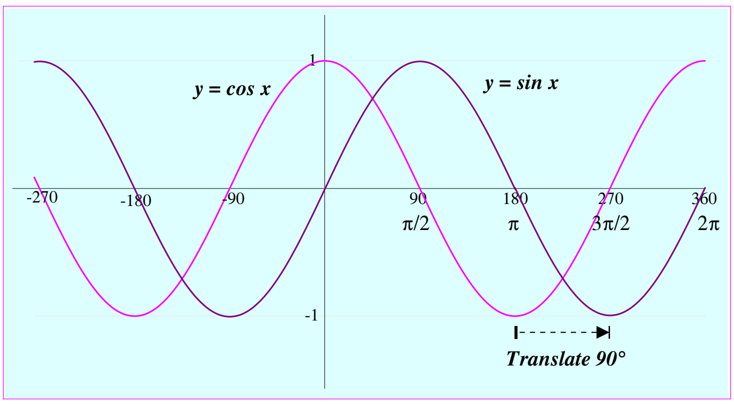
$$y = \tan\left(x + \tfrac{1}{4}\pi\right)$$

This means a translation to the LEFT!



Note that translating the cosine graph by 90°, gives a sine wave. Hence:

$$\cos(\theta - 90)° = \sin\theta°$$

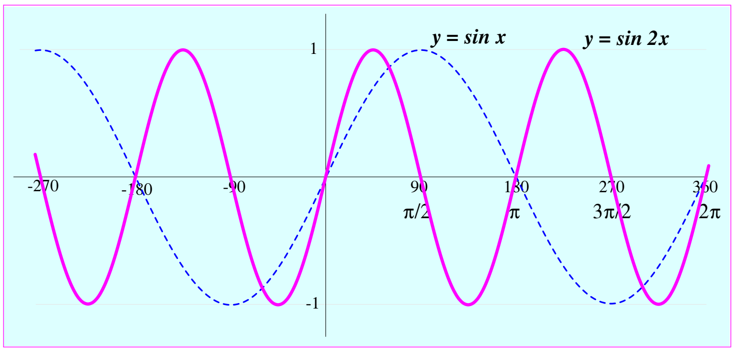


Stretches in the x axis.

Map $y = sin\,x$ to $y = sin\,2x$

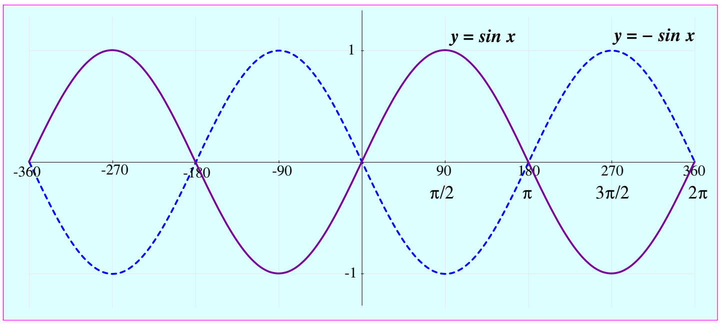
The graph is stretched parallel to the x-axis with a scale factor of ½.

Scale factor $= \times \frac{1}{a}$



Reflection in the x axis.

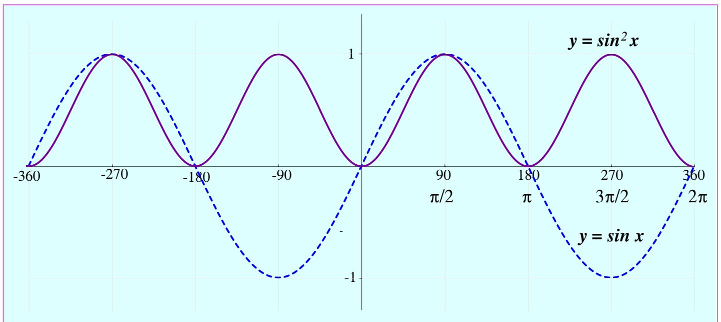
Map $y = sin\,x$ to $y = -sin\,x$



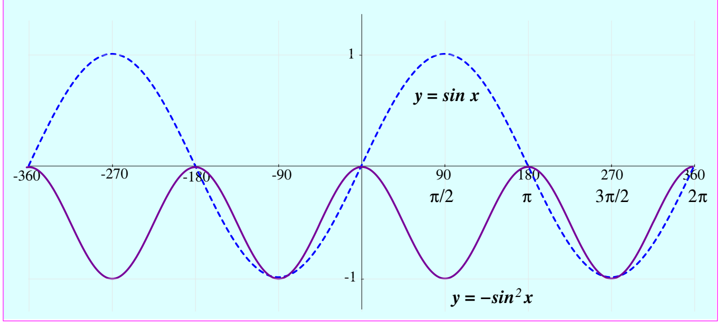
32.3 Graphs of Squared Trig Functions

It is worth being familiar with the graphs for squared trig functions. Note how the curves remain in positive territory, as you would expect when something is squared.

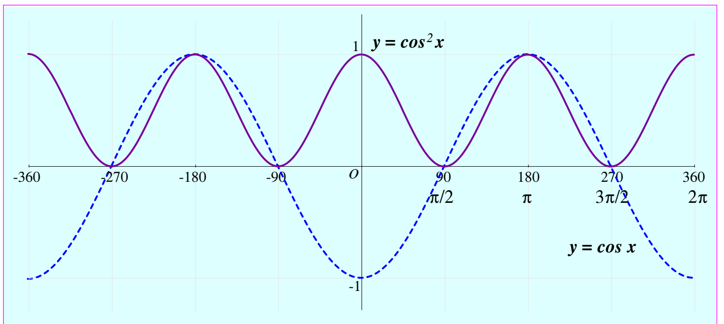
$y = sin^2 x$



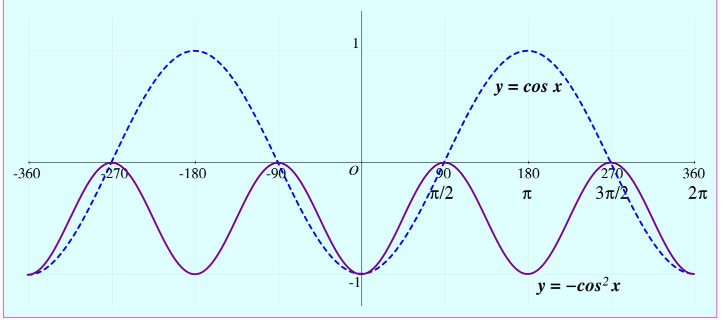
$y = -sin^2 x$



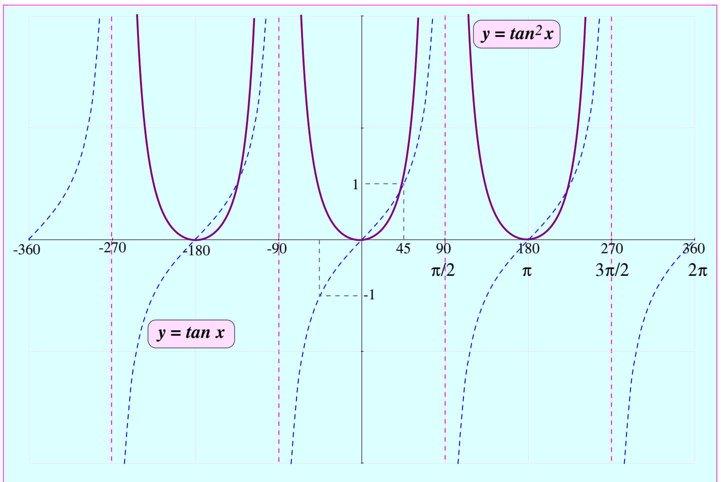
$y = cos^2 x$



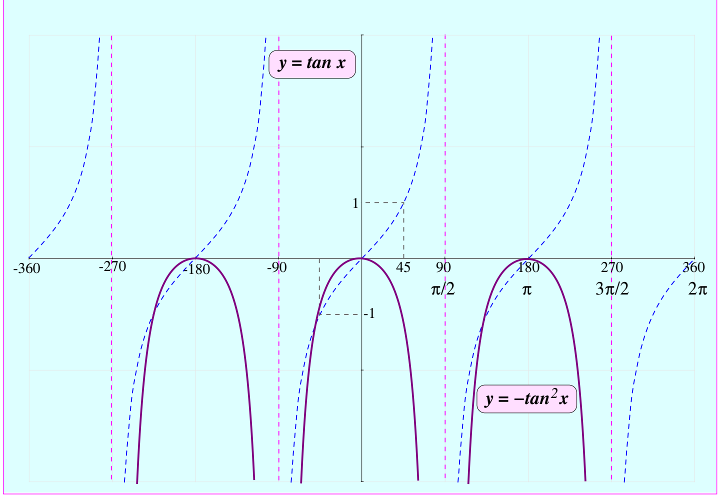
$y = -cos^2 x$



$y = tan^2 x$



$y = -tan^2 x$



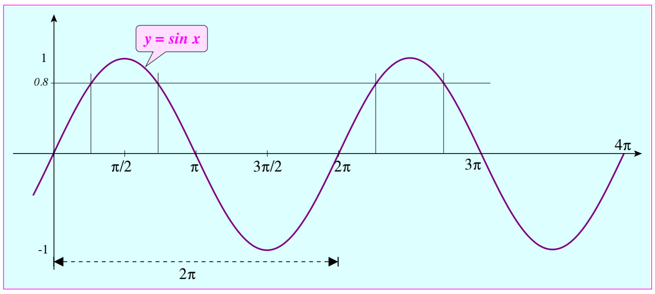
32.4 Worked Examples

32.4.1 Example:

Solve

$$\sin 2\theta = 0{\cdot}8$$

$$0 \leqslant \theta \leqslant 2\pi$$



Solution:

$$2\theta = \sin^{-1}(0{\cdot}8) \qquad \text{(radian mode set)}$$

$$2\theta = 0{\cdot}92729 \qquad \text{(principal value)}$$

Based on the period of 2π for a sine function, the values are:

$$2\theta = 0{\cdot}927,\ \pi - 0{\cdot}927,\ 2\pi + 0{\cdot}927,\ 3\pi - 0{\cdot}927,\ 4\pi + 0{\cdot}927$$

Since the limits have been defined as $0 \leqslant \theta \leqslant 2\pi$ then for 2θ, limits are $0 \leqslant 2\theta \leqslant 4\pi$

Hence: $\quad 2\theta = 0{\cdot}927,\ 2{\cdot}214,\ 7{\cdot}210,\ 8{\cdot}497$

$$\theta = 0{\cdot}464,\ 1{\cdot}11,\ 3{\cdot}61,\ 4{\cdot}25\ (2dp)$$

32.5 Transformation Summary

$$y = a \sin(bx - c) + k$$

- a: affects Amplitude (Vertical stretch)
- b: affects Period (Horizontal stretch)
- c: affects Horizontal translation
- k: affects Vertical translation

Amplitude is given by:

$|a| = amplitude$ (not applicable directly to $\tan x$)

$a \quad \Rightarrow \quad$ vertical stretch of any trig ratio by a factor of a

e.g. $\quad y = -5 \sin x \quad \Rightarrow$ vertical stretch by a factor of 5, a reflection in the x-axis & amplitude of 5

Period is given by:

For sin & cos $\quad Period = \dfrac{360°}{|b|} \quad or \quad \dfrac{2\pi}{|b|}$

e.g. $\; y = \sin(-4x) \Rightarrow$ horizontal stretch by a factor of 1/4, reflection in the y-axis & a period of 90° or $\dfrac{\pi}{2}$

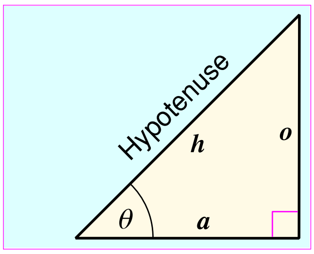
For tan $\quad Period = \dfrac{180°}{|b|} \quad or \quad \dfrac{\pi}{|b|}$

Changing a or b	$y = a \sin bx$	$y = a \cos bx$	$y = a \tan bx$	
$a > 1$	vertical stretch, (expansion) scale factor of a			y axis
$0 < a < 1$	vertical stretch, (compression) scale factor of $\frac{1}{a}$			y axis
$a < 0$	vertical stretch, scale factor of a, with reflection in the x-axis			y axis
$b > 1$	horizontal stretch, (compression) scale factor of $\frac{1}{b}$			x axis
$0 < b < 1$	horizontal stretch, (expansion) scale factor of b			x axis
$b < 0$	horizontal stretch, (compression) scale factor of $\frac{1}{b}$, with reflection in the y-axis			x axis
Amplitude	$\lvert a \rvert = \dfrac{max - min}{2}$		Not applicable	
Period (degrees)	$\dfrac{360°}{\lvert b \rvert}$	$\dfrac{360°}{\lvert b \rvert}$	$\dfrac{180°}{\lvert b \rvert}$	
Period (radians)	$\dfrac{2\pi}{\lvert b \rvert}$	$\dfrac{2\pi}{\lvert b \rvert}$	$\dfrac{\pi}{\lvert b \rvert}$	

33 • C2 • Trig Identities

33.1 Basic Trig Ratios

A reminder of the basic trig ratios:



$$sin\,\theta = \frac{opposite}{hypotenuse} = \frac{o}{h}$$

$$cos\,\theta = \frac{adjacent}{hypotenuse} = \frac{a}{h}$$

$$tan\,\theta = \frac{opposite}{adjacent} = \frac{o}{a} \text{ = gradient of hypotenuse}$$

Degrees	0	30	45	60	90
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*asymtote*

Recall that $(cos\,\theta)^2$ is written as $cos^2\,\theta$ etc., but $cos^{-1}\,\theta$ means the inverse of $cos\,\theta$ not the reciprocal $\frac{1}{cos\,\theta}$. Similarly for *sin* and *tan*.

$$tan\,\theta \equiv tan\,(\theta \pm 180)$$

$$cos\,\theta = cos\,(-\theta) \qquad \therefore \text{ even}$$

$$sin\,(-\theta) = -sin\,(\theta) \qquad \therefore \text{ odd}$$

33.2 Identity *tan x* ≡ *sin x* / *cos x*

From the basic definitions we have:

$$tan\,\theta = \frac{o}{a} = \frac{h\,sin\,\theta}{h\,cos\,\theta} = \frac{sin\,\theta}{cos\,\theta}$$

$$tan\,\theta \equiv \frac{sin\,\theta}{cos\,\theta}$$

(When $cos\,\theta = 0$, $tan\,\theta$ is not defined. i.e. when $\theta = (2n + 1)\frac{\pi}{2}$

33.3 Identity $sin^2x + cos^2x \equiv 1$

From pythag: $a^2 + o^2 = h^2$

$$(h\,cos\,\theta)^2 + (h\,sin\,\theta)^2 = h^2$$

$$h^2\,cos^2\theta + h^2\,sin^2\theta = h^2$$

$$cos^2\theta + sin^2\theta \equiv 1$$

≡ means true for all values of θ)

33.4 Solving Trig Problems with Identities

33.4.1 Problems of the form: $p \sin x \pm q \cos x = k$

Division by *sin* or *cos* will render the equation in terms of *tan*.

33.4.1.1 Example:

1 Solve $4 \sin\theta - \cos\theta = 0$ where $0° \leqslant \theta \leqslant 180°$

Solution:

$$4 \sin\theta - \cos\theta = 0$$

$$4 \sin\theta = \cos\theta$$

$$4 \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta}$$

$$4 \tan\theta = 1$$

$$\theta = \tan^{-1}\frac{1}{4}$$

$$\theta = 14° \ (2\ sf)$$

2

33.4.2 Problems of the form: $p \sin x \pm q \cos^2 x = k$

Change the equation to be in terms of *sin* or *cos* by using the identity $\cos^2\theta + \sin^2\theta \equiv 1$.

33.4.2.1 Example:

1 Solve $5\cos^2\theta + 4\sin\theta = 5$ where $0 \leqslant \theta \leqslant \pi$

Solution:

$$5\cos^2\theta + 4\sin\theta = 5$$

$$5(1 - \sin^2\theta) + 4\sin\theta - 5 = 0$$

$$5 - 5\sin^2\theta + 4\sin\theta - 5 = 0$$

$$-5\sin^2\theta + 4\sin\theta = 0$$

$$5\sin^2\theta - 4\sin\theta = 0$$

$$\sin\theta(5\sin\theta - 4) = 0$$

$$\sin\theta = 0 \text{ or } \sin\theta = \frac{4}{5}$$

$$\theta = 0,\ \pi,\ 0{\cdot}93,\ 2{\cdot}21$$

2 Show that $5\,tan\,\theta\,sin\,\theta = 24$ can be written as $5\,cos^2\,\theta + 24\,sin\,\theta - 5 = 0$ and solve $5\,tan\,\theta\,sin\,\theta = 24$ for $0° \leqslant \theta \leqslant 360°$.

Solution:

$$5\,tan\,\theta\,sin\,\theta = 24$$

$$5\frac{sin\,\theta}{cos\,\theta}sin\,\theta = 24$$

$$5\,sin^2\,\theta = 24\,cos\,\theta$$

$$5(1 - cos^2\theta) = 24\,cos\,\theta$$

$$5 - 5\,cos^2\theta - 24\,cos\,\theta = 0$$

$$5\,cos^2\theta + 24\,cos\,\theta - 5 = 0$$

$$(5cos\,\theta - 1)(cos\,\theta + 5) = 0$$

$$cos\,\theta = \frac{1}{5} \qquad (cos\,\theta = -5 \text{ not a valid solution})$$

$$\therefore \qquad \theta = 78{\cdot}5°,\ 281{\cdot}5$$

33.4.3 Proving Other Identities

The standard identities can be used to prove other identities. Usually proved by taking the more complex side if the identity and manipulating it to equal the simpler side.

33.4.3.1 Example:

1 Prove the identity $(sin\,\theta - cos\,\theta)^2 + (sin\,\theta + cos\,\theta)^2 = 2$

Solution:

$$LHS = (sin\,\theta - cos\,\theta)^2 + (sin\,\theta + cos\,\theta)^2$$

$$= (sin^2\theta - 2sin\,\theta\,cos\,\theta + cos^2\theta) + (sin^2\theta + 2sin\,\theta\,cos\,\theta + cos^2\theta)$$

$$= 2(sin^2\theta + cos^2\theta)$$

$$\text{now} \qquad (sin^2\theta + cos^2\theta) = 1$$

$$LHS = 2$$

$$LHS = RHS$$

2

33.5 Trig Identity Digest

33.5.1 Trig Identities

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\sin\theta \equiv \cos\left(\frac{1}{2}\pi - \theta\right) \qquad \sin x = \cos(90° - x)$$

$$\cos\theta \equiv \sin\left(\frac{1}{2}\pi - \theta\right) \qquad \cos x = \sin(90° - x)$$

33.5.2 Pythagorean Identities

$$\cos^2\theta + \sin^2\theta \equiv 1$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

33.5.3 General Trig Solutions

- Cosine
  - The principal value of $\cos\theta = k$ is as per your calculator where $\theta = \cos^{-1}k$
  - A second solution is found at $\theta = 360 - \cos^{-1}k \qquad (\theta = 2\pi - \cos^{-1}k)$
  - Thereafter, add or subtract multiples of 360° (or 2π)
  - k valid only for $-1 \leqslant k \leqslant 1$
- Sine
  - The principal value of $\sin\theta = k$ is as per your calculator where $\theta = \sin^{-1}k$
  - A second solution is found at $\theta = 180 - \sin^{-1}k \qquad (\theta = \pi - \sin^{-1}k)$
  - Thereafter, add or subtract multiples of 360° (or 2π)
  - k valid only for $-1 \leqslant k \leqslant 1$
- Tan
  - The principal value of $\tan\theta = k$ is as per your calculator where $\theta = \tan^{-1}k$
  - A second solution is found at $\theta = 180 + \tan^{-1}k \qquad (\theta = \pi + \tan^{-1}k)$
  - Thereafter, add or subtract multiples of 360° (or 2π)
  - k valid for $k \in \mathbb{R}$
- Complementary angles add up to 90°
  - $\sin(90 - \theta) = \cos\theta$
  - $\cos(90 - \theta) = \sin\theta$
  - $\tan(90 - \theta) = \cot\theta$
- Supplementary angles add up to 180°
  - $\sin(180 - \theta) = \sin\theta$
  - $\cos(180 - \theta) = -\cos\theta$
  - $\tan(180 - \theta) = -\tan\theta$

34 • C2 • Trapezium Rule

34.1 Estimating Areas Under Curves

Normally, areas under a curve are calculated by using integration, however, for functions that are really difficult to integrate, other methods have to be used to give a good approximation.

In the syllabus there are three methods you need to know:

- The Trapezium rule – covered here
- The Mid-ordinate Rule – C3 (AQA requirement)
- Simpson's Rule – C3

All these methods are based on the premise of dividing the area under the curve into thin strips, calculating the area of each strip and then summing these areas together to find an overall estimate. Clearly, the more strips that are used, the more accurate the answer, and in practise, many hundreds of strips would be chosen with results being calculated electronically. In the exam, calculations with up to 5 ordinates may be required.

Each method has its advantages and disadvantages.

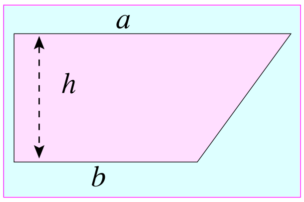
34.2 Area of a Trapezium

Recall that the area of a trapezium is given by:

$$\frac{1}{2}(a + b)h$$

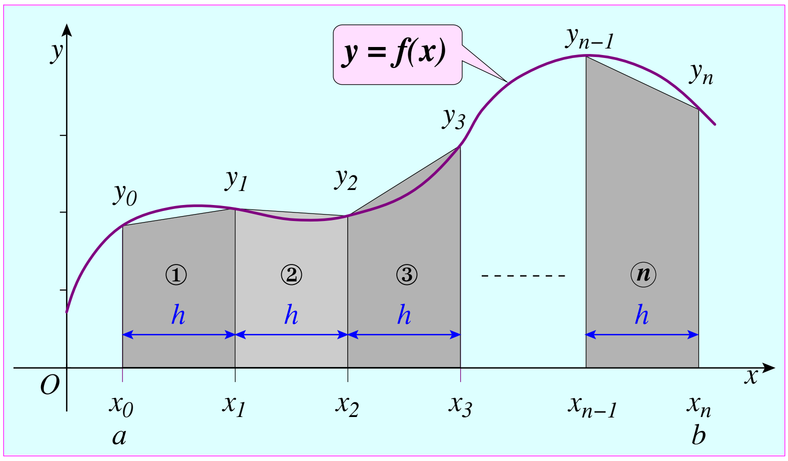
where a and b are the length of the parallel sides and h is the distance between the parallel lines of the trapezium.

A triangle can be considered a special trapezium, with one side length zero.



34.3 Trapezium Rule

An approximation of the area under a curve, between two values on the x-axis, can be found by dividing up the area into n strips, of equal length h. The lines dividing the strips are called **ordinates**, and for convenience are labelled x_0, x_1, x_2… x_n. The length of these lines represents the values of y. There are $n + 1$ ordinates.



Trapezium Rule

For a function $f(x)$ the approximate area is given by:

$$\int_a^b f(x)\, dx = \int_{x_0}^{x_n} f(x)\, dx \approx \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

where $h = \dfrac{b - a}{n}$ and n = number of strips

The value of the function for each ordinate is given by:

$$y_i = f(x_i) = f(a + ih)$$

and where i is the ordinate number.

In simpler terms:

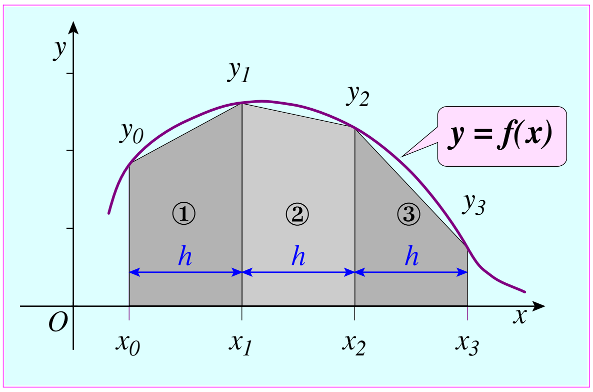
$$A \approx \frac{\text{width}}{2}\left[(\text{First} + \text{last}) + 2 \times \text{the sum of the middle } y \text{ values}\right]$$

To use the trapezium rule, ensure that the part of the curve of interest is either **all above** or **all below** the x-axis, such that y is either $y > 0$ *OR* $y < 0$.

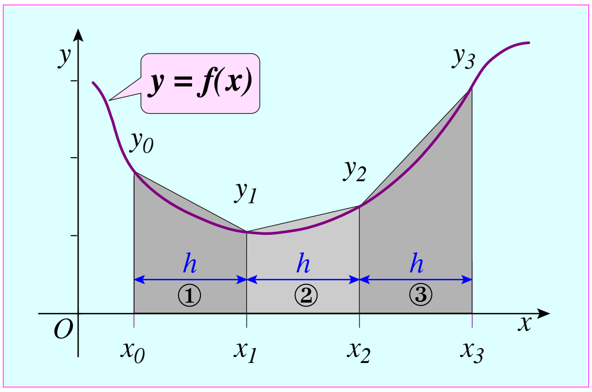
34.4 Trapezium Rule Errors

Depending on the shape of the original function, the trapezium rule may over or under estimate the true value of the area. The examples below show how the estimates may vary.

In this example, the area will be under estimated.



In this example, the area will be over estimated.



34.5 Trapezium Rule: Worked Examples

34.5.1 Example:

1 Use the trapezium rule, with 5 ordinates, to estimate

$$\int_1^3 2x^2 + 4$$

Solution:

First find the value of h from the given information, then set up a table to calculate the required values of y. Finally add the values together in the approved manner.

With 5 ordinates there are 4 strips.

$$h = \frac{b - a}{n} = \frac{3 - 1}{5 - 1} = \frac{1}{2}$$

Ordinate No	x_i	y_i
0	$x_0 = 1$	6·0
1	$x_1 = 1.5$	8·5
2	$x_2 = 2$	12·0
3	$x_3 = 2.5$	16·5
4	$x_4 = 3$	22·0

$$\int_1^3 2x^2 + 4\,dx \approx \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

$$\approx \frac{1}{2} \times \frac{1}{2}[6 + 22 + 2(8{\cdot}5 + 12 + 16{\cdot}5)]$$

$$\approx \frac{1}{4}[28 + 74]$$

$$\approx 25{\cdot}50 \ Sq\ units$$

Compare this answer with the fully integrated value which is 25·33

Hence, the trapezium rule gives a slight over estimate in this case.

2 Use the trapezium rule, with 4 intervals, to estimate

$$\int_{-4}^0 \frac{12}{x + 6}$$

Solution:

$$h = \frac{b - a}{n} = \frac{0 - (-4)}{4} = 1$$

Ordinate No	x_i	y_i
0	$x_0 = -4$	6·0
1	$x_1 = -3$	4·0
2	$x_2 = -2$	3·0
3	$x_3 = -1$	2·4
4	$x_4 = 0$	2·0

$$\int_{-4}^0 \frac{12}{x + 6} \approx \frac{1}{2}[6 + 2 + 2(4 + 3 + 2{\cdot}4)]$$

$$\approx \frac{1}{2}[8 + 2(9{\cdot}4)] = 13{\cdot}4 \ Sq\ units$$

3 Use the trapezium rule, with 2 strips and width 3, to estimate

$$\int_3^9 log_{10} x \, dx$$

Solution:

$$h = \frac{b - a}{n} = \frac{9 - 3}{2} = 3$$

Ordinate No	x_i	y_i
0	$x_0 = 3$	0·477
1	$x_1 = 6$	0·778
2	$x_2 = 9$	0·954

$$\int_3^9 log_{10} x \, dx \approx \frac{3}{2}[0{\cdot}477 + 0{\cdot}954 + 2(0{\cdot}778)]$$

$$\approx 4{\cdot}481 \; sq \; units$$

4 Use the trapezium rule, with 4 strips to estimate

$$\int_0^{\frac{\pi}{2}} cos \, x \, dx$$

Solution:

$$h = \frac{b - a}{n} = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

Ordinate No	x_i	y_i
0	$x_0 = 0$	1·000
1	$x_1 = \frac{\pi}{8}$	0·924
2	$x_2 = \frac{\pi}{4}$	0·707
3	$x_3 = \frac{3\pi}{8}$	0·383
4	$x_4 = \frac{\pi}{2}$	0·000

$$\int_0^{\frac{\pi}{2}} cos \, x \, dx \approx \frac{\pi}{16}[1{\cdot}0 + 0{\cdot}0 + 2(0{\cdot}924 + 0{\cdot}707 + 0{\cdot}383)]$$

$$\approx 0{\cdot}987 \; sq \; units$$

NOTE: as with all things to do with integrals, you must use radians when trig functions are discussed. The limits, in terms of π, will be a strong clue here.

34.6 Topical Tips

Exam hints:

- Always start the counting of the ordinates from zero, in which case the last ordinate will have a subscript value equal to the number of strips
- The number of strips is always one less that the number of ordinates. (Fence post problem!)
- Draw a sketch, even if you don't know what the function really looks like
- Ensure that the part of the curve of interest is either all above or all below the x-axis, such that y is either $y > 0$ *OR* $y < 0$.
- Don't use these numerical methods unless specified by the question or it is clear that no other method is available.
- In all other cases use the full integral methods.

35 • C2 • Integration I

AQA C1 / C2 / OCR C2

35.1 Intro: Reversing Differentiation

Once we have found the differential of a function the question becomes, ‘can we reconstruct the original function from the differential?’ The short answer is ‘yes’, but with a small caveat.

The process of reversing differentiation is called integration, and we find that differentiation & integration are inverse processes.

$$\text{If} \quad y = ax^n \quad \text{then} \quad \frac{dy}{dx} = anx^{n-1}$$

To reverse the process we would have to increase the power of x by 1, and divide by this new power.

$$\int anx^{n-1}\,dx = \frac{anx^{n-1+1}}{n-1+1} = ax^n$$

So far so good, but if our original function included a constant term, which differentiates to zero, how are we to reconstruct this constant. In short we can’t, unless we have some more information to hand. What we can do is add an arbitrary constant, c, which means that integration will give us not one reconstructed function, but a whole family of similar curves.

$$\text{If} \quad \frac{dy}{dx} = bx^n \quad \text{then} \quad \int bx^n\,dx = \frac{b}{n+1}x^{n+1} + c$$

$$\therefore \quad y = \frac{b}{n+1}x^{n+1} + c$$

This general form of integration, where c is not defined, is called **Indefinite Integration**.

There is one other restriction on integrating this form of function, which is that a value of $n = -1$ is not allowed, as it results in division by zero, as seen below:

$$\text{If} \quad \frac{dy}{dx} = 2x^{-1} \quad \text{then} \quad \int 2x^{-1}\,dx = \frac{2}{-1+1}x^{-1+1} + c$$

$$= \frac{2}{0}x^0 + c$$

This problem is tackled in later modules. Hence:

$$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} + c \qquad n \neq -1$$

Of course integration can also be used to find the gradient function from a second derivative.

$$\int f''(x)\,dx = f'(x) + c$$

35.2 Integrating a Constant

In integrating a constant, consider the constant to be $k = kx^0$, hence:

$$\int kx^0 = \frac{k}{0+1}x^{0+1} + c$$

$$= kx + c$$

35.3 Integrating Multiple Terms

Using function notation; the following is true:

If $\frac{dy}{dx} = f'(x) \pm g'(x)$ then $y = \int f'(x) \pm g'(x)\, dx = \int f'(x)\, dx \pm \int g'(x)\, dx$

In other words, we integrate each term individually. When integrating, you will need to put the function in the right form. Only one constant of integration is required.

- Terms have to be written as a power function before integrating, e.g. $\sqrt{x} = x^{\frac{1}{2}}$
- Brackets must be removed to provide separate terms before integrating, e.g. $(x - 4)(x - 1) \Rightarrow x^2 - 5x + 4$
- An algebraic division must be put into the form $ax^n + bx^{n-1} \ldots c$ e.g. $y = \frac{x^4 + 7}{x^2} = x^2 + 7x^{-2}$
- Only one constant of integration is required

35.3.1 Example:

1 Find $\int (3x - 1)^2\, dx$.

$$\int (3x - 1)^2\, dx = \int \left(9x^2 - 6x + 1\right) dx$$

$$= \int 9x^2\, dx - \int 6x\, dx + \int 1\, dx$$

$$= \frac{9}{3}x^3 - \frac{6}{2}x^2 + x + c$$

$$= 3x^3 - 3x^2 + x + c$$

35.4 Finding the Constant of Integration

The Constant of Integration can be found if a point on the original curve is known.

35.4.1 Example:

1 Find the equation of a curve, which passes through the point (1, 4) and which has the gradient function of $f'(x) = 9x^2 - 2x$

$$f'(x) = 9x^2 - 2x$$

$$f(x) = \int \left(9x^2 - 2x\right) dx$$

$$= \frac{9}{3}x^3 - \frac{2}{2}x^2 + c$$

$$\therefore \quad f(x) = 3x^3 - x^2 + c$$

To find c, substitute the value for x & y. Since $f(1) = 4$ then:

$$4 = 3 - 1 + c$$

$$c = 4 - 3 + 1$$

$$c = 2$$

$\therefore$ The original function is: $f(x) = 3x^3 - x^2 + 2$

35.5 The Definite Integral – Integration with Limits

By integrating between limits, we find a definite answer to the integral, rather than a generic family of curves, and hence this is called the **Definite Integral**.

The symbology is:

If $$\int f'(x)\, dx = f(x) + c$$

then $$\int_a^b f'(x)\, dx = [f(x)]_a^b = f(b) - f(a)$$

where a = the lower limit of x

b = the upper limit of x

dx is the operator which tells us what variable is being integrated, and which limits should be used.

35.5.1 Example:

1 Find $\int_0^2 3x^2\, dx$.

$$\int_0^2 3x^2\, dx = [x^3 + c]_0^2 = [x^3 + c]^2 - [x^3 + c]_0$$

$$= 8 + c - (1 + c)$$

$$= 8 - 1 + c - c$$

$$= 7$$

Note that the constant of integration, c, is cancelled out. So this is not required in definite integrals.

2 Find $\int_1^a 6x^{-2}\, dx$ and express the answer in terms of a. Deduce the value of $\int_1^\infty 6x^{-2}\, dx$

Solution:

$$\int_1^a 6x^{-2}\, dx = [-6x^{-1}]_1^a = \left[-\frac{6}{x}\right]_1^a$$

$$= \left[-\frac{6}{a}\right] - \left[-\frac{6}{1}\right]$$

$$= 6 - \frac{6}{a}$$

$$\int_1^\infty 6x^{-2}\, dx = 6 - \frac{6}{\infty}$$

$$= 6$$

If $a = \infty$, then $\frac{6}{a} \Rightarrow 0$

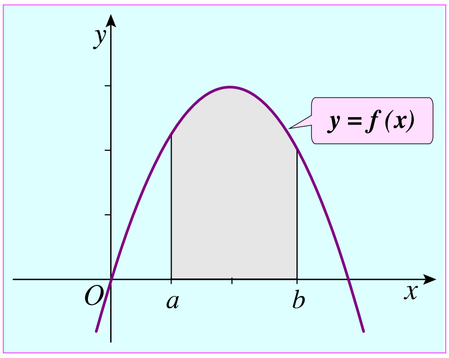
35.6 Area Under a Curve

As briefly explained in C1, integration is a way of finding the area under a curve, as well as finding the original function from the gradient function.

Limits are nearly always used in finding the area under a curve.

35.6.1 Area between the curve and the x-axis

The area under a curve, $y = f(x)$, and the x-axis, between the limits of $x = a$, & $x = b$ is given by

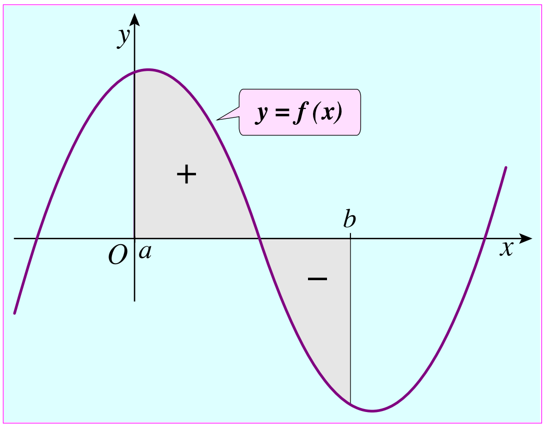


$$\int_a^b f(x)\ dx$$

In finding the area below the curve, the integral returns a +ve answer if the curve is above the x-axis, and a −ve answer if below the x-axis.

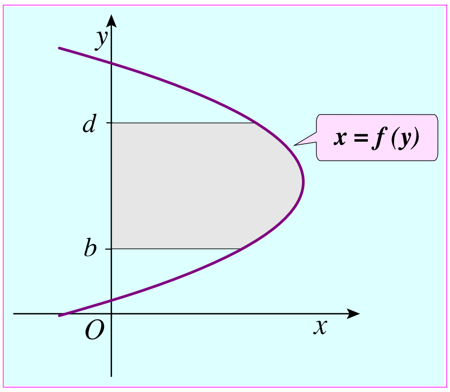
To find the total area, you need to split the areas into two regions and integrate separately and finally add the areas together.

If not split, and the function is integrated over the whole area of $a \to b$, some of the area will be cancelled out.



35.6.2 Area between the curve and the y-axis

The area under a curve, $x = g(y)$, and the y-axis, between the limits of $y = c$, & $y = d$ is given by



$$\int_c^d g(y)\ dy$$

35.6.3 Example:

1

Find $\int_0^{16} \frac{1}{\sqrt{x}}\, dx$

$$\int_0^{16} \frac{1}{\sqrt{x}}\, dx = \int_0^{16} x^{-\frac{1}{2}}\, dx$$

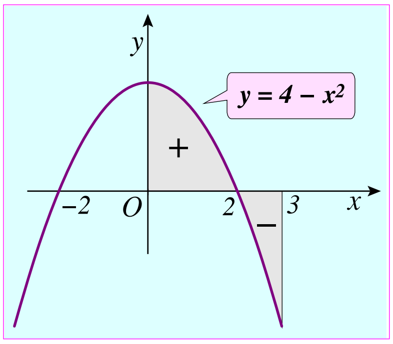
$$= \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{16} = \left[2\sqrt{x}\right]_0^{16} = 2\sqrt{16}$$

$$= 8$$

2

Find the area under the curve, $y = 4 - x^2$ between $x = 0$ & $x = 3$.

Draw a sketch to clarify your thinking, for which you will need to find the roots.



Solution:

The integral has to be taken in 2 parts. The positive part between $x = 0$ & $x = 2$ and the negative part between $x = 2$ & $x = 3$. Then the areas obtained can be added together.

Area 1 $\quad \int_0^2 4 - x^2\, dx = \left[4x - \frac{x^3}{3}\right]_0^2$

$$= \left[8 - \frac{8}{3}\right] - \left[0\right]$$

$$= 8 - \frac{8}{3}$$

$$= \frac{16}{3}$$

Area 2 $\quad \int_2^3 4 - x^2\, dx = \left[4x - \frac{x^3}{3}\right]_2^3$

$$= \left[12 - \frac{27}{3}\right] - \left[8 - \frac{8}{3}\right]$$

$$= 3 - 8 + \frac{8}{3} = -5 + \frac{8}{3}$$

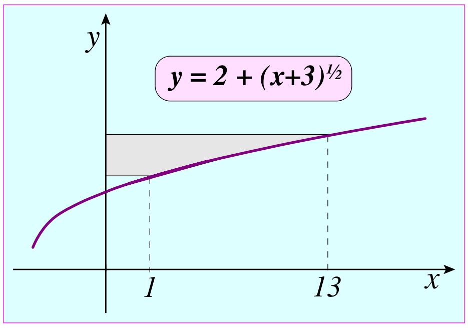
$$= -\frac{7}{3}$$

Total area is: $\quad \frac{16}{3} + \frac{7}{3} = \frac{23}{3} = 7\frac{2}{3}$ square units

3

A curve is given by $y = 2 + \sqrt{x + 3}$

Find the shaded area for limits of
$x = 1, \; x = 13$



Solution:

The plan here is to re-write the equation with x as the subject, and determine the limits on the y axis to use for integration.

$$y = 2 + \sqrt{x + 3} \qquad (1)$$

$$y - 2 = \sqrt{x + 3}$$

$$(y - 2)^2 = x + 3$$

$$x = (y - 2)^2 - 3$$

$$= y^2 - 4y + 4 - 3$$

$$x = y^2 - 4y + 1$$

Set the limits:

From (1) when $x = 1 \qquad y = 2 + \sqrt{1 + 3} = 2 + \sqrt{4}$

$y = 4$

From (1) when $x = 13 \qquad y = 2 + \sqrt{13 + 3} = 2 + \sqrt{16}$

$y = 6$

Set up the integral:

$$\int_4^6 y^2 - 4y + 1 \, dy = \left[\frac{y^3}{3} - \frac{4y^2}{2} + y\right]_4^6 = \left[\frac{y^3}{3} - 2y^2 + y\right]_4^6$$

$$= \left[\frac{6^3}{3} - 2 \times 6^2 + 6\right] - \left[\frac{4^3}{3} - 2 \times 4^2 + 4\right]$$

$$= [72 - 72 + 6] - \left[\frac{64}{3} - 32 + 4\right]$$

$$= [6] - \left[-6\frac{2}{3}\right]$$

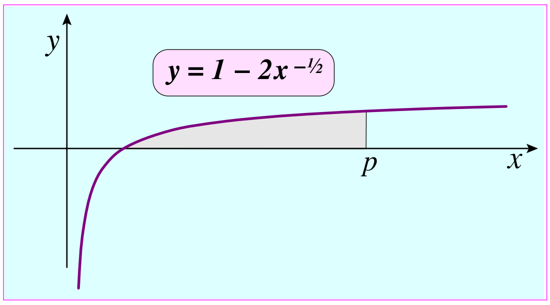
$$= 12\frac{2}{3} \; sq \; units$$

oecfrl

4

The curve is given by $y = 1 - 2x^{-\frac{1}{2}}$

Find the value of p given that the shaded area has an area of 4 square units.



Solution:

The first action is to determine the lower limit of the shaded area and find where the curve crosses the x-axis. Then set up the integration and work out the value of p.

$$y = 1 - 2x^{-\frac{1}{2}} \qquad (1)$$

$$0 = 1 - \frac{2}{\sqrt{x}}$$

$$1 = \frac{2}{\sqrt{x}}$$

$$\sqrt{x} = 2$$

$$x = 4$$

Set up the integral:

$$\int_4^p 1 - 2x^{-\frac{1}{2}}\,dy = \left[x - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^p = \left[x - 4\sqrt{x}\right]_4^p$$

$$= \left[x - 4\sqrt{x}\right]_4^p$$

$$= \left[p - 4\sqrt{p}\right] - \left[4 - 4\sqrt{4}\right]$$

$$= p - 4\sqrt{p} + 4$$

Give area is 4:

$$4 = p - 4\sqrt{p} + 4$$

$$0 = p - 4\sqrt{p}$$

$$p = 4\sqrt{p}$$

$$p^2 = 16p$$

$$p = 16$$

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35.7 Compound Areas

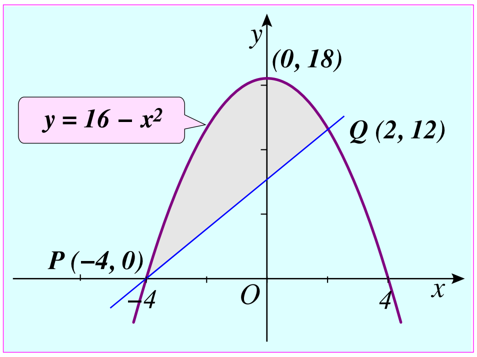
A common question is to find the area between two curves and not just between a curve and the one of the axes. Often there will be ancillary questions that require you to find the roots, or work out limits.

35.7.1 Example:

1

A function is given by $y = 16 - x^2$ and has the line $y = 2x + 8$ intersecting in two places, at points P (−4, 0) and Q (2, 12).

Find the shaded area as shown on the graph.



Solution:

The plan here is to find the area under the curve from $x = -4$ *to* $x = 2$, then subtract the area under the line for the same limits. Either use integration or calculate the area of the triangle formed.

$$\int_{-4}^{2} 16 - x^2\,dx - \frac{1}{2}bh = \left[16x - \frac{x^3}{3}\right]_{-4}^{2} - \frac{6 \times 12}{2}$$

$$= \left[32 - \frac{8}{3}\right] - \left[-64 + \frac{64}{3}\right] - 36$$

$$= 32 - \frac{8}{3} + 64 - \frac{64}{3} - 36$$

$$= 60 - \frac{72}{3} = 60 - 24$$

$$= 36$$

An alternative method is to combine the functions in one integral

$$\int_{-4}^{2} \left[\left(16 - x^2\right) - \left(2x + 8\right)\right] dx = \int_{-4}^{2} 8 - x^2 - 2x\,dx$$

$$= \left[8x - \frac{x^3}{3} - \frac{2x^2}{2}\right]_{-4}^{2}$$

$$= \left[8x - x^2 - \frac{x^3}{3}\right]_{-4}^{2}$$

$$= \left[16 - 4 - \frac{8}{3}\right] - \left[-32 - 16 + \frac{64}{3}\right]$$

$$= \left[12 - \frac{8}{3}\right] - \left[-48 + \frac{64}{3}\right]$$

$$= 12 - \frac{8}{3} + 48 - \frac{64}{3}$$

$$= 60 - \frac{72}{3} = 60 - 24$$

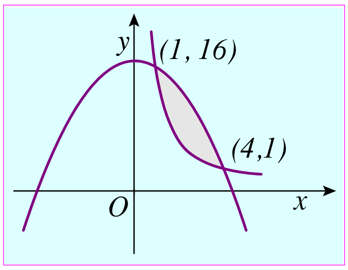
$$= 36$$

2 Find the area between the two curves, $y = 17 - x^2$, & $y = \frac{16}{x^2}$.

Solution:

First, find the intersection points of the two curves to establish the limits of integration.

Then find the area under both curves and subtract the values. A sketch is recommended.



$$17 - x^2 = \frac{16}{x^2}$$

$$17x^2 - x^4 = 16$$

$$17x^2 - x^4 - 16 = 0$$

$$x^4 - 17x^2 + 16 = 0$$

$$(x^2 - 1)(x^2 - 16) = 0$$

$$x^2 = 1 \quad \& \quad x^2 = 16$$

$$\therefore \quad x = 1 \quad \& \quad x = 4$$

$$\therefore \quad y = 16 \quad \& \quad y = 1$$

Intersection is therefore at $(1, 16)$ & $(4, 1)$ and the limits for integration are $x = 1$, $x = 4$

Curve 1 $$\int_1^4 \frac{16}{x^2}\, dx = \int_1^4 16x^{-2}\, dx$$

$$= \left[-16x^{-1}\right]_1^4 = \left[-\frac{16}{4}\right] - \left[-\frac{16}{1}\right]$$

$$= -4 + 16 = 12$$

Curve 2 $$\int_1^4 17 - x^2\, dx = \left[17x - \frac{x^3}{3}\right]_1^4$$

$$= \left[68 - \frac{64}{3}\right] - \left[17 - \frac{1}{3}\right]$$

$$= 68 - 17 - \frac{64}{3} + \frac{1}{3}$$

$$= 30$$

Area between the 2 curves: $30 - 12 = 18$ *sq units*

35.8 More Worked Examples

35.9 Topical Tips

It is usual to state any answer in the same form as the original function in the question. If asked for an exact answer, leave the answer in surd form or in terms of π or e.

Module C3

Core 3 Basic Info

Algebra and functions; Trigonometry; Differentiation and integration; Numerical Methods.

The C3 exam is 1 hour 30 minutes long and is in two sections, and worth 72 marks (75 AQA).
Section A (36 marks) 5 – 7 short questions worth at most 8 marks each.
Section B (36 marks) 2 questions worth about 18 marks each.

OCR Grade Boundaries.

These vary from exam to exam, but in general, for C3, the approximate raw mark boundaries are:

Grade	100%	*A* *	*A*	*B*	*C*
Raw marks	72	61 ± 2	54 ± 2	47 ± 3	40 ± 3
UMS %	100%	90%	80%	70%	60%

The raw marks are converted to a unified marking scheme and the UMS boundary figures are the same for all exams.

C3 Contents

C3 Assumed Basic Knowledge

Knowledge of C1 and C2 is assumed, and you may be asked to demonstrate this knowledge in C3.
You should know the following formulae, (which are NOT included in the Formulae Book).
Graphical calculators are allowed in C3/C4.

1 Trig

$$\sec\theta \equiv \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta \equiv \frac{1}{\sin\theta} \qquad \cot\theta \equiv \frac{1}{\tan\theta}$$

$$\sec^2 A \equiv 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A \equiv 1 + \cot^2 A$$

$$\sin 2A \equiv \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A \equiv \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin 2A \equiv 2\sin A\cos A \qquad \{A = B \text{ in } \sin(A + B)\}$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \qquad \{A = B \text{ in } \cos(A + B)\}$$

$$\cos 2A \equiv 2\cos^2 A - 1 \qquad \{\sin^2 A = 1 - \cos^2 A\}$$

$$\cos 2A \equiv 1 - 2\sin^2 A \qquad \{\cos^2 A = 1 - \sin^2 A\}$$

2 Differentiation and Integration

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\ln x$	$\frac{1}{x}$
e^{kx}	$k\,e^{kx}$
$u\,v$	$u'v + u\,v'$
$\frac{u}{v}$	$\frac{u'v - u\,v'}{v^2}$
	$\frac{dy}{dx} = 1 \div \frac{dx}{dy}$

Function $f(x)$	Integral $\int f(x)\,dx$	
x^n	$\frac{x^{n+1}}{n+1} + c$	$n \neq -1$
e^x	$e^x + c$	
e^{kx}	$\frac{1}{a}e^{kx} + c$	$k \neq 0$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$	
$\frac{1}{ax + b}$	$\frac{1}{a}\ln\lvert ax + b\rvert + c$	

Rates of change $\quad \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Volume of revolution about x axis $\quad V_x = \pi\int_a^b y^2\,dx$

Volume of revolution about y axis $\quad V_y = \pi\int_a^b x^2\,dy$

3 Other

$$R = k\ln(at + b) \quad \Leftrightarrow \quad e^{\frac{R}{k}} = at + b$$

C3 Brief Syllabus

1 Algebra and Functions

- understand the terms function, domain, range, one-one function, inverse function and composition of functions
- identify the range of a given function in simple cases, and find the composition of two given functions
- determine if a given function is one-one, and find the inverse of a one-one function in simple cases
- illustrate in graphical terms the relation between a one-one function and its inverse
- use and recognise compositions of transformations of graphs, such as the relationship between the graphs of $y = f(x)$ & $y = af(x + b)$ - See C2 notes. Combined translations.
- understand the meaning of $|x|$ and use relations such as $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x - a| < |b| \Leftrightarrow a - b < x < a + b$ in solving equations and inequalities
- understand the relationship between the graphs of $y = f(x)$ & $y = |f(x)|$
- understand the exponential & log function properties (e^x & $\ln x$) & their graphs, including their inverse functions
- understand exponential growth and decay.

2 Trigonometry

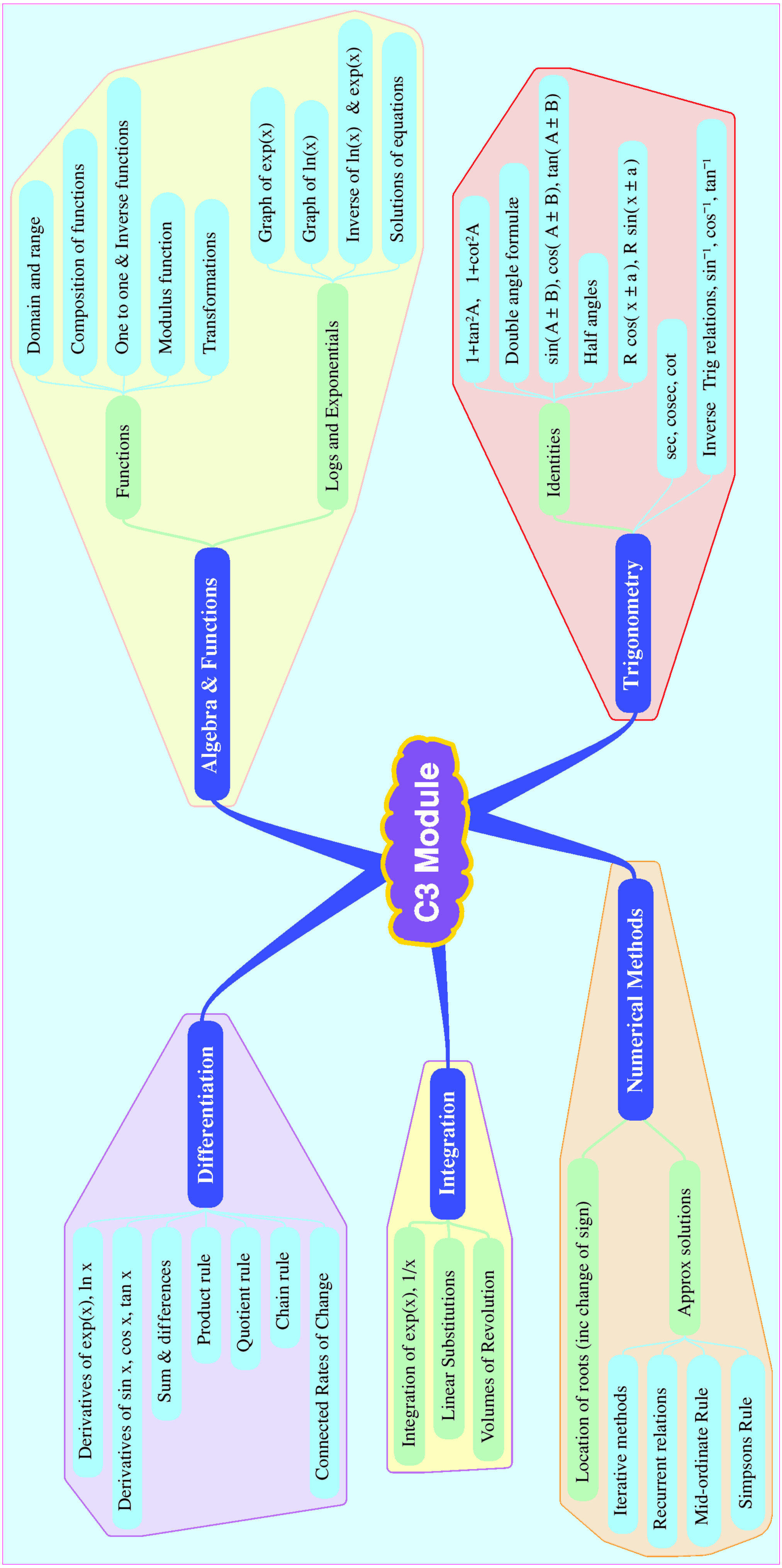
- use the notations $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ to denote the inverse trig relations, and relate their graphs (for the appropriate domains) to those of sine, cosine and tangent
- understand the relationship of the sec, cosec and cotan functions to cos, sin and tan, and use properties and graphs of all six trig functions for all angles
- use trig identities for the simplification and exact evaluation of expressions, and be familiar with the use of
  - $\sec^2 A \equiv 1 + \tan^2 A$ and $\mathrm{cosec}^2 A \equiv 1 + \cot^2 A$
  - the expansions of sin(A ± B), cos(A ± B) and tan(A ± B),
  - the formulae for sin 2A, cos2A and tan 2A,
  - the expression of $a \sin x + b \cos x$ in the forms $R \sin(x \pm \alpha)$ and $R \cos(x \pm \alpha)$.

3 Differentiation and Integration

- use the derivatives of e^x & $\ln x$, together with constant multiples, sums, and differences
- differentiate composite functions using the chain rule
- differentiate products and quotients
- understand and use the relation $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$
- apply differentiation to connected rates of change (chain rule)
- integrate e^x and $\frac{1}{x}$, together with constant multiples, sums, and differences
- integrate expressions involving a linear substitution, e.g. $(3x - 1)^8$, e^{3x+1}
- use definite integration to find a volume of revolution about one of the coordinate axes (including, for example, the region between the curves $y = x^2$ & $y = \sqrt{x}$, rotated about the x-axis.

4 Numerical Methods

- locate approximately a root of an equation, using graphical means and/or searching for a sign-change
- understand the idea, and the notation for a sequence of approximations which converges to a root of an equation
- understand how a simple iterative formula of the form $x_{n+1} = f(x_n)$ relates to an equation being solved, and use a given iteration, or one based on a given rearrangement of an equation, to find a root to a given degree of accuracy (the condition for convergence is not required, but know that an iteration may fail to converge)
- carry out numerical integration of functions by means of Simpson's rule, and mid ordinate rule.


C3 Module
Algebra & Functions
Functions
Domain and range
Composition of functions
One to one & Inverse functions
Modulus function
Transformations
Logs and Exponentials
Graph of exp(x)
Graph of ln(x)
Inverse of ln(x) & exp(x)
Solutions of equations
Trigonometry
Identities
1+tan²A, 1+cot²A
Double angle formulæ
sin(A ± B), cos(A ± B), tan(A ± B)
Half angles
R cos(x ± a), R sin(x ± a)
sec, cosec, cot
Inverse Trig relations, sin⁻¹, cos⁻¹, tan⁻¹
Differentiation
Derivatives of exp(x), ln x
Derivatives of sin x, cos x, tan x
Sum & differences
Product rule
Quotient rule
Chain rule
Connected Rates of Change
Integration
Integration of exp(x), 1/x
Linear Substitutions
Volumes of Revolution
Numerical Methods
Location of roots (inc change of sign)
Approx solutions
Iterative methods
Recurrent relations
Mid-ordinate Rule
Simpsons Rule

36 • C3 • Functions

36.1 Function Intro

Previously, we have glibly used the term 'function' without really defining what a function really is. You may even recall being taught about function machines at primary school!

A function is just a rule that we apply to a number and which generates another number as an answer.

We often talk about a 'function of x' which we take to mean $y = (\textit{something to do with } x)$.

$$\text{e.g.} \quad y = x^2 + 3x + 4$$

We say y is a function of x, where y depends on the value of x, and so we call y the **dependent variable** and x the **independent variable**. To find the value of y, we plug in some selected value of x into our equation and calculate the result. We see that x is the input and y is the output.

In function terminology we replace y with $f(x)$, where $f(x)$ means 'the value of our function f at the point x'. Hence:

$$y = x^2 + 3x + 4$$

$$f(x) = x^2 + 3x + 4$$

$$\text{At the point } x = 2 \qquad f(2) = 2^2 + 3 \times 2 + 4$$

$$= 14$$

$$f(\textit{input}) = (\textit{output})$$

We read $f(x)$ as 'f of x' or $f(2)$ as 'f of 2'.

Note how the x in $f(x)$ is replaced by the value of x. In fact x is acting here as a place holder. We could substitute any symbols we like here; e.g. $f(\nabla) = \nabla^2 + 3\nabla + 4$.

Often we use $f(x)$ to mean the 'function of x', where strictly speaking f is the function and x is the input. We can also say that $f(x)$ means 'the rule of our function, f, applied to the value of x'.

To be considered a true function, our equation must give rise to one, and only one, value in the **output**. If the input gives us two or more values in the output, then it is not a function (more below).

A function can also be written as:

$$f : x \mapsto x^2 + 3x + 4$$

This can be read as 'the function f such that x maps onto $x^2 + 3x + 4$'. In set terminology ':' reads as 'such that'.

(Check your exam board as to how they represent functions).

Although any letter could be used to represent a function, convention is that the letters used are generally restricted to f, g and h, or their corresponding capital letters.

36.2 Mapping Relationships, Domain & Ranges

It is difficult to really talk about functions without a little bit of set theory. You might want to review that elsewhere, (I may put something in the appendices later).

A **relation** is a pair of two numbers (ordered pairs), connected via the function, like the x & y co-ordinates of a graph.

There are four types of relationship to consider, and these are illustrated below, using some simple Venn diagrams. These relationships are:

- Relationships that are functions
  - One to one relationship: where one value in the domain maps to one and only one value in the range. An inverse relationship also exists.
  - Many to one relationship: where more than one value in the domain maps to one and only one value in the range. No inverse relationship exists, (except in cases where the domain is restricted).
- Relationships that are NOT functions
  - One to many relationship: where one value in the domain maps to more than one value in the range.
  - Many to many relationship: where more than one value in the domain maps to more than one value in the range.

A function is just a special sort of relation. In function terminology we talk about functions that map a set of input values to a set of output values.

For any function, the set of values that the input is allowed to take is called the **domain** of the function, and the output is the **range** of the function.

$$f\,(domain) = (range)$$

Note the terminology: The domain is all the input values, the co-domain is all the possible values that could be mapped to, and the range is the actual output values. The range is a subset of the co-domain. Each element of the domain maps to an image in the range. The images are also called the image set.

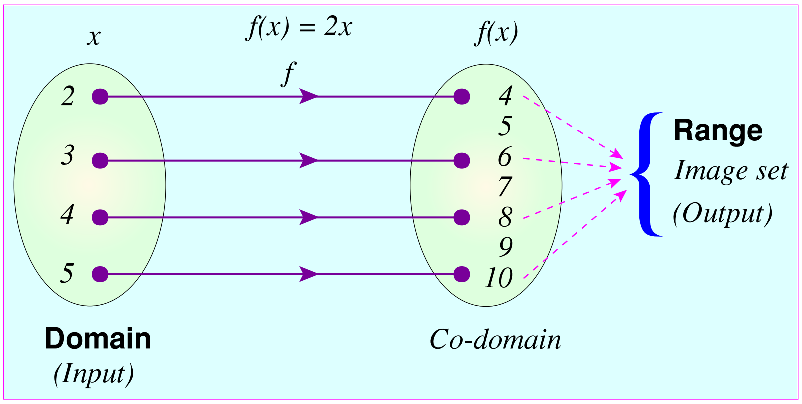
One to one relationship:

showing that one element in the domain maps to one element in the range or image set.

We can say that '4 is the image of 2 under f'

Domain: $x \in \{2, 3, 4, 5\}$
Range: $f(x) \in \{4, 6, 8, 10\}$

This is a function.

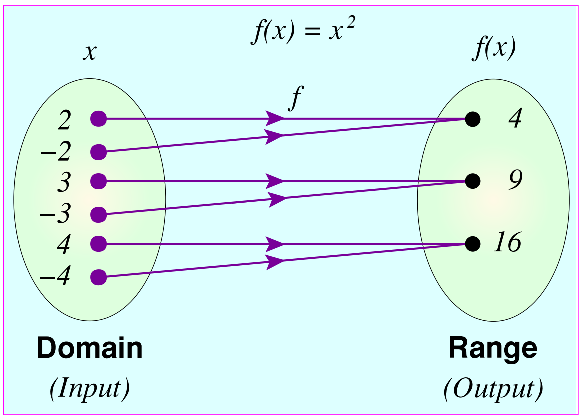


Many to one relationship:

Two or more elements of the domain map to one element in the range.

Domain: $x \in \{-4, -3, -2, 2, 3, 4\}$
Range: $f(x) \in \{4, 9, 16\}$

This is a function.

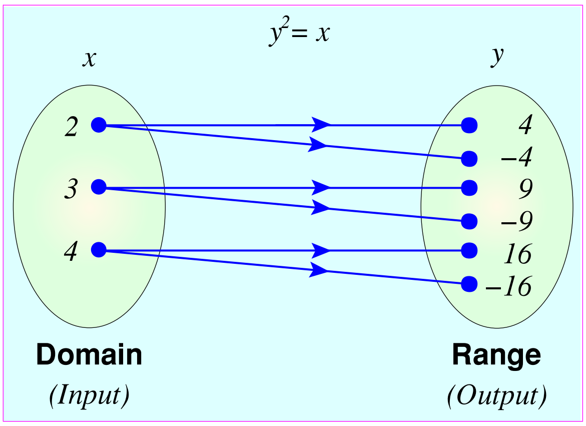


One to many relationship:

One element of the domain maps to two or more elements in the range.

This is NOT a function, (although x is a function of y).

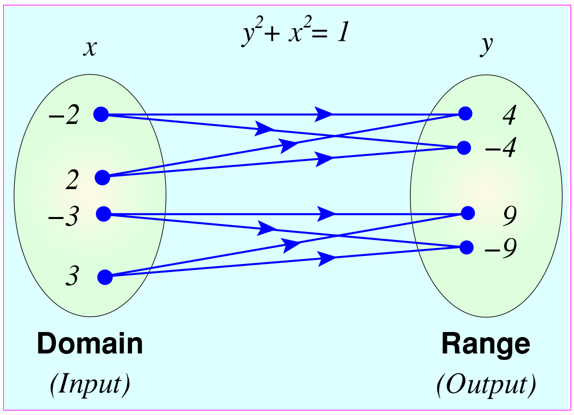
(This type of relationship is very important in database design).



Many to many relationship:

Two or more elements of the domain map to two or more elements in the range.

This is NOT a function.



In the diagrams above, the domains have been artificially restricted for the sake of clarity, but in reality we use a much larger sets of numbers, usually the set of all real numbers. The following table summarises some of the standard sets used and their notation:

Domain Notation	Range Notation	Meaning
$x \in \mathbb{R}$	$f(x) \in \mathbb{R}$	x or $f(x)$ is a member of the set of all real numbers
$\{x : x \in \mathbb{R}, x \neq 2\}$ $x \in \mathbb{R}, x \neq 2$	$\{f(x) : f(x) \in \mathbb{R}, f(x) \neq 2\}$ $f(x) \in \mathbb{R}, f(x) \neq 2$	Read this as 'The set of values such that x or $f(x)$ belongs to $\mathbb{R}$ and is not 2'. The ':' is read as 'such that'. A simpler alternative notation.
	$y \in \mathbb{R},\ 2 \leqslant y < 4$	y is a real number, and greater or equal to 2 and less than 4.

If no domain is specified assume the largest set available, usually $x \in \mathbb{R}$.

Domains can be restricted to anything we want or need, but some restrictions are imposed just from a purely algebraic point of view.

Division by zero.

Since division by zero is not possible, any equation that is a quotient (fraction), must exclude values of x which make the denominator zero.

E.g. $f(x) = \dfrac{5}{6 - x}$ has a domain of $\{x : x \in \mathbb{R}, x \neq 6\}$

Even Roots

Even roots of –ve real numbers cannot be evaluated, so the domain of any function must exclude these values.

E.g. $f(x) = \sqrt{5x - 3}$ has a domain of $\left\{x : x \in \mathbb{R}, x \geqslant \dfrac{3}{5}\right\}$

(Make $5x - 3 \geqslant 0$ & solve for x)

36.3 Vertical Line Test for a Function

The Vertical Line test is a relatively simple test to see if a graph is a function or not.

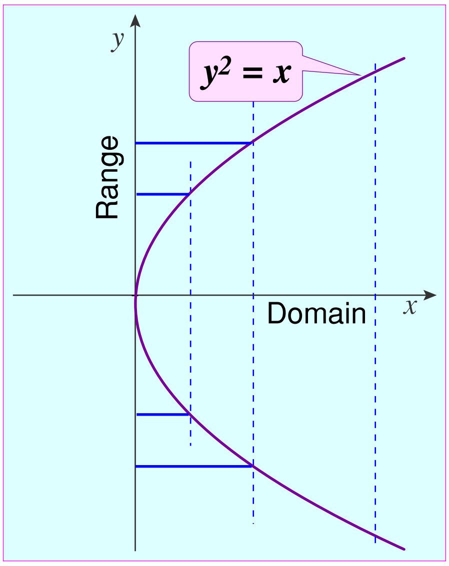
Draw a series of vertical lines on the chart and if any one of the vertical lines crosses the curve at more than one place, then the equation is not a function.

The curve is only a function if every element in the domain is mapped to one and only one element in the range, in which case the vertical line will cross the curve only once.

In this example of $y^2 = x$, each vertical line crosses the curve twice, hence each value of x gives two values for y.

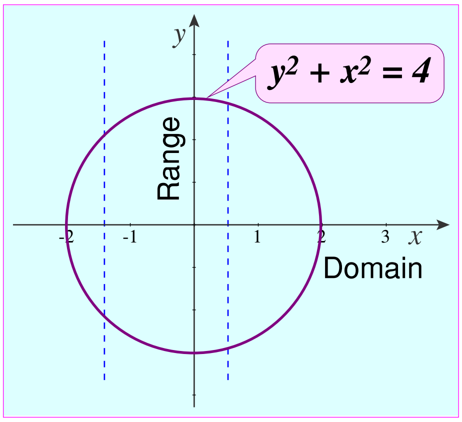
Domain: $x \in \mathbb{R},\ x \geqslant 0$
Range: $f(x) \in \mathbb{R},$

Therefore, this is not a function of x, (although x is a function of y).



The equation of a circle is not a function.

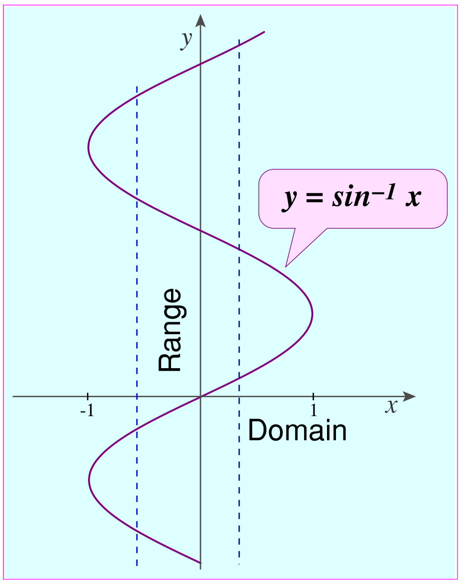
Domain: $x \in \mathbb{R},\ -2 \leqslant x \leqslant 2$
Range: $f(x) \in \mathbb{R},\ -2 \leqslant y \leqslant 2$



In this example, $y = \sin^{-1}x$ can be seen to fail the vertical line test. As illustrated, this is NOT a function of x.

Domain: $x \in \mathbb{R},\ -1 \leqslant x \leqslant 1$
Range: $f(x) \in \mathbb{R}$

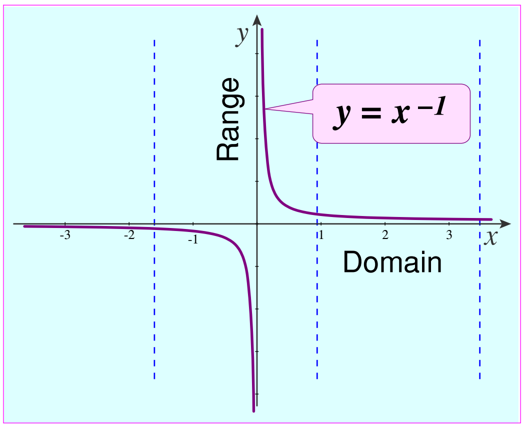
Restricting the range will ensure it can be regarded as a inverse function (see later).



With the function $y = x^{-1}$, at first site this appears to pass the test, but when $x = 0$, the equation is not determined.

If we exclude the value for $x = 0$, then this can be considered as a function.

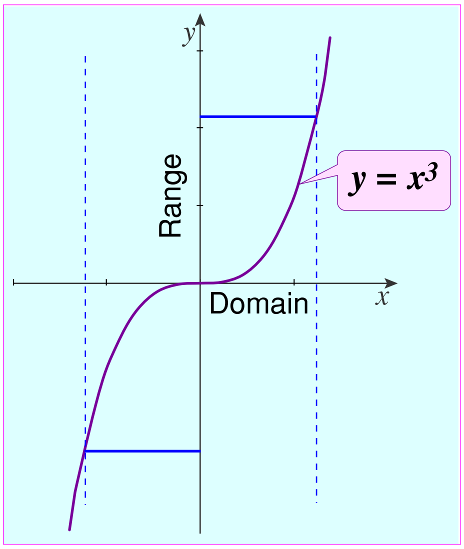
Domain: $x \in \mathbb{R},\ x \neq 0$
Range: $f(x) \in \mathbb{R},\ f(x) \neq 0$



Finally, $y = x^3$.

This passes the vertical line test and is a function.

Domain: $x \in \mathbb{R}$
Range: $f(x) \in \mathbb{R}$



36.4 Compound or Composite Functions

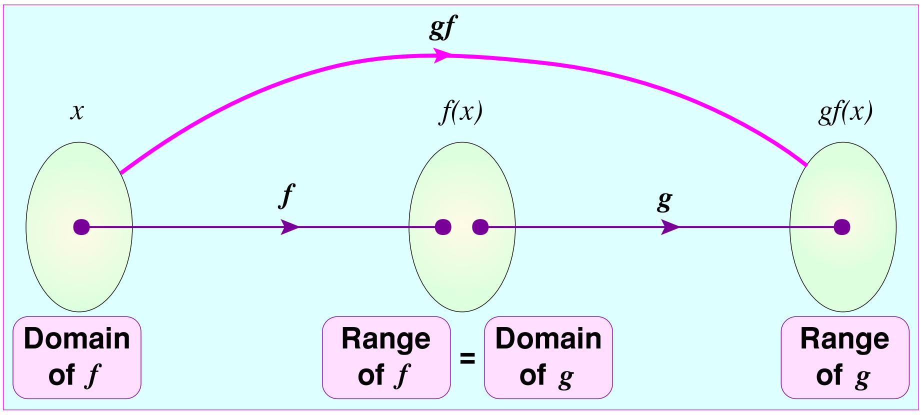
Composite Functions are a bit like Russian dolls, with one doll inside another. They describe the combined effect of two of more functions that are done in order, one after the other. This is not the same as functions being multiplied together.

The function $gf(x)$, often referred to as a function of a function. and is read as 'g of f of x' which means do $f(x)$ first, then $g(x)$ second, by substituting $f(x)$ into $g(x)$. Note that since $f(x)$ is done first, it is written closer to (x) than the g.

For gf to exist, the range of f must be a subset of the domain of g.

Composite functions can be written in a number of ways:

$$gf(x) = g(f(x)) = (g \circ f)(x)$$

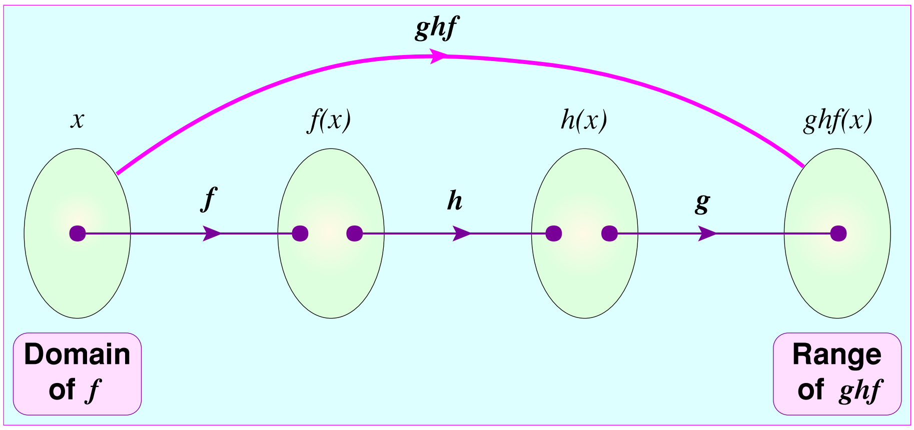


Function of a Function

One important result about composite functions is that generally:

$$gf(x) \neq fg(x)$$

A composite function $ghf(x)$, with three functions of x, is shown. $f(x)$ is the first, $h(x)$ is 2nd, and $g(x)$ is 3rd.



Function of a Function of a Function

36.4.1 Example:

1 Evaluate $fg(x)$ & $gf(x)$ when:

$f(x) = \cos x \qquad g(x) = x^2 + 3x - 1$

Solution:

$$fg(x) = f(x^2 + 3x - 1)$$
$$= \cos(x^2 + 3x - 1)$$

$$gf(x) = g(\cos x)$$
$$= (\cos x)^2 + 3(\cos x) - 1$$

2 If $f(x) = 2x + 8, \quad x \in \mathbb{R}$, evaluate $ff(x) = 8$

Solution:

$$ff(x) = 8$$
$$f(2x + 8) = 8$$
$$2(2x + 8) + 8 = 8$$
$$4x + 16 = 0$$
$$x = -4$$

3 If $f(x) = 1 - 2x, \quad x \in \mathbb{R}, g(x) = x^2 + 5, h(x) = \frac{x + 6}{2}$. Evaluate hg$f(3)$

Solution:

$$f(3) = 1 - 2 \times 3 = -5$$
$$g(-5) = (-5)^2 + 5 = 30$$
$$h(30) = \frac{30 + 6}{2} = 18$$

4 If $f(x) = x^2 - 9$, and $g(x) = \sqrt{9 - x^2}$, find the domain of $fg(x)$.

Solution:

$$fg(x) = f(g(x))$$
$$= f\left(\sqrt{9 - x^2}\right)$$
$$= \left(\sqrt{9 - x^2}\right)^2 - 9$$
$$= 9 - x^2 - 9$$
$$= -x^2$$

Domain of $f(x) = x \in \mathbb{R}$

Now $9 - x^2 \geqslant 0$ to avoid a –ve square root

$\Rightarrow \quad -x^2 \geqslant -9 \quad \Rightarrow \quad x^2 \leqslant 9 \quad \Rightarrow \quad x \leqslant \pm 3$

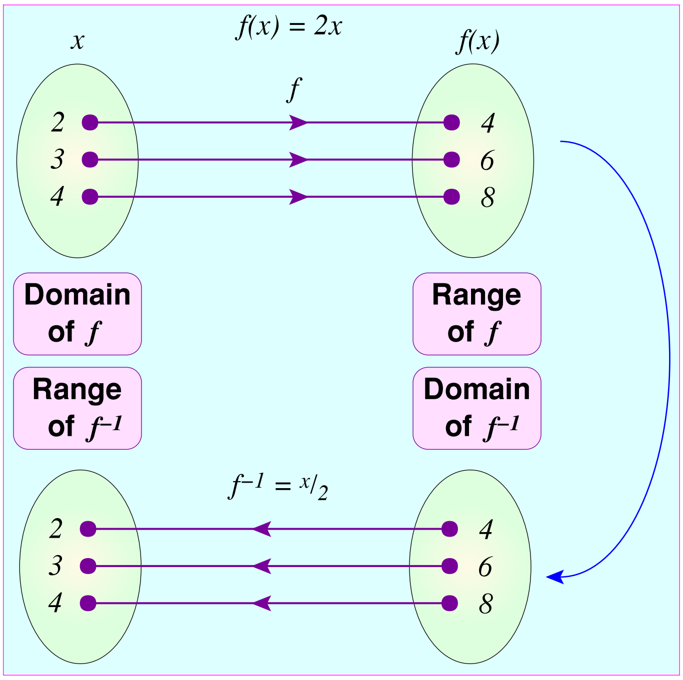
$\therefore \quad$ Domain of $g(x) = -3 \leqslant x \leqslant 3$

$\therefore \quad$ Domain of $fg(x) = -3 \leqslant x \leqslant 3$

36.5 Inverse Functions

Inverse functions are written as $f^{-1}(x)$ (not to be confused with the reciprocal of a function).

An inverse function is one that is reversible (one undoes the other) in that the range of *f(x)* acts as the domain of $f^{-1}(x)$, and the range of $f^{-1}(x)$ equals the domain of *f(x)*.



Inverse Function

We can therefore write:

$$y = f(x) \qquad \Rightarrow \qquad x = f^{-1}(y)$$

E.g. In simple terms, a function such as $f(x) = 3x - 2$ means multiply x by 3 and subtract 2. The inverse means add 2 and divide by 3.

$$x \to \boxed{\times 3} \to \boxed{-2} \to 3x - 2$$

$$x \leftarrow \boxed{\div 3} \leftarrow \boxed{+2} \leftarrow \;\lrcorner$$

It should be clear that for an inverse to exist the function needs to have a 'one to one' relationship.
A 'many to one' function would have a 'one to many' inverse relationship, which is not a function. However, if the domain is restricted, then a 'many to one' function can be changed to a 'one to one' function.

E.g. The function $f(x) = 4x^2$ is a 'many to one' function. Restricting the functions domain to $x \in \mathbb{R},\ x \geqslant 0$ then the inverse can be found as

$$f^{-1}(x) = 2\sqrt{x}$$

Note that $\sqrt{\quad}$ means take the +ve square roots.

A '**self inverse**' function is one that is its own inverse, such that $f(x) = f^{-1}(x)$.

In general we find that if:

$$ff^{-1}(x) = f^{-1}f(x) = x$$

the function is self inverse. So finding $ff(x)$ should determine if the function is self inverse.

Reciprocal functions are self inverse.

To find the inverse of a function, use this procedure:

- Replace $f(x)$ with y
- Make x the subject of the equation
- Swop x and y, since their roles are reversed when taking the inverse function
- Replace y with $f^{-1}(x)$

36.5.3 Example:

1 Find the inverse of $f(x) = \dfrac{1}{4 - 3x}$

Solution:

$$y = \frac{1}{4 - 3x}$$ Make x the subject

$$4 - 3x = \frac{1}{y}$$ cross multiply

$$-3x = \frac{1}{y} - 4$$

$$3x = 4 - \frac{1}{y}$$ $\times -1$

$$x = \frac{1}{3}\left(4 - \frac{1}{y}\right)$$

$$y = \frac{1}{3}\left(4 - \frac{1}{x}\right)$$ reverse x & y

$$\therefore \quad f^{-1}(x) = \frac{1}{3}\left(4 - \frac{1}{x}\right)$$

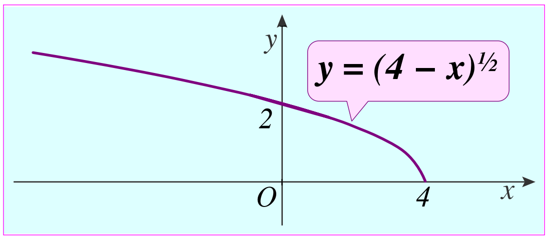
2 Find the inverse of $f(x) = \sqrt{4 - x}$

Solution:

Domain of $f(x) = \sqrt{4 - x}$ is:

$$4 - x \geqslant 0$$
$$-x \geqslant -4$$
$$x \leqslant 4$$

Domain is $x \leqslant 4$



$$y = \sqrt{4 - x}$$
$$y^2 = 4 - x$$
$$x = 4 - y^2$$
$$y = 4 - x^2$$ reverse x & y
$$f^{-1}(x) = 4 - x^2$$

Domain of $f(x) : x \in \mathbb{R},\ x \leqslant 0 \quad \Rightarrow \quad$ Range of $f(x) : x \in \mathbb{R},\ x \geqslant 0$

Range of $f(x) : x \in \mathbb{R},\ x \geqslant 0 \quad \Rightarrow \quad$ Domain of $f^{-1}(x) : x \in \mathbb{R},\ x \geqslant 0$

Range of $f^{-1}(x) : x \in \mathbb{R},\ x \leqslant 0$

3 Find the inverse of $f(x) = \sqrt{4 - x^2} \quad x \in \mathbb{R},\ 0 \leqslant x \leqslant 2$ and show that it is self inverse.

Solution 1:

$$y = \sqrt{4 - x^2}$$
$$y^2 = 4 - x^2$$
$$x^2 = 4 - y^2$$
$$y^2 = 4 - x^2 \qquad \text{reverse } x \text{ \& } y$$
$$y = \sqrt{4 - x^2}$$
$$\therefore \quad f^{-1}(x) = \sqrt{4 - x^2}$$

Since $f(x) = f^{-1}(x)$ the function is self inverse.

Solution 2:

Find the value of $ff(x)$ and test to see if $ff(x) = x$

$$f(x) = \sqrt{4 - x^2}$$
$$\therefore \quad ff(x) = \sqrt{4 - \left(\sqrt{4 - x^2}\right)^2}$$
$$= \sqrt{4 - (4 - x^2)}$$
$$= \sqrt{4 - 4 + x^2}$$
$$= \sqrt{x^2}$$
$$= x$$

$\therefore$ the function is self inverse.

4 Show that the function $f(x) = \dfrac{x}{x - 1} \quad x \in \mathbb{R},\ x \neq 1$ is self inverse.

Solution:

The function is self inverse if the value of the function for a given value of x, is the same when the function is applied to that answer.

Let $x = 4$ (say)

$$\therefore \quad f(4) = \frac{4}{4 - 1} = \frac{4}{3}$$

Apply the function to the answer for $f(4)$

$$\therefore \quad f\left(\frac{4}{3}\right) = \frac{\frac{4}{3}}{\frac{4}{3} - 1} = \frac{\frac{4}{3}}{\frac{1}{3}}$$
$$= \frac{4}{3} \times \frac{3}{1} = 4$$
$$\therefore \quad f(4) = \frac{4}{3} \quad \& \quad f\left(\frac{4}{3}\right) = 4$$

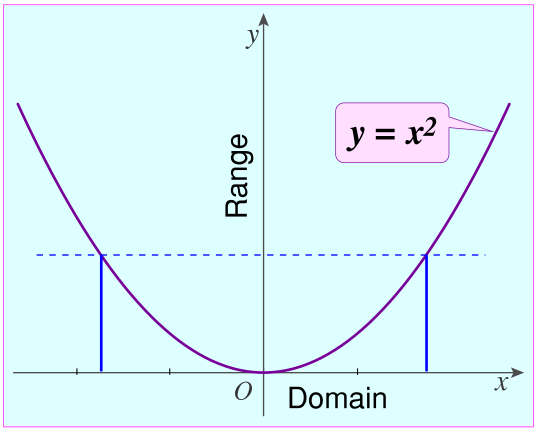
Hence the function is self inverse

36.6 Horizontal Line Test for an Inverse Function

The Horizontal Line test is another simple test, this time to see if a graph has a one to one relationship, and hence find if it has an inverse function or not.

In a similar manner to the vertical line test, draw a series of horizontal lines on the chart and if any one of the horizontal lines crosses the curve at more than one place, then the equation is a ‘many to one’, or even a ‘many to many’ relationship.

The curve is only a one to one function if only one element in the domain is mapped to one, and only one, element in the range, in which case the horizontal line will cross the curve only once.



Horizontal Line Test, showing a ‘Many to One’ relationship

36.7 Graphing Inverse Functions

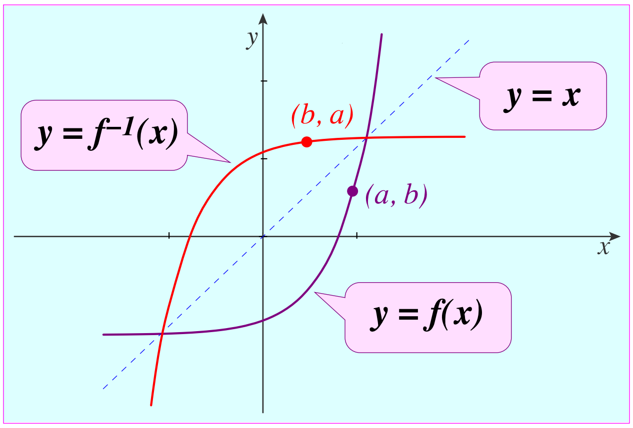
In graphical terms, the role of x and y are reversed, and a reflection is the line $y = x$ is created.

A 'self inverse' function is its own reflection in the line $y = x$.

On a practical note, the x and y axes should have the same scales and the $y = x$ line should be at a 45° angle, otherwise the image may look distorted.

Asymptotes are also reflected

For every point (a, b) on the function curve, there is a corresponding point (b, a) on the inverse function curve.



The standard trig functions can be made into one to one functions by restricting the domain.

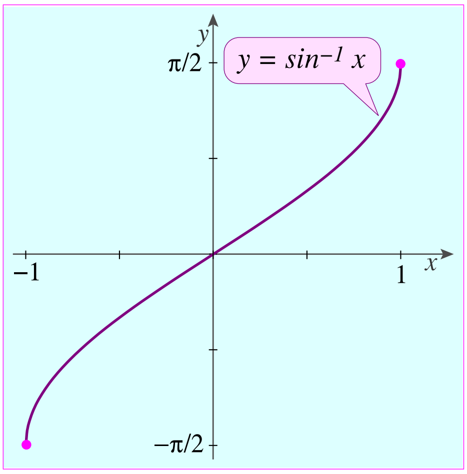
The inverse can then be found.

For $f^{-1}(x) = sin^{-1}x$

Restricted Domain: $-1 \leqslant x \leqslant 1$

Range: $-\frac{\pi}{2} \leqslant sin^{-1} \leqslant \frac{\pi}{2}$

(More later)



36.8 Odd, Even & Periodic Functions

The concept of odd and even functions is all about the symmetry of the function.

- The function and graph is said to be '**even**' if it is symmetrical about the y-axis and:

$$f(-x) = f(x)$$

(This is a transformation with a reflection in the y-axis).

- The function and graph is said to be '**odd**' if it has rotational symmetry, order 2, (180° rotation) about the origin and:

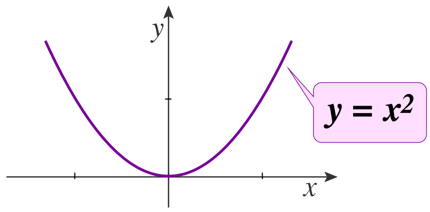
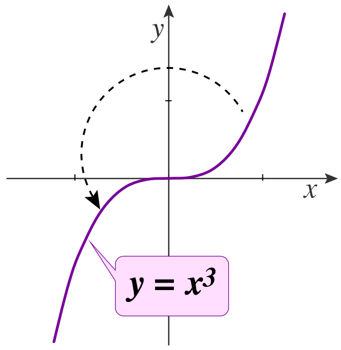
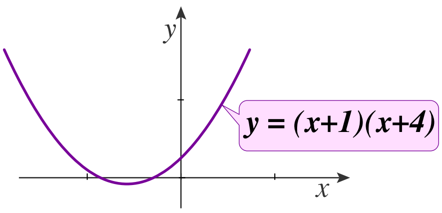
$$f(-x) = -f(x)$$

or

$$f(x) = -f(-x)$$

(This is equivalent to two transformations with a reflection in both the x-axis and y-axis).

- Many functions are neither odd nor even.

Type of Function	Symmetry	Algebraic Definition	Case study
Even	Line symmetry about the y-axis.	$f(-x) = f(x)$ example: $(-a)^{even} = (a)^{even}$	
Odd	Rotational symmetry about the origin. Hence, must go through origin to be ODD.	$f(-x) = -f(x)$ example: $(-a)^{odd} = -(a)^{odd}$	
Even & Odd	Both types of symmetry.	$f(x) = 0$	Only example of both
Not even or odd	No symmetry.	N/A	

36.9 Functions: Worked Examples

36.9.1 Example:

1 Find the inverse of $f(x) = \dfrac{4x + 3}{x + 2} \quad x \in \mathbb{R},\ x > -2$. Calculate the co-ordinates of the points of intersection of $f(x)$ & $f^{-1}(x)$.

Solution:

$$y = \frac{4x + 3}{x + 2} \qquad \text{Make } x \text{ the subject}$$

$$y(x + 2) = 4x + 3$$

$$xy + 2y = 4x + 3$$

$$xy - 4x = 3 - 2y$$

$$x(y - 4) = 3 - 2y$$

$$x = \frac{3 - 2y}{y - 4} \qquad \text{reverse } x \text{ \& } y$$

$$\therefore \quad y = \frac{3 - 2x}{x - 4}$$

$$\therefore \quad f^{-1}(x) = \frac{3 - 2x}{x - 4}$$

The intersection of $f(x)$ & $f^{-1}(x)$ occurs on the line $y = x$, hence solve for:

$$y = \frac{3 - 2x}{x - 4} \quad \& \quad y = x$$

$$x = \frac{3 - 2x}{x - 4}$$

$$x(x - 4) = 3 - 2x$$

$$x^2 - 4x + 2x - 3 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$\therefore$ co-ordinates are $(3,\ 3)$ & $(-1, -1)$

2

3

36.10 Heinous Howlers

The notation for the inverse function can be easily confused with the notation for a reciprocal function. Note the following:

$$f^{-1}(x) \text{ is the inverse of } f(x)$$

$$\cos^{-1}(x) \text{ is the inverse of } \cos(x)$$

$$x^{-1} = \frac{1}{x} \qquad \text{(The reciprocals)}$$

$$(\cos x)^{-1} = \frac{1}{\cos x}$$

$$[f(x)]^{-1} = \frac{1}{f(x)}$$

Take care in substituting the values for x:

1 Give that $f(x) = \frac{1}{1 - 3x}$, evaluate $f(x + a)$.

$$f(x + a) \neq \frac{1}{1 - 3x} + a \qquad ✘$$

$$f(x + a) = \frac{1}{1 - 3(x + a)} \qquad ✔$$

37 • C3 • Modulus Function & Inequalities

37.1 The Modulus Function

The **Modulus Function** has the symbol $|x|$, and is called the 'modulus of x', the 'absolute value of x' or more generally the 'magnitude of x'. The modulus disregards the sign of the function, and is always positive.

We are now only interested in the size of the function, not the sign. On the calculator it is labelled 'Abs'.

The definition is:

$$|x| = \begin{cases} x & \text{for all real } x \geqslant 0 \\ -x & \text{for all real } x < 0 \end{cases}$$

E.g.

$|x| = x$

$|-x| = x$

$|f(x)|$ is always positive or zero hence:

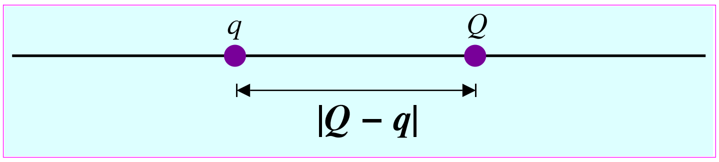
$|f(x)| = f(x)$ when $f(x)$ is +ve

$|f(x)| = -f(x)$ when $f(x)$ is −ve

This short hand way of representing numbers means that we can express the difference between two numbers, without saying which number is the larger one.

Hence: $|Q - q|$ is the same, whether $Q > q$ or $q > Q$ or even when $Q = q$

Illustrated on a number line thus:



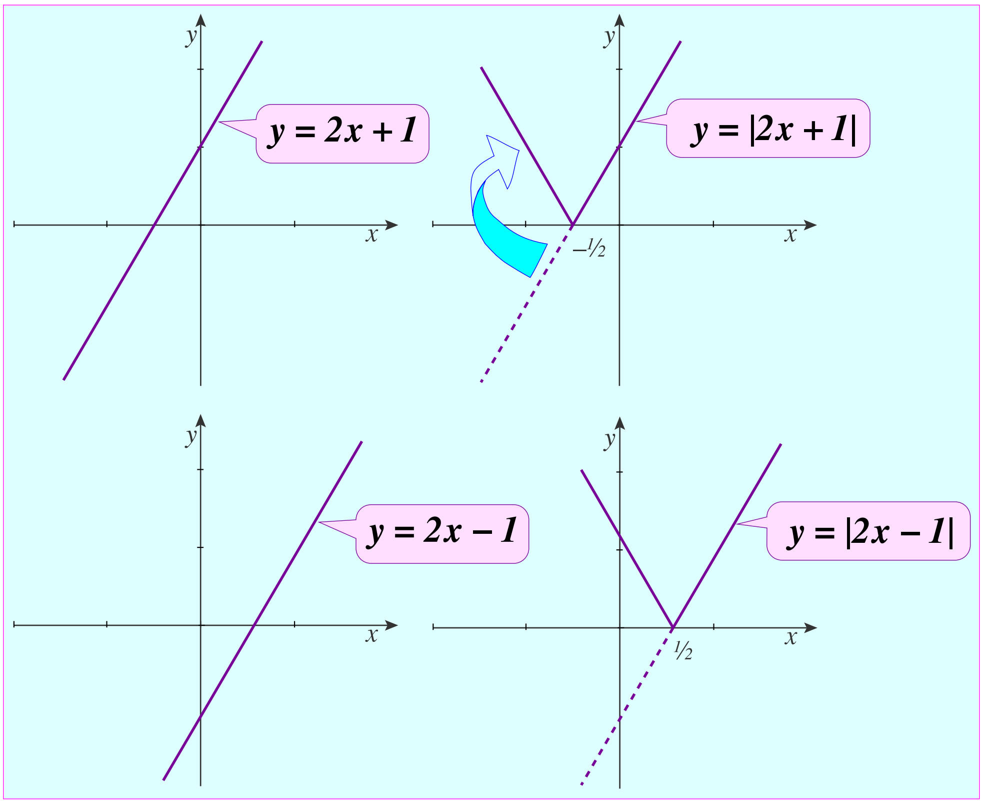
If x is +ve $\qquad \sqrt{x^2} = x$

If x is −ve $\qquad \sqrt{x^2} = -x$

so from the definition $\qquad |x| = \sqrt{x^2}$

37.2 Graphing $y = \left| f(x) \right|$

A graph of $y = \left| f(x) \right|$ is plotted in the same way as $y = f(x)$, except that any values below the x-axis are reflected in the x-axis.



From the above, it can be seen that the modulus function $\left| f(x) \right|$ is always a positive quantity or zero.

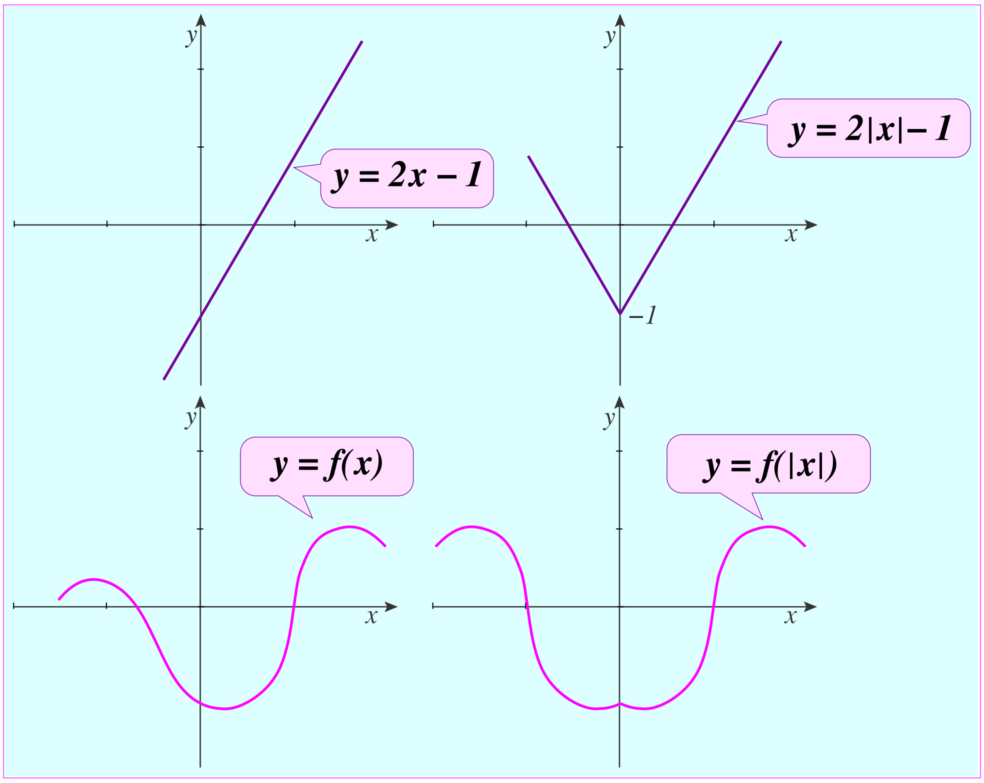
37.3 Graphing $y = f(|x|)$

In the case of $y = f(|x|)$ we find that because $|x| = |-x|$ then $f|x|$ will have the same values irrespective of the sign of x.

This means that $y = f(|x|)$ is symmetrical about the y-axis, and hence it is an even function.

$$y = f(|x|) = f(x) \text{ when } x \geqslant 0$$
$$y = f(|x|) = f(-x) \text{ when } x < 0$$

From the section on transformations recall that $f(-x)$ is a reflection in the y-axis.



37.3.1 Summary

Sketching $y = |f(x)|$

- Sketch $y = f(x)$
- Any part of $y = f(x)$ that is below the x-axis is reflected in the x-axis.

Sketching $y = f(|x|)$

- Sketch $y = f(x)$
- Any part of $y = f(x)$ that is to the right of the y-axis is reflected in the y-axis.

37.4 Inequalities and the Modulus Function

Inequalities such as: $-2 < x < 2$ can be written as $|x| < 2$

and: $-5 \leqslant x \leqslant 5$ can be written as $|x| \leqslant 5$

In other words:

$$|x| < a \quad \Leftrightarrow \quad -a < x < a$$

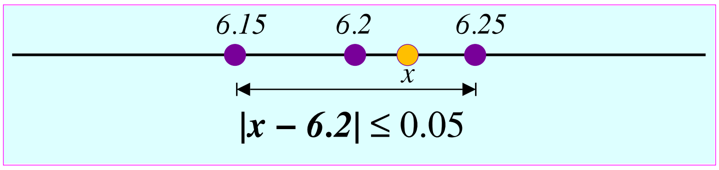
(Note that we don't write: $-a > x > a$)

We can also say:

$$|x - k| \leqslant a \quad \Leftrightarrow \quad k - a < x < k + a$$

and $$|a| = |b| \quad \Leftrightarrow \quad a^2 = b^2$$

E.g. If $x = 6.2$ to 1 dp, what is the range of values for x?



$$|x - 6{\cdot}2| \leqslant 0{\cdot}05 \quad \Leftrightarrow \quad 6{\cdot}2 - 0{\cdot}05 < x < 6{\cdot}2 + 0{\cdot}05$$

$$\Leftrightarrow \quad 6{\cdot}15 < x < 6{\cdot}25$$

37.5 Algebraic Properties

A summary of the algebraic properties:

$$|a \times b| = |a| \times |b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

but:

$$|a + b| \neq |a| + |b|$$

$$|a - b| \neq |a| - |b|$$

37.6 Solving Equations Involving the Modulus Function

There are a number of ways of solving equations involving the modulus function:

- Critical values
- Squaring
- Graphing
- Geometrical

In solving these types of question, it is always advisable to draw a sketch.

37.7 Solving Modulus Equations by Critical Values

37.7.1 Example:

1 Solve for x:

$|x - 4| < 5$

Solution:

Let: $(x - 4) = 0 \quad \Rightarrow \quad \therefore x = 4$ when line crosses the x-axis. i.e. when $y = 0$

Critical value when $x < 4$ is $-(x - 4) = 5 \quad \Rightarrow \quad x = -1$

Critical value when $x \geqslant 4$ is $(x - 4) = 5 \quad \Rightarrow \quad x = 9$

$\therefore \quad |x - 4| < 5$ when $-1 < x < 9$

2 Solve for x:

$|2x + 1| \leqslant 3$

Solution:

Let: $(2x + 1) = 0 \quad \Rightarrow \quad \therefore x = -½$ i.e. when $y = 0$

Critical value when $x < -½$ is $-(2x + 1) = 3 \quad \Rightarrow \quad x = -2$

Critical value when $x \geqslant -½$ is $(2x + 1) = 3 \quad \Rightarrow \quad x = 1$

$\therefore \quad |2x + 1| \leqslant 3$ when $-2 \leqslant x \leqslant 1$

3 Solve for x:

$4 - |2x| = x$

Solution:

This equivalent to solving where $4 - |2x|$ and $y = x$ intersect.

Solve $4 - 2x = x \quad \Rightarrow \quad x = \frac{4}{3}$

Solve $4 + 2x = x \quad \Rightarrow \quad x = -4$

$\therefore \quad 4 - |2x| = x$ when $x = -4$ or $x = \frac{4}{3}$

37.8 Squares & Square Roots Involving the Modulus Function

For any value of x, x^2 will always be a positive number.

Mathematically we write:

$$x \in \mathbb{R} \quad \text{then} \quad x^2 \geqslant 0$$

E.g. $(-9)^2 = 81$

From the algebraic rules we find that:

$$|a \times b| = |a| \times |b|$$

$$\therefore \quad |a^2| = |a|^2 = a^2$$

Taking the square root:

$$\sqrt{a^2} = |a|$$

If $x \geqslant 0$ then $\sqrt{a^2} = x$

If $x < 0$ then $\sqrt{a^2} = -x$

37.8.2 Example:

1 Solve for x:

$$|x - 3| = |3x - 1|$$

Solution:

$(x - 3)^2 = (3x - 1)^2$ — Square both sides

$x^2 - 6x + 9 = 9x^2 - 6x + 1$

$8x^2 - 8 = 0$

$8(x + 1)(x - 1) = 0$

$x = -1 \quad$ *or* $\quad x = 1$

Note: only true for $|f(x)| = |g(x)|$ or $|f(x)| < |g(x)|$

2 Solve for x:

$$|x - 4| = 5$$

Solution:

$(x - 4)^2 = 25$ — Square both sides

$x^2 - 8x + 16 = 25$

$x^2 - 8x - 9 = 0$

$(x + 1)(x - 9) = 0$

$x = -1 \quad$ *or* $\quad x = 9$

3 Solve for x:

$$|2x + 1| \leqslant 3$$

Solution:

$$(2x + 1)^2 \leqslant 9$$

$$4x^2 + 4x + 1 - 9 \leqslant 0$$

$$4x^2 + 4x - 8 \leqslant 0$$

$$4(x^2 + x - 2) \leqslant 0$$

$$4(x + 2)(x - 1) \leqslant 0$$

Critical values: $x = -2, \quad x = 1$

$$-2 \leqslant x \leqslant 1$$

4 A function is defined as:

$$f(x) = (x + 2)(x - 4)$$

Solve: $|f(x)| = 8$

Solution:

$$|f(x)| = \pm 8$$

$$(x + 2)(x - 4) = -8$$

$$x^2 - 2x - 8 = -8$$

$$x^2 - 2x = 0$$

$$x(x - 2) = 0$$

$$x = 0, \quad x = 2$$

$$(x + 2)(x - 4) = 8$$

$$x^2 - 2x - 8 = 8$$

$$x^2 - 2x - 16 = 0$$

$$(x - 1)^2 - 1 - 16 = 0$$

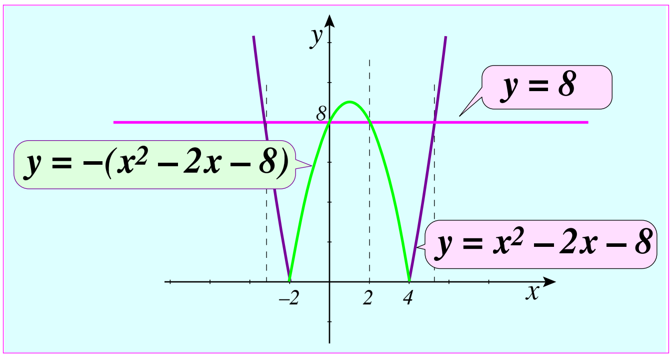
$$(x - 1)^2 = 17$$

$$x - 1 = \pm\sqrt{17}$$

$$x = 1 \pm \sqrt{17}$$

$$x = -3{\cdot}12, \quad x = 5{\cdot}12$$

Use a diagram to sketch the layout.



5 Solve for x:

$$4 - |2x| = x$$

Solution:

Rearrange to keep the modulus on the LHS, then square both sides.

$$|2x| = 4 - x$$

$$(2x)^2 = (4 - x)^2$$

$$4x^2 = 16 - 8x + x^2$$

$$3x^2 + 8x - 16 = 0 \qquad \Rightarrow \qquad \left(x - \frac{4}{3}\right)\left(x + \frac{12}{3}\right) = 0$$

$$(3x - 4)(x + 4) = 0$$

$$x = -4 \quad \text{or} \quad x = \frac{4}{3}$$

37.9 Solving Modulus Equations by Graphing

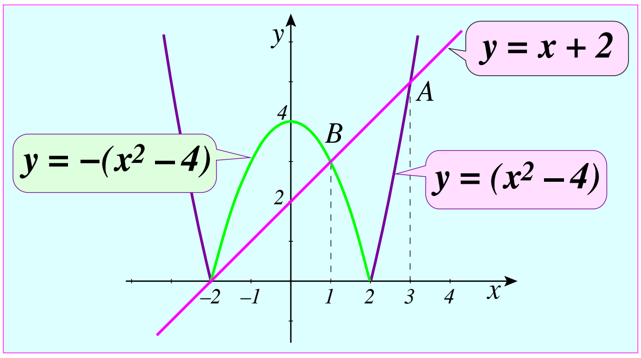
37.9.1 Example:

1 Solve for x:

$$y = x + 2 \quad \text{and} \quad y = |x^2 - 4|$$

Solution:

Drawing a graph of the equations to aid the metal picture:



Solve the following equations:

$$x + 2 = x^2 - 4 \qquad (A)$$

and

$$x + 2 = -(x^2 - 4) \qquad (B)$$

At point A

$$x + 2 = x^2 - 4$$

$$x^2 - x - 2 - 4 = 0$$

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

Since A is +ve $\quad x = 3$

At point B

$$x + 2 = -(x^2 - 4)$$

$$x^2 + x + 2 - 4 = 0$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

Since B is +ve $\quad x = 1$

2 Solve for x:

$$|2x + 1| \leqslant 3$$

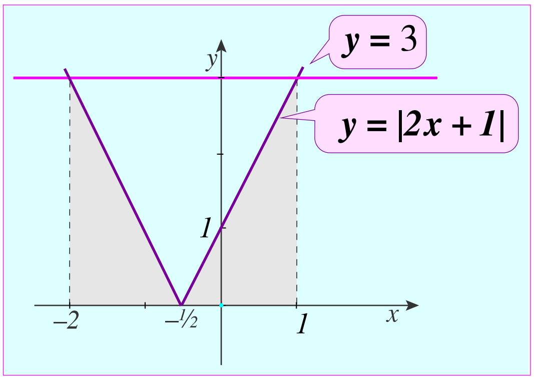
Solution:

Drawing a graph of

$y = |2x + 1|$

and $y = 3$ gives a visual representation of the inequality.

$-2 \leqslant x \leqslant 1$



37.10 Solving Modulus Equations by Geometric Methods

37.10.1 Example:

1 Solve for x:

$$|x - 4| = 5$$

Solution:

$|x - 4|$ represents the distance of x from 4.

$\therefore \quad |x - 4| = 5$ then $x - 4 = \pm 5 \quad \Rightarrow \quad x = 4 \pm 5$

$\therefore \quad x = -1, \ or\ x = 9$

2

37.11 Heinous Howlers

In trying to solve a problem like:

$$|x - 3| + |3x - 1| = 0$$

You cannot move the right hand modulus to the other side of the equals sign, and so:

$$|x - 3| \neq -|3x - 1|$$

(See algebraic rules)

This is because we don't know if x is negative or positive.

You can only multiply or divide.

37.12 Modulus Function Digest

37.12.1 Gradient not defined

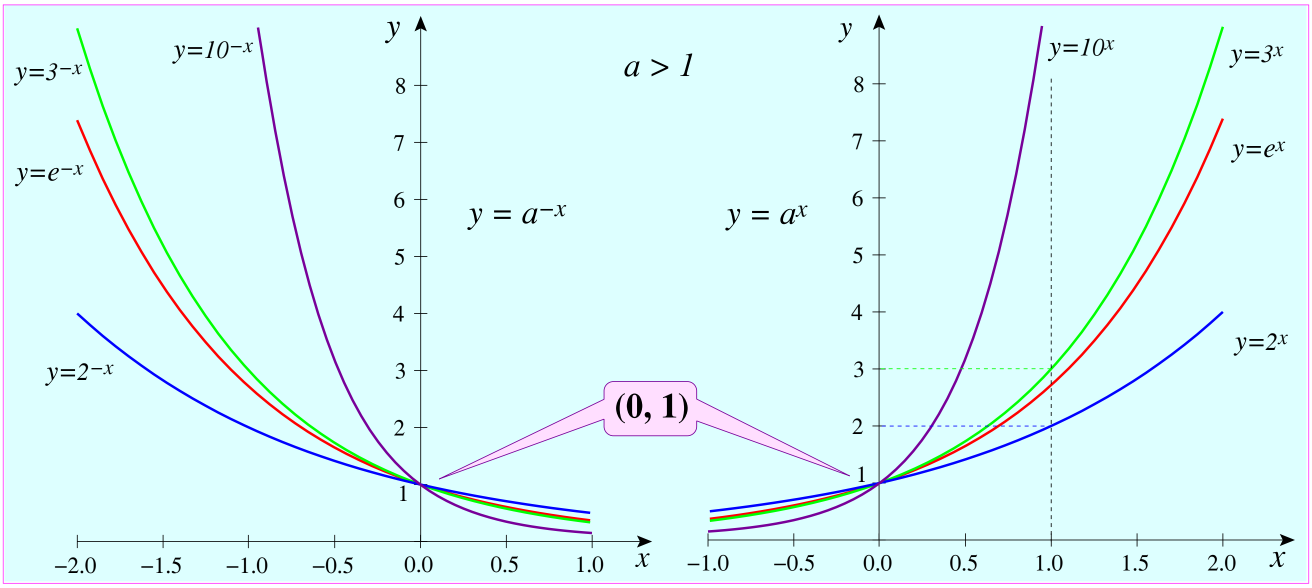
One obvious feature of any graph of a modulus function are the sharp corners generated by the function. These sharp points do not have a tangent and so $\frac{dy}{dx}$ is meaningless and has no solution.

Nevertheless, these sharp points may still represent turning points of some sort, such as a max or min. Hence, we can say that any turning point occurs when either $\frac{dy}{dx} = 0$ or where $\frac{dy}{dx}$ is not defined.

38 • C3 • Exponential & Log Functions

38.1 Exponential Functions

Recall from C2, that Exponential functions have the following properties:



Graphs for $y = a^x$ *and* $y = a^{-x}$*, all with* $a > 1$

- An **exponential function** has the form:
  $f(x) = a^x$ or $y = a^x$ where a is the base and is a positive constant.
- The value of a is restricted to $a > 0$ and $a \neq 1$
  - Note that when $a = 0$, $a^x = 0$, and when $a = 1$, $a^x = 1$, hence the restrictions above
  - The function is not defined for negative values of a. (e.g. $-1^{0.5} \equiv \sqrt{-1}$)
- All exponential graphs have similar shapes
- All graphs of $y = a^x$ and $y = a^{-x}$ pass through co-ordinates (0, 1)
- Graphs pass through the point $(1, b)$ where b is the base
- The larger the value of a, the steeper the curve
- Graphs with a negative exponent are mirror images of the positive ones, being reflected in the y-axis
- For $a > 1$ and +ve x, the gradient is always increasing and we have exponential growth
  For $a > 1$ and −ve x, the gradient is always decreasing and we have exponential decay
  For $0 < a < 1$ and +ve x, the gradient is always decreasing and we have exponential decay
- For +ve values of x, the gradient is always increasing as x increases, i.e. the rate of change increases (exponential growth)
- The x-*axis* of a exponential graph is an asymptote to the curve hence:
  - The value of y never reaches zero
  - and is always positive
- For exponential graphs, the gradient divided by its y value is a constant
- Recall that $a^0 = 1$, for +ve values of a, and that $a^{-3} \equiv \frac{1}{a^3}$

For $y = a^x$ $\qquad x \to +\infty \Rightarrow y \to +\infty$

$\qquad x \to -\infty \Rightarrow y \to 0$

For $y = a^{-x}$ $\qquad x \to +\infty \Rightarrow y \to 0$

$\qquad x \to -\infty \Rightarrow y \to +\infty$

38.2 THE Exponential Function: e

Whereas a^x is an exponential function, there is one special case which we call THE exponential function.

By adjusting the value of the base a, we can make the gradient at the co-ordinate (0, 1) anything we want. If the gradient at (0, 1) is adjusted to 1 then our base, a, is found to be 2·71828…

The function is then written as:

$$y = e^x \qquad \text{where } e = 2{\cdot}718281828 \ (9\,dp)$$

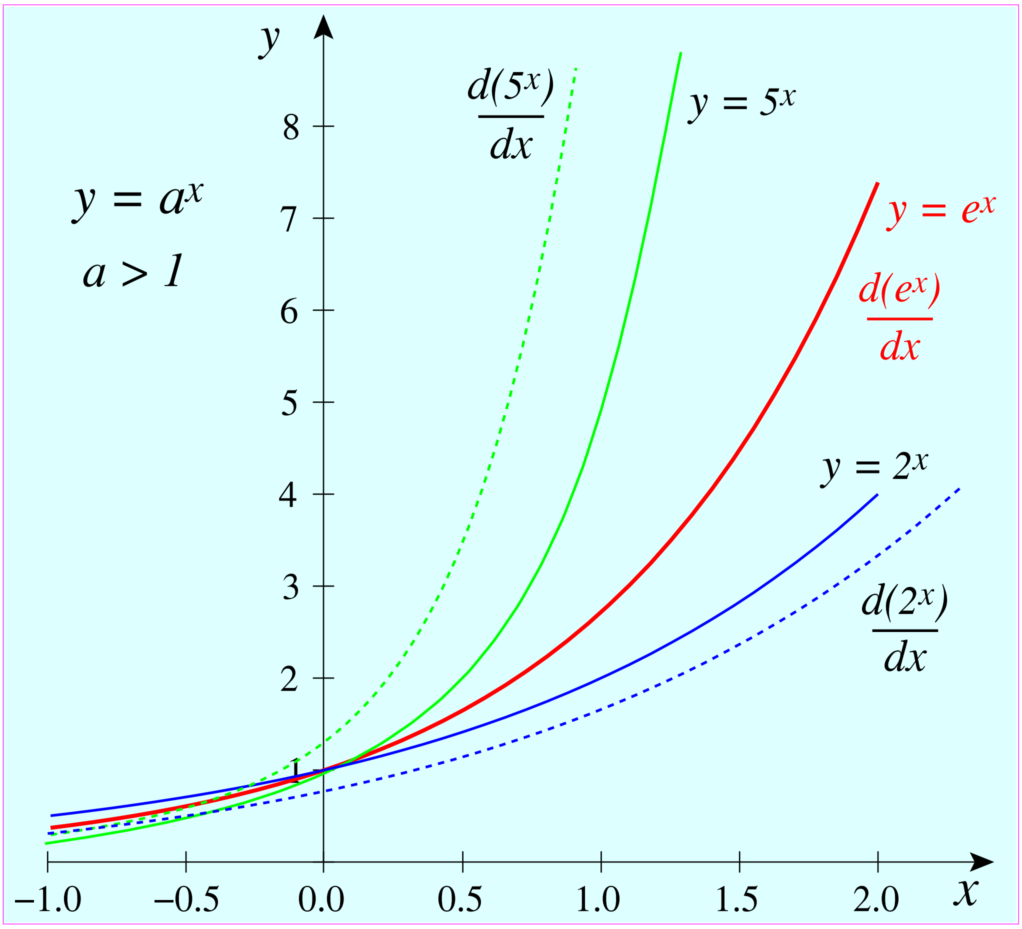
Like the number for π, e is an irrational number and never repeats, even though the first few digits may look as though they make a recurring pattern.

THE exponential function can also be found from the exponential series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \ldots$$

To find the value of e, set $x = 1$:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n!} + \ldots$$



Exponential Gradient Functions

In the illustration above, the gradient function of $y = 2^x$ and $y = 5^x$ are shown (dotted lines). The value of e is chosen such that the gradient function of $y = e^x$ is the same as the original function.
Therefore, in exponential graphs, the gradient divided by the y value ($^{dy}/_{dx} \div y$) is a constant. For e^x this value is 1, and we find that the gradient at any point is equal to y. Hence $\frac{dy}{dx} = e^x$.

$$\frac{dy/dx}{y} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = y$$

$$\text{but} \quad y = e^x \quad \Rightarrow \quad \therefore \frac{dy}{dx} = e^x$$

38.3 Natural Logs: $ln\ x$

The functions $y = a^x$ and $y = log_a x$ are inverse functions, i.e. the processes are reversible — one undoes the other.

$$y = 3^x \Leftrightarrow x = log_3 y$$

The exponential function, $y = e^x$ is the basis for natural logs, written log_e or ln

$$y = e^x \Leftrightarrow x = log_e y$$

$$y = e^x \Leftrightarrow x = ln\, y$$

Recall that $ln\, 1 = 0$, $ln\, e = 1$

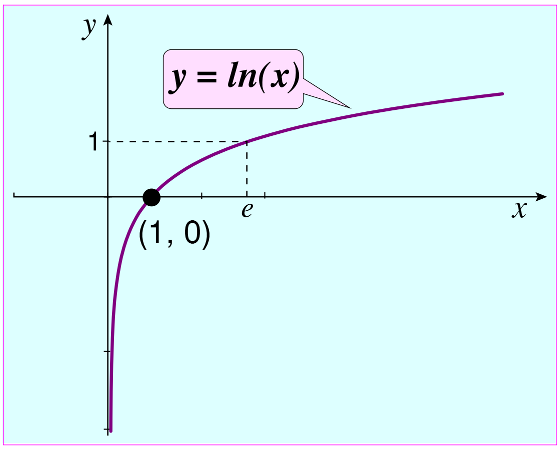
The natural log is used extensively in calculus, because the differential of e^x is e^x. Differentiating logs to other bases is more complicated.

Note that all log functions are undefined for $x \leqslant 0$ and therefore have a domain of $x > 0$

When x equals the base of the log, $y = 1$

i.e. $log_x x = 1$

$log_e e = 1 \quad \therefore \quad ln\, e = 1$

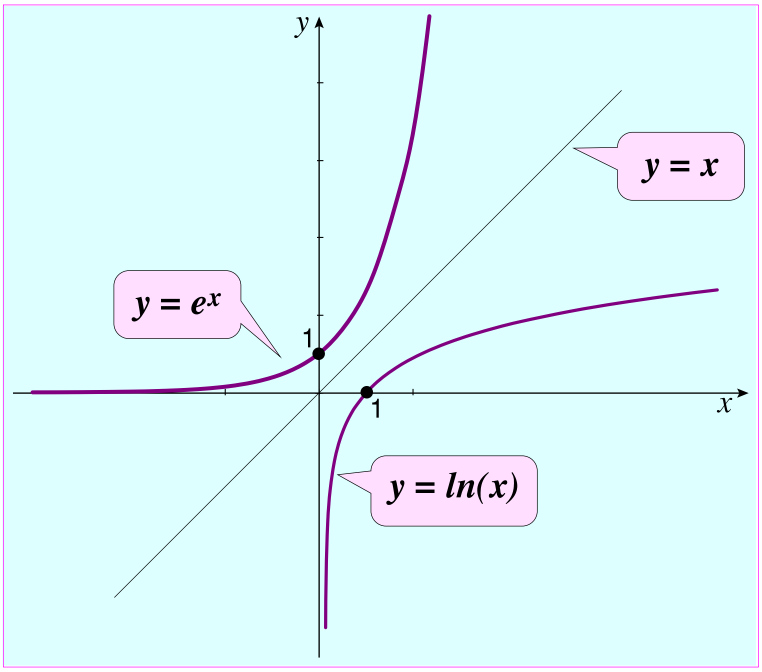


From the definition of the log we have:

$$\begin{aligned} ln\, e^x &= x\, ln\, e \\ &= x\, log_e\, e \\ &= x \times 1 \\ \therefore \quad ln\, e^x &= x \\ e^{ln\, x} &= x \end{aligned}$$

38.4 Graphs of e^x and $ln\ x$

As with other inverse functions these two functions, when plotted, are mirror images of each other in the line $y = x$.



- The gradient of $y = e^x$ at any point is equal to y.
  $i.e.$ at $y = e^3$, the gradient is e^3
- At the point (0, 1), the gradient of $y = e^x$ is 1
- $ln\ (1) = 0$
- Domain = set of values that x can take (input)
  Range = set of values that y can take (output)
- Domain of e^x is in the range of $ln\, x$, i.e. all the real numbers: $x \in \mathbb{R}$
  Range of e^x is in the domain of $ln\, x$, i.e. all the +ve numbers: $y \in \mathbb{R}, y > 0$
- Domain of $ln\, x$ is in the range of e^x, i.e. all the +ve numbers: $x \in \mathbb{R}, x > 0$
  Range of $ln\, x$ is in the domain of e^x, i.e. all the real numbers: $y \in \mathbb{R}$
- The graph of $ln\, x$ shows that you cannot have the ln of a −ve number

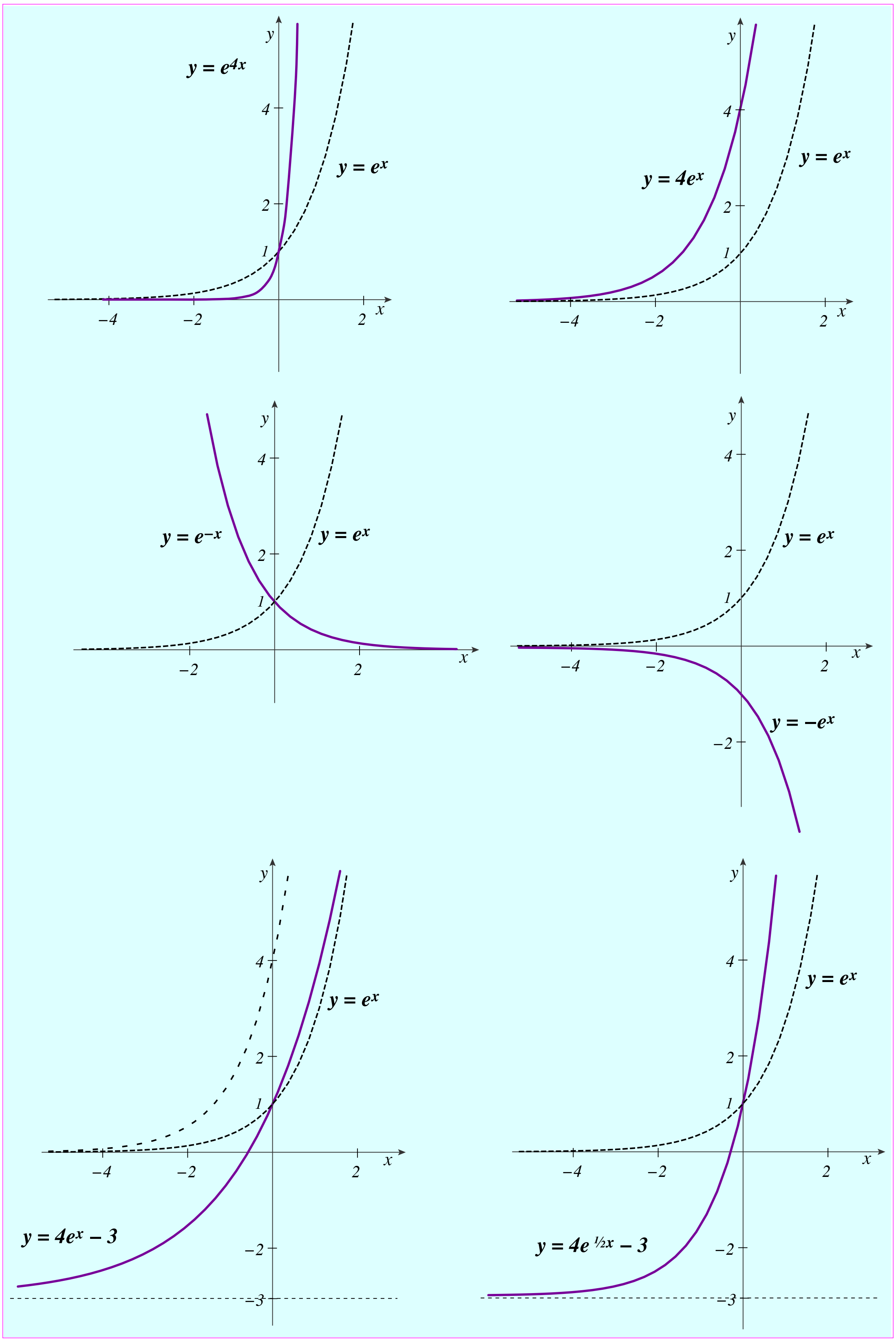
38.5 Graph Transformations of The Exponential Function

Some transformations showing

$$y = e^x \Rightarrow y = e^{4x} \qquad y = e^x \Rightarrow y = 4e^x$$

$$y = e^x \Rightarrow y = e^{-x} \qquad y = e^x \Rightarrow y = -e^x$$

$$y = e^x \Rightarrow y = 4e^x - 3 \qquad y = e^x \Rightarrow y = 4e^{\frac{x}{2}} - 3$$



Graph Transformations

38.6 Solving Exponential Functions

Some general tips on solving exponential functions:

- You need to use the log and indices laws
- Know that the $ln\, x$ and e^x functions have an inverse relationship
  e.g. if $e^x = 6$ then $x = ln\, 6$
  if $ln\, x = 4$ then $x = e^4$
  i.e. change the subject of the equation such that $x = something$
- Solving equations of the form: $ln\,(ax + b) = p$
  Rewrite equation such that: $ax + b = e^p$
- Solving equations of the form: $e^{ax+b} = q$
  Take natural logs both sides: $ax + b = ln\, q$
- Look for questions that allow for substitution, creating a quadratic or cubic equation
- Calculators: some calculators have a function button to allow calculations of logs to any base
- Otherwise use the change of base calculation.

38.6.1 Example:

1

$$10^{3x} = 270 \quad \therefore \quad 3x = log_{10} 270 \Rightarrow x = 0{\cdot}810$$

$$6^x = 78$$

$$log_{10} 6^x = log_{10} 78$$

$$x\, log_{10} 6 = log_{10} 78$$

$$x = \frac{log_{10} 78}{log_{10} 6} = \frac{1{\cdot}892}{0{\cdot}778} = 2{\cdot}432$$

2 Converting equations of type $y = ab^x$ to base e is required if a differential or integral is to be taken. The equation should be of the form: $y = ae^{kt}$.

$$y = ab^x$$

$$y = a \times e^{ln\,(b^x)}$$

$$y = a \times e^{x\, ln\,(b)}$$

$$y = a \times e^{kx} \quad \text{where} \quad k = ln\, b$$

Recall that $something = e^{ln\,(something)}$

38.7 Exponential Growth & Decay

Exponentials allow real world events to be modelled.

Exponential growth is modelled by the equation with the form:

$$N = Ae^{kt} \quad \text{where A, k are constants and } k > 0$$

This applies to investments, population growth, and heating to name a few.

Exponential decay is modelled by the equation with the form:

$$N = Ae^{-kt} \quad \text{where A, k are constants and } k > 0$$

This applies to radioactive decay, population falls, and cooling to name a few.

38.7.1 Example:

1 An oil bath is heated and the temperature of the oil, $T°C$, after t hours of heating is given by:

$$T = 28 + 100e^{-\frac{t}{20}} \qquad t > 0$$

Give the temp at the moment the heating is removed:

$$T = 28 + 100e^{-\frac{0}{20}}$$
$$= 28 + 100 \times 1$$
$$= 128°C$$

Give the temp 5 hours after the heating is removed:

$$T = 28 + 100e^{-\frac{5}{20}}$$
$$= 28 + 100e^{-\frac{1}{4}}$$
$$= 28 + 100 \times 0{\cdot}7788\ldots$$
$$= 105{\cdot}88°C$$

Find the time taken for the temp to fall to 64°C:

$$64 = 28 + 100e^{-\frac{t}{20}}$$
$$64 - 28 = 100e^{-\frac{t}{20}}$$
$$\frac{36}{100} = e^{-\frac{t}{20}}$$
$$ln\left(\frac{36}{100}\right) = ln\, e^{-\frac{t}{20}}$$
$$ln\, 0{\cdot}36 = -\frac{t}{20}\, ln\, e \qquad \text{but } ln\, e = 1$$
$$\therefore \qquad t = -20\, ln\, 0{\cdot}36$$
$$= -20 \times (-1{\cdot}022)$$
$$= 20{\cdot}43\ hrs$$

2 Plutonium decay is represented by:

$$P = 10\left(\frac{1}{2}\right)^{\frac{t}{24100}}$$

Where P = amount left after time t, starting with 10Kgs in this example:

After 241 years $\quad P = 10\left(\frac{1}{2}\right)^{\frac{241}{24100}}$

$$P = 10\left(\frac{1}{2}\right)^{\frac{1}{100}} = 10 \times 0{\cdot}933$$
$$P = 9{\cdot}33\ Kgs$$

38.8 Differentiation of e^x and $ln\ x$

e^x when differentiated is e^x. This is the only function to be its own derivative.

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x$$

This is one of its most useful properties as it can be used with the chain, product & quotient rules.

Differentiating $ln\ x$ gives:

$$y = ln\, x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

38.9 Integration of e^x and $ln\ x$

See later sections.

38.10 Heinous Howler

Don't make the mistake of trying to differentiate $y = e^x$ 'normally':

Note that if $y = e^x$ then $\frac{dy}{dx} = e^x$ and NOT $\frac{dy}{dx} \neq xe^{x-1}$

39 • C3 • Numerical Solutions to Equations

39.1 Intro to Numerical Methods

Most equations covered so far have been relatively easy to solve by algebraic means, leading to exact answers, even if the solutions are in surd form.

Now we consider equations that cannot be solved algebraically, which means finding other methods to estimate the solutions to the required degree of accuracy. Typical equations that require numerical solutions are:

$$x^3 - 4x + 3 = 0 \qquad e^x - 6x = 0 \qquad x^4 + 3x^2 - 2 = 0 \qquad x^3 - \sin(x) - 5 = 0$$

Recall that solving an equation starts by setting the equation to zero and finding all the values of x, for which, $y = 0$ and which we call the real roots of the equation.

E.g.

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

Roots are: $x = 2$ & $x = 4$

This is the same as finding all the values of x for which the curve $y = x^2 - 6x + 8$ intersects the line $y = 0$.

In function notation, the real roots are found when $f(x) = 0$.

There are three main numerical methods which can be used to estimate the solution of an equation:

- Graphical methods: Draw a sketch or use a graphical calculator. Use the change of sign methods to refine the solution.
- Change of Sign methods: Locate a real root between two points by detecting a change of sign in $f(x)$.
- Iterative formulae: Set up and use a formula that converges on a solution.

  Illustrate with staircase or cobweb graphs.

In using these methods note the following:

- The accuracy of each solution should be stated, usually to the required number of decimal places (dp).
- Be aware of the limitations of each of these methods.
- If available, use algebraic methods to give an exact solution.
- A sketch is worth a 1000 numbers! You should be familiar with the various standard graphs,

  see 68 • Apdx • Catalogue of Graphs

39.2 Locating Roots Graphically

There are two ways of locating the real roots graphically. The traditional method is to set the function to zero and plot the function directly.

For example if $f(x) = g(x)$, then rearrange to give $f(x) - g(x) = 0$ and plot $y = f(x) - g(x)$

However, some functions are too complicated to sketch directly, and it becomes simpler if our function $f(x) = g(x)$ is plotted as two separate curves, where the intersection of the two functions $f(x)$ and $g(x)$ will give the required solutions.

This method also makes it easier if the combined function crosses the x-axis at a very shallow angle making it difficult to read the actual root from the graph.

E.g. Consider the function: $e^x = 4x + 8$

Since e^x and $4x + 8$ are standard curves, it is easier to sketch them separately and observe the intersection of the two curves.

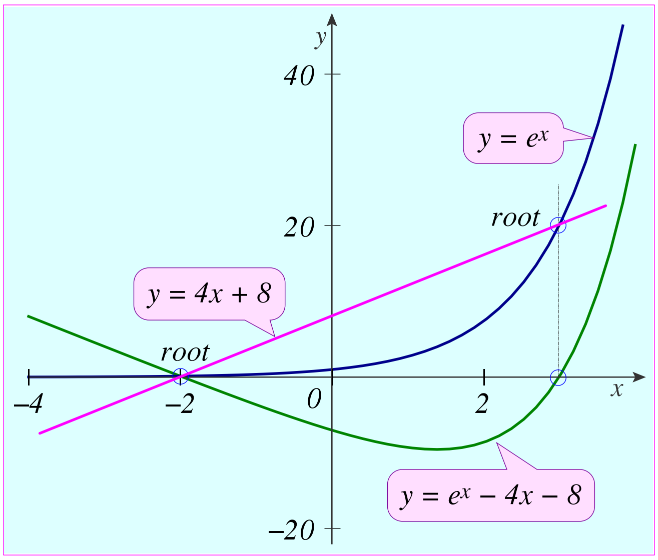
Sketch the LHS and RHS of the equation thus:

$y = e^x$ and
$y = 4x + 8$

The curve for $y = e^x - 4x - 8$ is shown for comparison.

Note how the roots of the original function are the same as the x values of the intersection of the two separate functions plotted.

Roots are located at:
$x \approx -2{\cdot}0, \quad x \approx 3{\cdot}0$



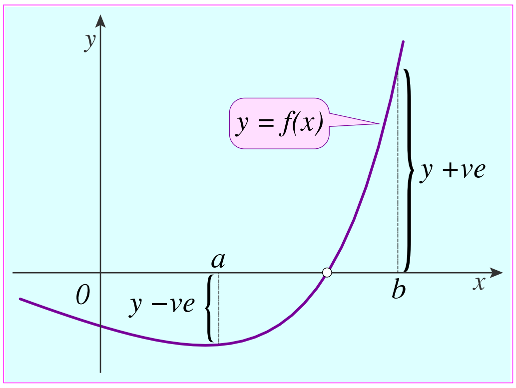
39.3 Change of Sign in $f(x)$

As seen from the diagram, right, as the curve crosses the x-axis (at the root), the value of $f(x)$ changes sign.

In this example, testing the function at points a and b, will show that the curve changes from −ve at point a, to +ve at point b.

We can then say that a root lies between $x = a$ and $x = b$, provided the function is continuous.

This is known as the interval, $a < x < b$



The change of sign is only valid if the function is set to zero and the function is continuous.

In the case of comparing two functions, say, $f(x)$ and $g(x)$, at an intersection, then set the equation to be $f(x) - g(x) = 0$.

39.4 Locating Roots Methodically

The following methods rely on choosing a range of values of x and testing them to see if $f(x)$ changes sign.

Sketching a graph is a first step in solving many of these numerical type problems, as this will often tell you how many solutions there are, and roughly what values of x to choose for testing. A graphing calculator is a useful tool for this.

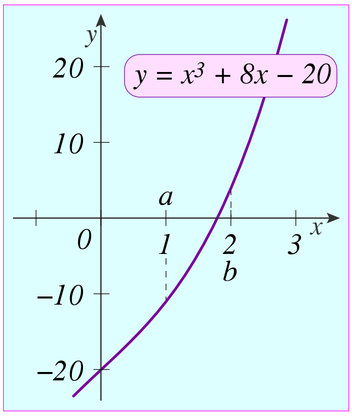
There are three alternate methods available, and you may wish to mix and match according to the situation presented in the question. Assuming a root is found in the interval of, say, $1 < x < 2$; a search for a change of sign in $f(x)$ is made by selecting values for x chosen thus:

- **Decimal search**: Use regularly spaced decimal values, such as $x = 1.1, 1.2, 1.3 \ldots 1.7, 1.8, 1.9$
  Once a change of sign is found, do another decimal search, but this time with smaller interval steps of 0·01, then steps of 0·001 and so on until the required accuracy is achieved.
  Use this method if an accurate graph is not available to you.
- **Interval bisection**: Bisect the interval and test for a change of sign, and keep on bisecting the subsequent intervals until the level of accuracy is achieved. Start with $x = 1.5, 1.75, 1.875$ as appropriate. A change of sign will govern which values to bisect.
- **Linear interpolation**: In this case, interpolate the probable value of the root from the values of $f(x)$ at the interval values, i.e. $f(1)$ and $f(2)$. You can then interpolate a new value of x based on the first interpolated value and so on.
  In practice it might be easier to do a simple interpolation on the first interval, then use either of the two methods above for further refinement. A certain amount of caution is required because the curve is not linear, so do not expect an accurate answer on the first interpolation – it is only a 'starter for 10'

E.g. Solve the equation $x^3 + 8x - 20 = 0$ accurate to 2 dp.

Solution:

Draw a sketch.



From this it can be see there is a solution in the interval $1 < x < 2$

Substituting these values in $f(x)$ and we observe a change of sign, which confirms a root in the given interval.

$f(1) = 1 + 8 - 20 = -11$

$f(2) = 8 + 16 - 20 = +4$

Decimal Search:

To speed up the calculation, observe that the root appears closer to $x = 2$ than $x = 1$. Start at $x = 2$ and initially use a difference of 0·2 between each x value:

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	−11		− 6·056	−3·104	+ 0·232	+4

Refine using steps of 0·05

x	1.60	1.65	1.70	1.75	1.80
$f(x)$	−3·1040		−1·4870	− 0·6406	+ 0·2320

Refine using steps of 0·01

x	1.75	1.76	1.77	1.78	1·785	1·79	1.80
$f(x)$	− 0·6406		− 0·2947	− 0·1202	− 0·3259	+ 0·0553	+ 0·2320

With an interval of $1{\cdot}785 < x < 1{\cdot}79$, we can say that the root is approximately 1·79 (2 dp), (note the extra column for $x = 1.785$ to help determine that 1.79 is the correct root to 2 dp).

Interval Bisection:

Bisect the interval given interval, $1{\cdot}0 < x < 2{\cdot}0$ giving $x = 1.5$

x	1.0	1.5	2.0
$f(x)$	−11	− 4·6250	+ 4

Bisect the new interval $1{\cdot}5 < x < 2{\cdot}0$, giving $x = 1.75$

x	1.50	1.75	2.00
$f(x)$	− 4·6250	− 0·6406	+ 4

Bisect the new interval, $1{\cdot}75 < x < 2{\cdot}0$ giving $x = 1.875$

x	1.750	1.875	2.000
$f(x)$	− 0·6406	+1·5918	+ 4

Bisect the new interval, $1{\cdot}75 < x < 1{\cdot}875$ giving $x = 1.8125$

x	1.750	1.8125	1.875
$f(x)$	− 0·6406	+ 0·4523	+1·5918

Bisect the new interval, $1{\cdot}75 < x < 1{\cdot}8125$

x	1.750	1.796875	1.8125
$f(x)$	− 0·6406	+ 0·1768	+ 0·4523

Bisect the new interval, $1{\cdot}75 < x < 1{\cdot}7969$

x	1.750	1.7734375	1.796875
$f(x)$	− 0·6406	− 0·2349	+ 0·1768

Bisect the new interval, $1{\cdot}7734 < x < 1{\cdot}7969$

x	1.7734375	1.7851562	1.796875
$f(x)$	− 0·2349	− 0·0298	+ 0·1768

Bisect the new interval, $1{\cdot}7852 < x < 1{\cdot}7969$

x	1.7851562	1.7910156	1.796875
$f(x)$	− 0·0298	+ 0·0732	+ 0·1768

With the new interval, $1{\cdot}7852 < x < 1{\cdot}7910$,
we can say that the root is approximately 1·79 (2 dp).

Linear Interpolation:

Using the values of $f(1)$ and $f(2)$ estimate the value of the root:

x	1.0	1.733	2.0
$f(x)$	−11	− 0·9259	+ 4

$$2 - \frac{4}{4+11}$$

Using the values of $f(1{\cdot}733)$ and $f(2)$ estimate the value of the root:

x	1.733	1.7834	2.0
$f(x)$	− 0·9259	− 0·0599	+ 4

$$2 - (2 - 1{\cdot}733)\left(\frac{4}{4+0{\cdot}9259}\right)$$

Using the values of $f(1{\cdot}7834)$ and $f(2)$ estimate the value of the root:

x	1.7834	1.7866	2.0
$f(x)$	− 0·0599	− 0·0038	+ 4

$$2 - (2 - 1{\cdot}7834)\left(\frac{4}{4+0{\cdot}0599}\right)$$

Using the values of $f(1{\cdot}7866)$ and $f(2)$ estimate the value of the root:

x	1.7866	1.78684	2.0
$f(x)$	− 0·0038	− 0·0002	+ 4

$$2 - (2 - 1{\cdot}7866)\left(\frac{4}{4+0{\cdot}0038}\right)$$

Once again, we can say that the root is approximately 1·79 (2 dp), particularly since the value of $f(1{\cdot}78684)$ is very small.

In Practise:

In this example, linear interpolation gave a more accurate answer in the least number of steps. However, it does require some extra maths to work our each new value of x. This is not a problem with a spreadsheet, but in exam conditions this may not be so easy.

Perhaps the easiest option is to use linear interpolation as the initial first step and then use a decimal search.

Using interpolation, the root is approximately:

$$b - (b - a)\left(\frac{f(b)}{f(b) + f(a)}\right) \quad \Rightarrow \quad 2 - \frac{4}{4 + 11} \approx 1.73$$

Since $f(1.73)$ is –ve, then the root must be in the interval $1{\cdot}73 < x < 2$

Set up some suitable values for a decimal search:

x	1.73	1.75	1.77	1.79	1.81	1.83
$f(x)$	−0·9822	−0·6406	−0·2947	+0·0553		

The interval is now: $1{\cdot}77 < x < 1{\cdot}79$. Refining the search with smaller increments:

x	1.770	1.775	1.780	1.785	1.787	1.79
$f(x)$	−0·2947		−0·1202	−0·0325	+0·0025	+0·0553

The interval is now: $1{\cdot}785 < x < 1{\cdot}787$. We can say the root is 1.79 to 2 dp.

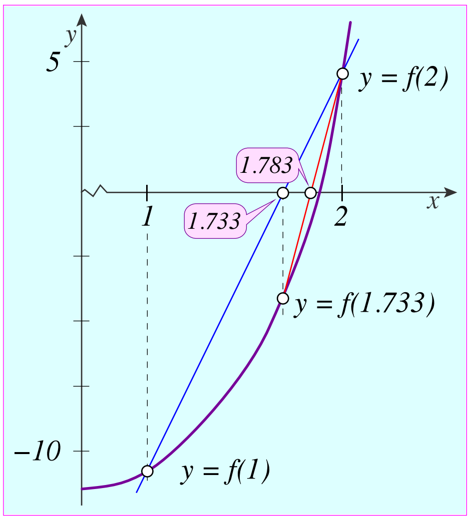
As you can see, the 'starter for 10' was not wholly accurate, but gave a very good starting point.

For interest, the most accurate figure found for the root is 1·78685492 (8 dp)

How Linear Interpolation works:

The blue line illustrates the first straight line interpolation between $f(1)$ and $f(2)$, giving the interception of the x–axis at 1.733.

This is followed by the second line in red between $f(1{\cdot}733)$ and $f(2)$, giving a new x value of 1.783. and so on.



DP accuracy:

If asked to find a root accurate to 2 dp, you need to work with values of x to 3 dp as a minimum. If our answer is 1.79 (2 dp), then you need to use an interval of $1{\cdot}785 < x < 1{\cdot}795$ to ensure the solution is within the prescribed accuracy.

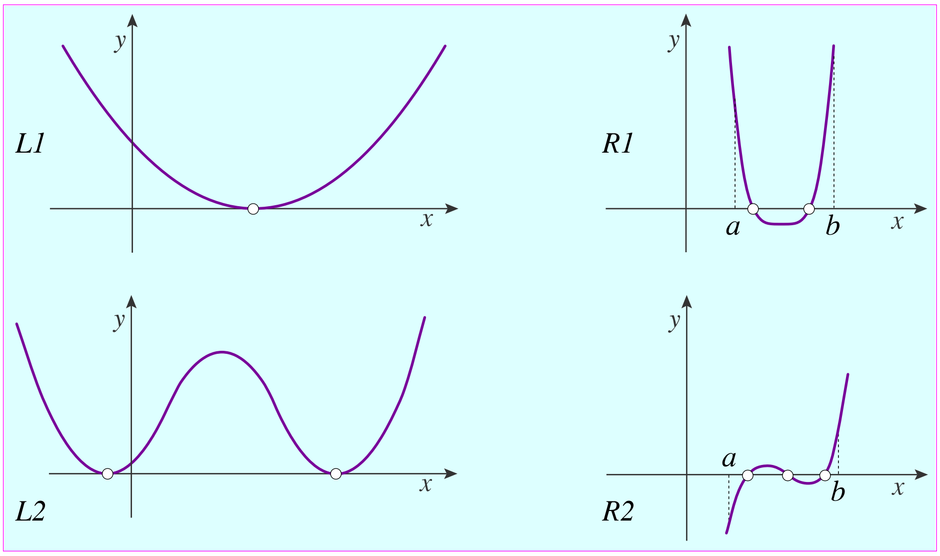
39.5 Limitations of the Change of Sign Methods

There are a few disadvantages with the change of sign methods. Notably it is time consuming and open to error when making several similar calculations, even with a half decent calculator. This method is not the best method to choose if a high degree of accuracy is required, in which case the iterative approach should be used.

There are three other traps for unwary players. These are:

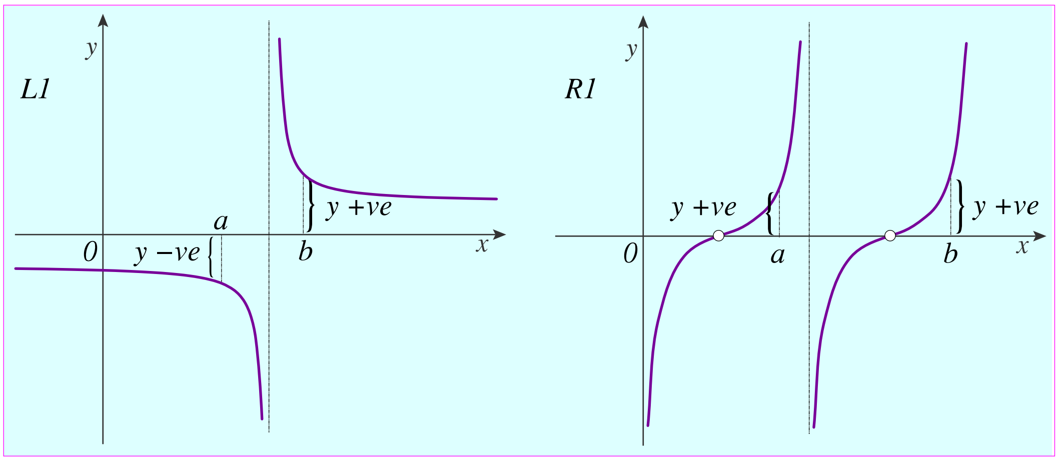
- The curve may touch the x-axis but not cross it, (repeated roots possibly).
- The chosen values of x for the interval search may be too course to find all the roots.
- The function may contain a discontinuity, such as an asymptote.

In the diagram below are two curves (L1 & L2) that touch the x-axis but do not cross it, and on the right hand side, are two curves that cross the x-axis between $x = a$ and $x = b$.
In R1, $f(a)$ and $f(b)$ are both +ve, therefore no root is detected.
In R2, $f(a)$ is −ve and $f(b)$ is +ve, indicating a root has been found, however, there are three roots in the interval, so potentially two roots may be missed.



Limitations of Change of Sign Methods

Not all functions are continuous, particularly functions like $f(x) = \tan x$, and $f(x) = \frac{1}{x-k}$. Functions that have asymptotes or other discontinuities may give a false indication of a root if the interval straddles the discontinuity and a change of sign is detected, see L1 below. On the other hand, R1 shows a curve in which $f(a)$ and $f(b)$ are both +ve, and no root is detected, missing the real root.



False Indication of a Root

39.6 Iteration to find Approximate Roots

This method uses an iterative formula in which the output of the first calculation is fed back into the same formula to find a second value of x, which, if the formula is chosen wisely, will lead to a series of x values that converge on the root. This is also know as a recurrence relationship, (see Sequences & Series).

This requires that you to rewrite a function $f(x)$ as:

$$x = g(x)$$

This can then be used as the basis of an iterative formulae such that:

$$x_{n+1} = g(x_n)$$

If the iteration converges it will approach some limit, r, such that:

$$r = g(r)$$

This limit will be the root of the original equation $f(x) = 0$

In a graphical sense, we are being asked to find the intersection points of $y = g(x)$ and $y = x$

The first step is to rearrange the function to make x the subject. There are many ways to rearrange a function, for example:

E.g.

$$x^4 - 10x + 9 = 0$$

- $x^4 = 10x - 9 \quad \Rightarrow \quad x = \sqrt[4]{10x - 9}$
- $10x = x^4 + 9 \quad \Rightarrow \quad x = \dfrac{x^4 + 9}{10}$
- $x(x^3 - 10) = -9 \quad \Rightarrow \quad x = \dfrac{-9}{(x^3 - 10)}$

In order to converge, the function $g(x)$ needs to be chosen such that the gradient of $g(x)$, as it crosses the line $y = x$, is less than the gradient of the line, which is 1. Which gives the rule that:

$$-1 < g'(x) < 1$$

or

$$|g'(x)| < 1$$

Having found a possible root, the change of sign method should be used to **prove** the result.

E.g. Take our function above:

$$x^4 - 10x + 9 = 0$$

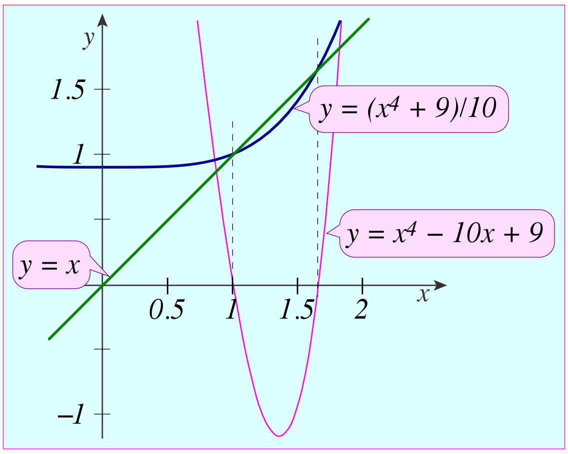
Rearrange to give:

$$x = \frac{x^4+9}{10}$$

Sketch $y = \frac{x^4+9}{10}$ and $y = x$

Also plotted as a comparison is:

$$y = x^4 - 10x + 9$$



From the graph we see that there are two roots of approximately $x \approx 1.0$ and $x \approx 1.7$

We can set up the iterative formulae as:

$$x_{n+1} = \frac{x_n^{\,4} + 9}{10}$$

Start with $x_0 = 0{\cdot}5$ and find the first root:

$$x_1 = \frac{(0{\cdot}5)^4 + 9}{10} = 0{\cdot}90625$$ Feed this answer back into the formula to give:

$$x_2 = \frac{(0{\cdot}90625)^4 + 9}{10} = 0{\cdot}96745$$ Again feed back the answer and so on…

$$x_3 = \frac{(0{\cdot}96745)^4 + 9}{10} = 0{\cdot}98760$$ *etc*

$$x_4 = 0{\cdot}99513$$

$$x_5 = 0{\cdot}99806$$

$$x_6 = 0{\cdot}99922$$

After just 6 iterations it can be seen the first root is in fact 1.00 (2 dp)

After 14 iterations the value of x is 0·9999995.

The iterative process can also be used with $x_0 = 1{\cdot}5$ as a starting value to give:

$$x_1 = \frac{(1{\cdot}5)^4 + 9}{10} = 1{\cdot}40625$$

$$x_2 = \frac{(1{\cdot}40625)^4 + 9}{10} = 1{\cdot}29107$$

$$x_3 = \frac{(1{\cdot}29107)^4 + 9}{10} = 1{\cdot}17784$$

$$x_4 = 1{\cdot}09246$$

$$x_5 = 1{\cdot}04244$$

$$x_6 = 1{\cdot}01809$$

$$x_7 = 1{\cdot}00743$$

This time the value converges from the other side — provided you choose a value below the second root of $x \approx 1{\cdot}66$ (found graphically). If a value of $x_0 > 1{\cdot}67$ is chosen, the iterations diverge very quickly.

To prove the result, find $f(0{\cdot}99922) = +0{\cdot}00468$ and $f(1{\cdot}00743) = -0{\cdot}04424$

A change of sign proves the root.

Of course the root could have been found by inspection, since $f(1) = 0$ and $(x - 1)$ is a factor, but this does illustrate that these numerical methods only give close approximations to the answer.

39.7 Staircase & Cobweb Diagrams

The iterative process can be illustrated with a staircase or cobweb diagram depending on the gradient of the curve as it crosses the line $y = x$. In drawing the x_n lines, always start with x_0 and draw a vertical line to the **curve**, then move across to the straight line. Use the straight line as a 'transfer' line to find the next value of x.

Convergence is only possible if: $-1 < g'(x) < 1$, as the curve crosses the straight line, $y = x$.

E.g. Find a positive root for $x^3 - 8x + 3 = 0$ using an iterative formula with a starting value of $x_0 = 1{\cdot}0$:

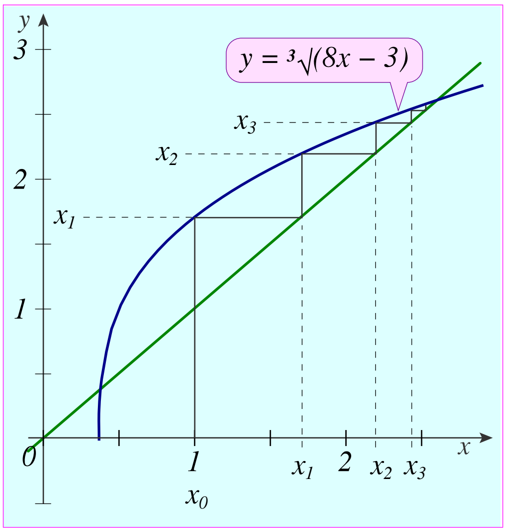
Take our function above:

$$x^3 - 8x + 3 = 0$$

Rearrange to give:

$$x = \sqrt[3]{8x - 3}$$

Sketch $y = \sqrt[3]{8x - 3}$ and $y = x$



From the sketch we see that there are two positive roots of approximately 0·4 and 2·6.

There is also a root at $x \approx -3{\cdot}0$

Set up the iterative formulae as:

$$x_{n+1} = \sqrt[3]{8x_n - 3}$$

Note that the gradient of $y = \sqrt[3]{8x - 3}$ is > 1 at the first root and the iterative process will not work on this root. The gradient of the line at the second root is positive and < 1, so this will produce a staircase diagram.

Start with $x_0 = 1{\cdot}0$ and find the root:

$x_1 = \sqrt[3]{(8 \times 1.00) - 3} = 1{\cdot}70997$ Feed this answer back to give:

$x_2 = \sqrt[3]{(8 \times 1{\cdot}70997) - 3} = 2{\cdot}20219$ Again feed back the answer and so on…

$x_3 = \sqrt[3]{(8 \times 2{\cdot}20219) - 3} = 2{\cdot}44507$

$x_4 = \sqrt[3]{(8 \times 2{\cdot}44507) - 3} = 2{\cdot}54893$

$x_5 = \sqrt[3]{(8 \times 2{\cdot}54893) - 3} = 2{\cdot}59087$

$x_6 = \sqrt[3]{(8 \times 2{\cdot}59087) - 3} = 2{\cdot}60742$

$x_7 = \sqrt[3]{(8 \times 2{\cdot}60742) - 3} = 2{\cdot}61389$

$x_8 = \sqrt[3]{(8 \times 2{\cdot}61389) - 3} = 2{\cdot}61642$

After 8 iterations it can be seen that the second root is 2·62 (2 dp).

You might try a starting value of $x_0 = 3{\cdot}0$ and note how the values converge from the other side.

E.g. Find the root for $\frac{1}{e^x} - x = 0$, using an iterative formula with a starting value of $x_0 = 0{\cdot}25$

Take our function above:

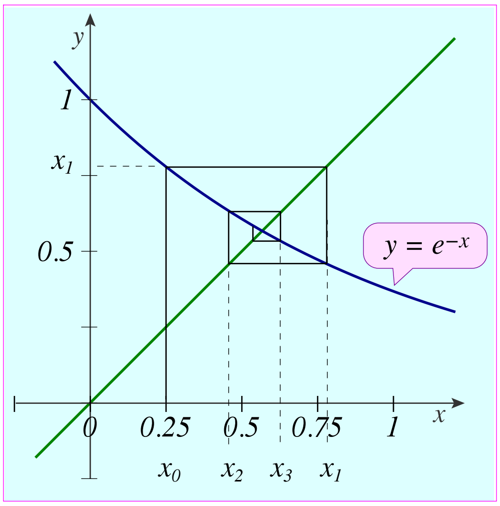
$$\frac{1}{e^x} - x = 0$$

Rearrange to give:

$$x = e^{-x}$$

Sketch $y = e^{-x}$ and $y = x$

From the sketch we see that there is a root of approximately 0·55



Set up the iterative formulae as:

$$x_{n+1} = e^{-x_n}$$

The gradient of the line at the intersection is negative and < 1, so this will produce a cobweb diagram.
Start with $x_0 = 0{\cdot}25$ and find the root:

$x_1 = e^{-0{\cdot}25} = 0{\cdot}77880$ Feed this answer back to give:

$x_2 = e^{-0{\cdot}77880} = 0{\cdot}45896$ Again feed back the answer and so on…

$x_3 = e^{-0{\cdot}45896} = 0{\cdot}63194$

$x_4 = e^{-0{\cdot}63194} = 0{\cdot}53156$

$x_5 = e^{-0{\cdot}53156} = 0{\cdot}58769$

$x_6 = e^{-0{\cdot}587689} = 0{\cdot}55561$

$x_7 = e^{-0{\cdot}55561} = 0{\cdot}57372$

$x_8 = e^{-0{\cdot}57372} = 0{\cdot}56342$

After 8 iterations it can be seen that the root is 0·56 (2 dp).
In these types of iterations, the values of x oscillate around the root.

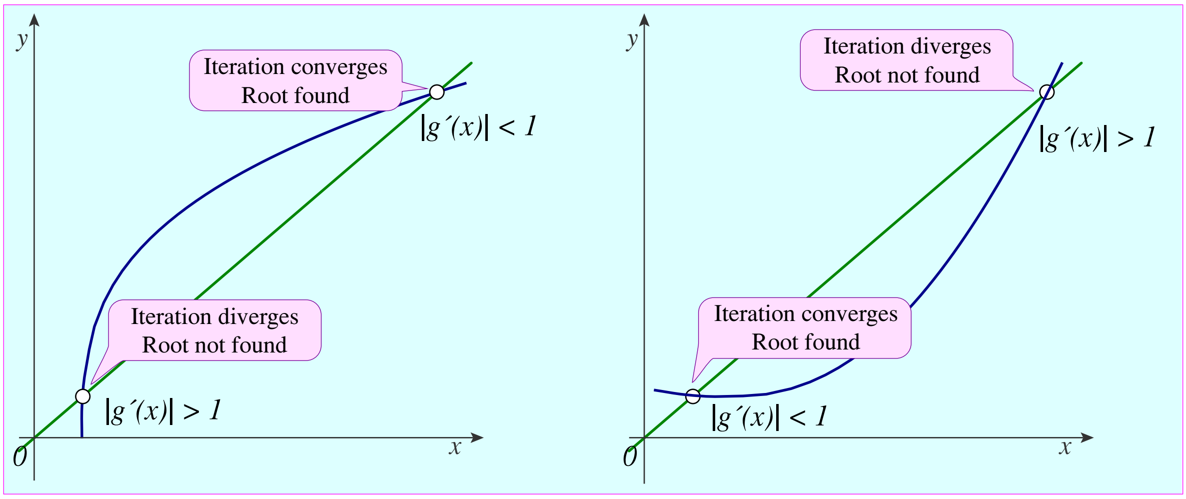
39.8 Limitations of the Iterative Methods

The iterative method will fail if the modulus of the gradient of $g(x)$ is greater than 1, as it crosses the line $y = x$. This leads to a diverging series of x values.

For convergence the rule is:

$$-1 < g'(x) < 1$$

or $$|g'(x)| < 1$$



Gradient Rules for Convergence

The rules can be summarised thus:

- If $|g'(x)|$ is small, the series converges quickly.
- If the gradient is positive, the series approaches the root from one side, or the other, and produces a staircase diagram.
- If the gradient is negative, the series alternates above and below the root and produces a cobweb diagram.

39.9 Choosing Convergent Iterations

Not every arrangement of $x = g(x)$ leads to an iterative formula that converges. In which case another rearrangement of $f(x)$ needs to be found.

Note that two different arrangements will be inverses, therefore, the curves will be reflections in the line $y = x$. This means that if one fails, the other one will provide a solution. (See Fig above).

39.10 Numerical Solutions Worked Examples

39.10.1 Example:

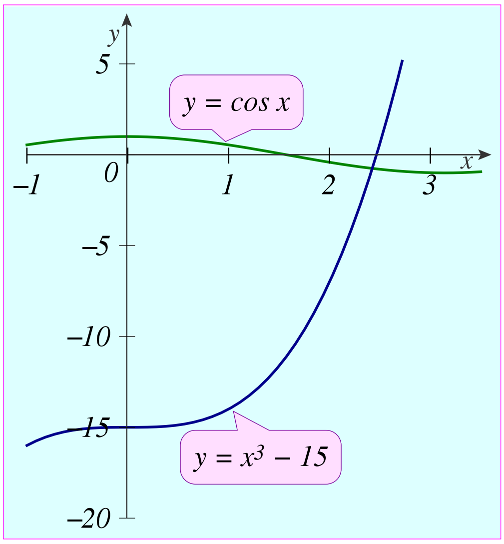
1 Show that $x^3 - \cos x - 15 = 0$ has only one root, and find the root to 2 dp.

Solution:

Rearrange the function to be:

$x^3 - 15 = \cos x$

Draw a sketch of $y = x^3 - 15$ and $y = \cos x$



There is only one intersection of the two lines and hence only one root, in the interval:

$$2 < x < 3$$

Tip: set calculator to radians.

Halving the interval and finding $f(x)$, between 2 and 2·5:

x	2·1	2·2	2·3	2·4	2·5	2·6
$f(x)$	−5.2342	−3.7635	−2.1667	−0.4386	+1.42614	

Refining with a decimal search

x	2·40	2·42	2·43	2·44	2·46	2·48	2·50
$f(x)$	−0.4386	−0.0768	+0·1062	+0.2906			+1.42614

Last search

x	2·420	2·422	2·424	2·426	2·428	2·430
$f(x)$	−0.0768	−0.0403	−0.0037	+0.0329		+0·1062

The root is in the interval: $2{\cdot}424 < x < 2{\cdot}426$

Hence root is 2.42 (2 dp)

2 Show that there is an intersection between the functions:

$$y = e^{\frac{1}{6}x} \qquad y = \sqrt[3]{3x + 5}$$

which has an x coordinate between 6 and 7.
Show that the two equations can be written in the form:

$$x = 2\ln(3x + 5)$$

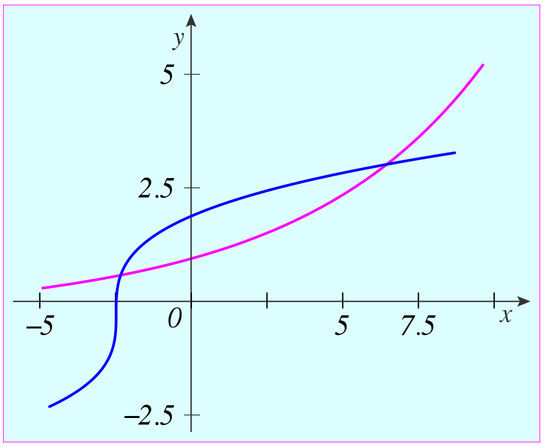
and using a suitable iterative formula, find the value of the x coordinate to 3 dp.

Solution:

To test for a root in the interval

$$6 < x < 7$$

substitute both values into each equation and compare results to see if there is a change of sign.



x	6	7
$y = e^{\frac{1}{6}x}$	2·718	3·211
$y = \sqrt[3]{3x + 5}$	2·844	2·962
$e^{\frac{1}{6}x} - \sqrt[3]{3x + 5}$	− 0·126	+ 0·249

Set $f(x) - g(x) = 0$

From the table, you can see a change of sign when the functions are compared.

To show the equations can be written in the given manner, equate both functions.:

$$e^{\frac{1}{6}x} = \sqrt[3]{3x + 5}$$

$$\frac{1}{6}x = \ln(3x + 5)^{\frac{1}{3}}$$

$$x = \frac{6}{3}\ln(3x + 5)$$

$$x = 2\ln(3x + 5)$$

The iterative formula becomes:

$$x_{n+1} = 2\ln(3x_n + 5)$$

Using $x_n = 6{\cdot}0$

$$x_1 = 2\ln(3 \times 6 + 5) = 6{\cdot}2710$$

$$x_2 = 2\ln(3 \times 6{\cdot}2710 + 5) = 6{\cdot}3405$$

$$x_3 = 2\ln(3 \times 6{\cdot}3405 + 5) = 6{\cdot}3579$$

$$x_4 = 2\ln(3 \times 6{\cdot}3579 + 5) = 6{\cdot}3622$$

$$x_5 = 2\ln(3 \times 6{\cdot}3622 + 5) = 6{\cdot}3633$$

$$x_6 = 6{\cdot}3636$$

$$x_7 = 6{\cdot}3637$$

Root is 6·364 to 3 dp.

3 Show that the function $f(x) = x^3 - 5x^2 - 6$ has a real root in the interval $5 < x < 6$.
Rearrange the function in the form: $x = \sqrt{\frac{c}{x+b}}$, where c and b are constants.
Using this form, write a suitable iterative formula and say whether it converges or diverges.

Solution:

Look for a change of sign in the interval $5 < x < 6$:

x	5	5.3	5.5	5.7	6
$f(x)$	−6·0	+2·427			+30

Change of sign, therefore a root exists.

Rearranging the function:

$$x^3 - 5x^2 - 6 = 0$$

$$x^2(x - 5) = 6$$

$$x^2 = \frac{6}{(x - 5)}$$

$$x = \sqrt{\frac{6}{(x - 5)}}$$

Making the iterative formula:

$$x_{n+1} = \sqrt{\frac{6}{(x_n - 5)}}$$

Let $x_0 = 5$

$$x_1 = \sqrt{\frac{6}{(5 - 5)}} = \text{no solution}$$

Let $x_0 = 6$

$$x_1 = \sqrt{\frac{6}{(6 - 5)}} = 2{\cdot}44948$$

$$x_1 = \sqrt{\frac{6}{(2{\cdot}44948 - 5)}} = \text{no solution}$$

The iterative formula does not converge.
However, this one below does, very slowly:

$$x^3 = 5x^2 + 6$$

$$x = \sqrt[3]{5x^2 + 6}$$

$$x_{n+1} = \sqrt[3]{5x_n^2 + 6}$$

$$x_1 = \sqrt[3]{5 \times 5^2 + 6} = 5{\cdot}07875$$

$$x_2 = 5{\cdot}1295$$

$$x_3 = 5{\cdot}1621$$

$$\ldots$$

$$x_{20} = 5{\cdot}2201$$

Root = 5·220 (3dp) (Note gradient at this point ≈ 0·64)

4

39.11 Numerical Solutions Digest

The accuracy of each solution should be stated, usually to the required number of decimal places (dp).

Iterative method:

Rewrite the function $f(x)$ as:

$$x = g(x)$$

The iterative formula is:

$$x_{n+1} = g(x_n)$$

For convergence the rule is:

$$-1 < g'(x) < 1$$

$$\text{or} \qquad |g'(x)| < 1$$

Calculator work:

On a calculator, with an iterative formula of, say, $\sqrt[3]{28 - 5x_n}$ and an $x_0 = 2$, place 2 into the 'Ans' field, then enter: $(28 - 5Ans)^{1 \div 3}$

Each press of the '=' key will give the next iteration.

40 • C3 • Estimating Areas Under a Curve

40.1 Estimating Areas Intro

This is part of the Numerical Methods section of the syllabus.

Normally, areas under a curve are calculated by using integration, however, for functions that are really difficult to integrate, numerical methods have to be used to give a good approximation. In reality, you will only need these methods for those hard cases and when told to use these methods in an exam!

In the syllabus there are three methods you need to know:

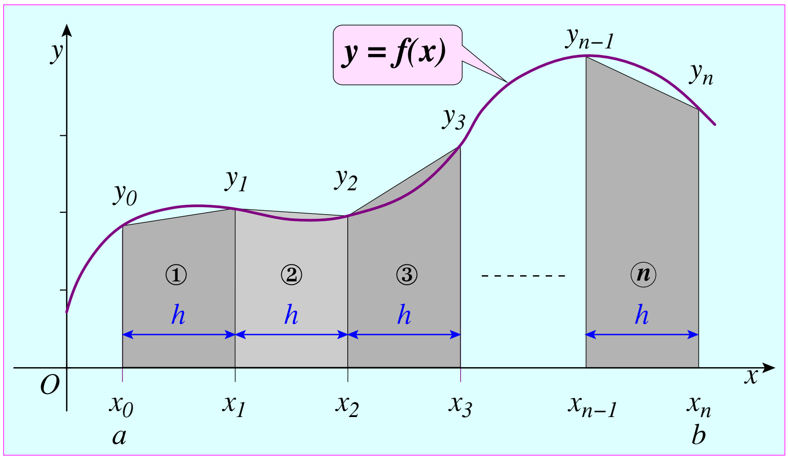
- The Trapezium rule – covered in C2
- The Mid-ordinate Rule – C3 (AQA requirement)
- Simpson's Rule – C3

All these methods are based on the premise of dividing the area under the curve into thin strips, calculating the area of each strip and then summing these areas together to find an overall estimate. Clearly, the more strips that are used, the more accurate the answer, and in practise, many hundreds of strips would be chosen with results being calculated electronically.

Each method has its advantages and disadvantages.

Exam hint: always start the counting of the ordinates from zero, and draw a diagram, even if you don't know what the function really looks like.

40.2 Trapezium Rule – a Reminder



The Trapezium Rule

For a function $f(x)$ the approximate area is given by:

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

where $\quad h = \dfrac{b-a}{n} \quad$ and $\quad n =$ number of strips

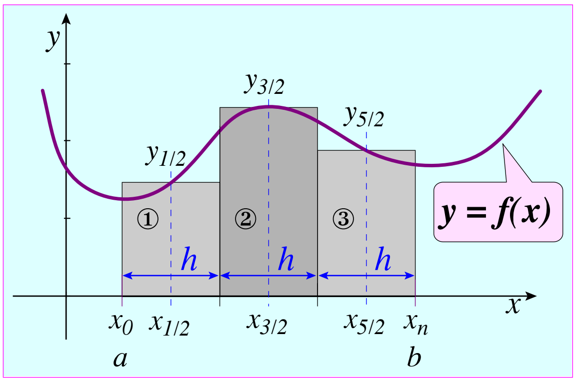
Recall that the disadvantage of the trapezium rule is that the space between the trapezium and the curve is either an under or over estimate of the real area, although this is offset if a large number of strips is used.

To use the trapezium rule, ensure that the part of the curve of interest is either all above or all below the x-axis, such that y is either $y > 0$ *OR* $y < 0$.

See the C2 section on the Trapezium Rule for more.

40.3 Mid-ordinate Rule

Both the Trapezium rule and the Mid-ordinate rule use straight lines to approximate the curve of the function. With the mid-ordinate rule, a line is drawn through the midpoint of the curve cut out by each strip, which attempts to average out the area.



Mid-ordinate rule

For a function $f(x)$ the approximate area is given by:

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx \approx h\left[y_{1/2} + y_{3/2} + \ldots + y_{n-3/2} + y_{n-1/2}\right]$$

where $h = \dfrac{b-a}{n}$ and $n =$ number of strips

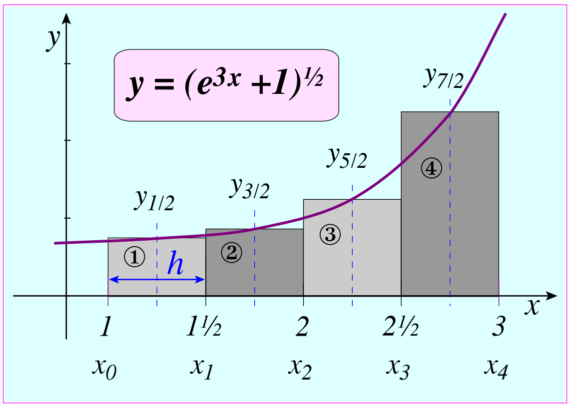
40.3.1 Example:

1 Use the mid-ordinate rule with 4 strips (5 ordinates) to estimate the area given by

$$\int_1^3 \left(e^{3x} + 1\right)^{1/2} dx$$

Solution:

Draw a sketch, even if you are not sure of the exact shape of the function, although in this case it is bound to be an exponential curve of some sort.



Then calculate h:

$$h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$

Set up a table to tabulate the results:

$x_{mid-ord}$	x		$f(x)$
$x_{1/2}$	1·25	$\Rightarrow f(x_{1/2})$	6·5970
$x_{3/2}$	1·75	$\Rightarrow f(x_{3/2})$	13·8407
$x_{5/2}$	2·25	$\Rightarrow f(x_{5/2})$	29·2414
$x_{7/2}$	2·75	$\Rightarrow f(x_{7/2})$	61·8759

$$\text{Area} \approx \frac{1}{2}\left[6{\cdot}5970 + 13{\cdot}8407 + 29{\cdot}2414 + 61{\cdot}8759\right] \Rightarrow \frac{111{\cdot}555}{2} = 55{\cdot}78 \text{ sq units (2 dp)}$$

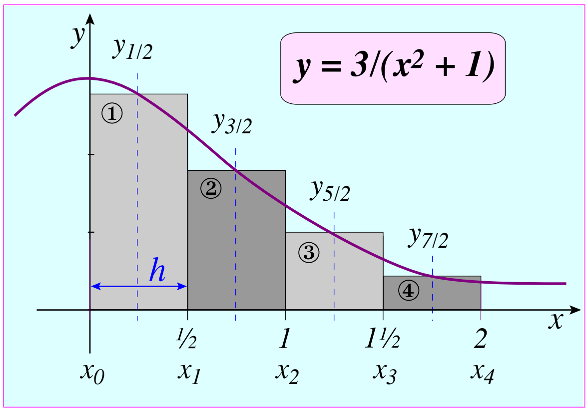
Compare this with the proper integrated value of 57·10 sq units.

2 Use the mid-ordinate rule with 4 strips (5 ordinates) to estimate the area given by

$$\int_0^2 \frac{3}{x^2 + 1}\, dx$$

Solution:

Draw a sketch.



Then calculate h:

$$h = \frac{b - a}{n} = \frac{2 - 0}{4} = \frac{1}{2}$$

Set up a table to tabulate the results:

$x_{mid-ord}$	x			$f(x)$
$x_{1/2}$	0·25	$\Rightarrow$	$f(x_{1/2})$	2·8235
$x_{3/2}$	0·75	$\Rightarrow$	$f(x_{3/2})$	1·9200
$x_{5/2}$	1·25	$\Rightarrow$	$f(x_{5/2})$	1·1707
$x_{7/2}$	1·75	$\Rightarrow$	$f(x_{7/2})$	0·7385

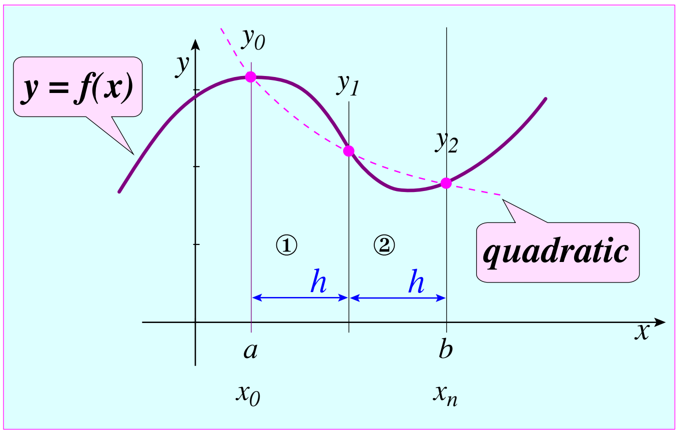
$$\text{Area} \approx \frac{1}{2}[2{\cdot}8235 + 1{\cdot}92 + 1{\cdot}1707 + 0{\cdot}7385] \Rightarrow \frac{6{\cdot}6527}{2} = 3{\cdot}33 \text{ sq units (2 dp)}$$

Compare this with the proper integrated value of 3·3214 sq units.

40.4 Simpson's Rule

In this case, the Simpson's rule finds a better fit with the function curve by using a series of quadratic curves instead of a straight lines. Each quadratic curve is made to fit between two strips and therefore this method requires an even number of strips.

The diagram illustrates this with an exaggerated function curve, and shows a quadratic curve used to fit the mid point and end points of the two strips.



Simpson's Rule showing two strips

For a function $f(x)$ the approximate area is given by:

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right]$$

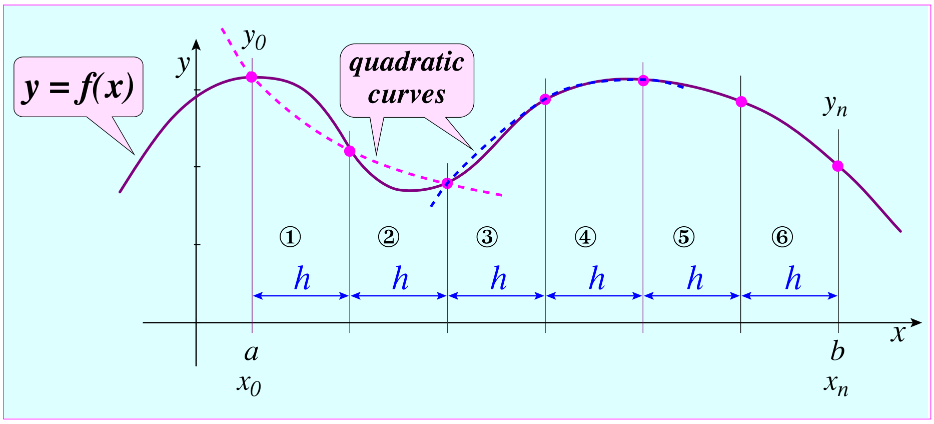
where $h = \dfrac{b-a}{n}$ and $n =$ an EVEN number of strips

In simpler terms:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[(\text{first + last ordinate}) + 4(\text{sum of odd ordinates}) + 2(\text{sum of even ordinates})\right]$$

The advantages of using Simpson's rule are:

- Accurate for any cubic graph, but less accurate for higher order functions
- More accurate than the other two methods discussed.



Simpson's Rule showing six strips

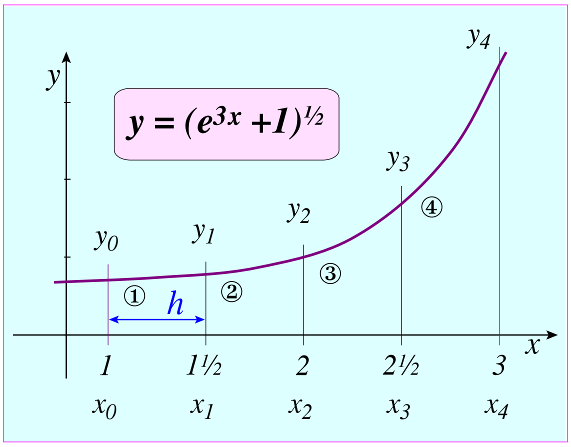
40.4.1 Example:

1 Use Simpson's rule with 4 strips (5 ordinates) to estimate the area given by

$$\int_1^3 \left(e^{3x} + 1\right)^{1/2} dx$$

Solution:

Draw a sketch, even if you are not sure of the exact shape of the function, although in this case it is bound to be an exponential curve of some sort.



Then calculate h:

$$h = \frac{b - a}{n} = \frac{3 - 1}{4} = \frac{1}{2}$$

Set up a table to tabulate the results:

$x_{ordinate}$	x			$f(x)$	
x_0	1·0	$\Rightarrow$	$f(x_0)$	4·5919	*First*
x_1	1·5	$\Rightarrow$	$f(x_1)$	9·5403	*Odd*
x_2	2·0	$\Rightarrow$	$f(x_2)$	20·1104	*Even*
x_3	2·5	$\Rightarrow$	$f(x_3)$	42·5328	*Odd*
x_4	3·0	$\Rightarrow$	$f(x_4)$	90·0227	*Last*

$$\text{Area} \approx \frac{1}{2} \times \frac{1}{3}\left[(4{\cdot}5919 + 90{\cdot}0227) + 4(9{\cdot}5403 + 42{\cdot}5328) + 2(20{\cdot}1104)\right]$$

$$\text{Area} \approx \frac{1}{6} \times 343{\cdot}115 \approx 57{\cdot}16 \text{ sq units (2 dp)}$$

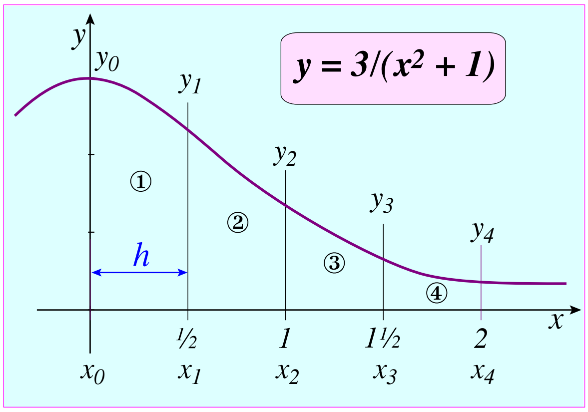
This compares with the previous calculation by the mid-ordinate rule of 55·78 and a fully integrated value of 57·10 sq units.

2 Use Simpson's rule with 4 strips (5 ordinates) to estimate the area given by:

$$\int_0^2 \frac{3}{x^2 + 1}\, dx$$

Solution:

Draw a sketch.



Then calculate h:

$$h = \frac{b - a}{n} = \frac{2 - 0}{4} = \frac{1}{2}$$

Set up a table to tabulate the results:

$x_{ordinate}$	x			$f(x)$	
x_0	0·0	$\Rightarrow$	$f(x_0)$	3·000	*First*
x_1	0·5	$\Rightarrow$	$f(x_1)$	2·400	*Odd*
x_2	1·0	$\Rightarrow$	$f(x_2)$	1·500	*Even*
x_3	1·5	$\Rightarrow$	$f(x_3)$	0·923	*Odd*
x_4	2·0	$\Rightarrow$	$f(x_4)$	0·600	*Last*

$$\text{Area} \approx \frac{1}{2} \times \frac{1}{3}\left[(3{\cdot}000 + 0{\cdot}600) + 4(2{\cdot}400 + 0{\cdot}923) + 2(1{\cdot}50)\right]$$

$$\text{Area} \approx \frac{1}{6} \times 19{\cdot}892 \approx 3{\cdot}32 \text{ sq units (2 dp)}$$

This compares with the previous calculation by the mid-ordinate rule of 3·33 and a fully integrated value of 3·3214 sq units.

i.e. $$\int_0^2 \frac{3}{x^2 + 1}\, dx = 3\left[\tan^{-1} x\right]_0^2 = 3{\cdot}3214$$

40.5 Relationship Between Definite Integrals and Limit of the Sum

Consider the function $y = f(x)$ as shown in the diagram below. In this case the width of each strip is a small value of x, called δx.

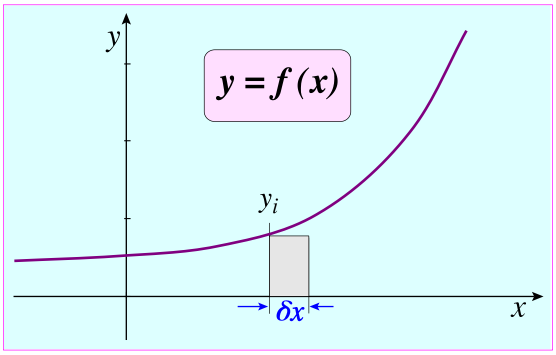
The height of each strip is the value of y at the start of each strip. For the ith strip, $y = y_i$

Hence, the area of the ith strip is given by:

$$A = y_i \, \delta x$$

But $$y = f(x)$$

Hence $$A = f(x_i) \, \delta x$$



The area under the curve is the summation of all these strips, therefore the area is given approximately by:

$$A \approx \sum_{i=1}^{n} f(x_i) \, \delta x$$

If δx is very, very small, the accuracy of the calculation improves such that, as y tends towards zero, then:

$$A = \lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i) \, \delta x$$

Hence, the limit of the sum becomes the equivalent of the definite integral thus:

$$\lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i) \, \delta x = \int_a^b f(x) \, dx$$

41 • C3 • Trig: Functions & Identities

41.1 Degrees or Radians

Generally the use of degrees or radians in a question is self explanatory, but the general terms, use of degrees will be made clear by using the degree symbol.

All the trig identities work for either degrees or radians.

41.2 Reciprocal Trig Functions

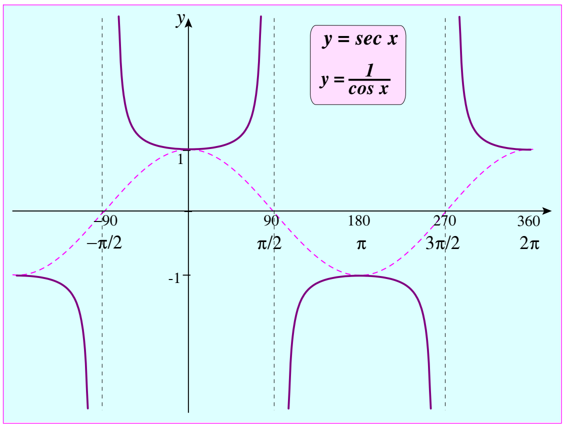
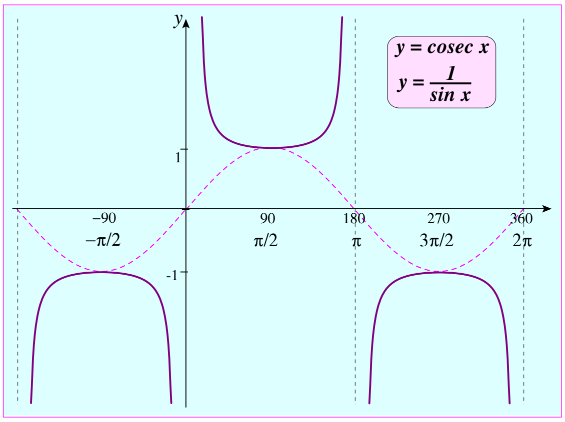
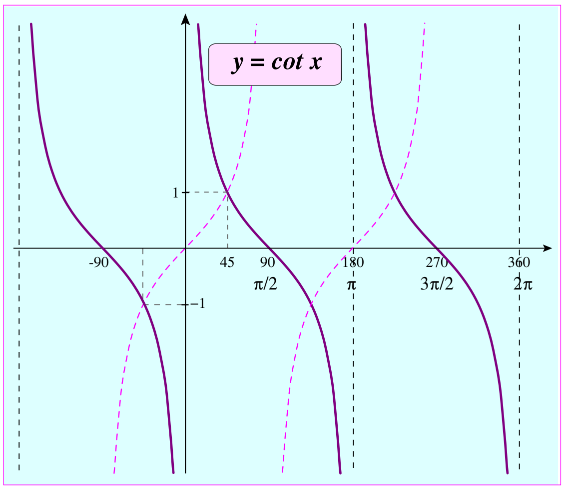
From earlier work we know about $sin\,\theta,\ cos\,\theta, and\ tan\,\theta$, not forgetting that $tan\,\theta = \frac{sin\,\theta}{cos\,\theta}$.

Three more ratios are generated when taking the reciprocal of these trig functions.

Full Name	Short Name	Definition	Limitations
$secant\,\theta$	$sec\,\theta$	$sec\,\theta = \frac{1}{cos\,\theta}$	$cos\,\theta \neq 0$
$cosecant\,\theta$	$cosec\,\theta$	$cosec\,\theta = \frac{1}{sin\,\theta}$	$sin\,\theta \neq 0$
$cotangent\,\theta$	$cot\,\theta$	$cot\,\theta = \frac{1}{tan\,\theta} = \frac{cos\,\theta}{sin\,\theta}$	$tan\,\theta \neq 0$; $sin\,\theta \neq 0$

Note that the above ratios are undefined if $sin\,\theta,\ cos\,\theta,\ or\ tan\,\theta$ are 0.

41.3 Reciprocal Trig Functions Graphs

Function	Properties	Illustration
$y = sec\,x$	**Secant Function:** Even function Domain: $x \in \mathbb{R},\ x \neq \frac{\pi}{2} + n\pi$ Range: $-1 \geqslant f(x) \geqslant 1$ $\lvert sec\,x \rvert \geqslant 1$ Periodic function, period 2π y-intercept: $(0,\ 1)$ Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$ where $cos\,x$ crosses the x-axis at odd multiples of ½π ($cos\,x = 0$) Line symmetry about the y-axis and every vertical line passing through each vertex.	
$y = cosec\,x$	**Cosecant Function:** Odd function Domain: $x \in \mathbb{R},\ x \neq n\pi$ Range: $-1 \geqslant f(x) \geqslant 1$ $\lvert cosec\,x \rvert \geqslant 1$ Periodic function, period 2π No x or y intercepts Vertical asymptotes: $x = n\pi$ where $sin\,x$ crosses the x-axis at any multiples of π ($sin\,x = 0$) Rotational symmetry about the origin - order 2. Line symmetry about every vertical line passing through each vertex.	
$y = cot\,x$	**Cotangent Function:** Odd function Domain: $x \in \mathbb{R},\ x \neq n\pi$ Range: $f(x) \in \mathbb{R}$ Periodic function, period π x-intercepts: $\left(\frac{\pi}{2} + n\pi,\ 0\right)$ where $tan\,x$ has asymptotes Vertical asymptotes: $x = n\pi$ where $tan\,x$ crosses the x-axis at any multiples of π ($tan\,x = 0$) Rotational symmetry about the origin - order 2.	

41.4 Reciprocal Trig Functions Worked Examples

To solve problems involving the reciprocal trig ratios, first solve for $sin\,\theta$, $cos\,\theta$, $and\ tan\,\theta$.

41.4.1 Example:

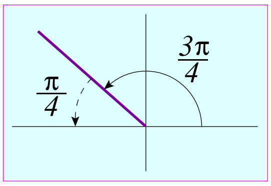
1 Find the exact value of $sec\,\dfrac{3\pi}{4}$.

Solution:

As $sec\,\theta = \dfrac{1}{cos\,\theta}$ we first solve $cos\,\dfrac{3\pi}{4}$

$$cos\,\frac{3\pi}{4} = -\,cos\,\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$sec\,\frac{3\pi}{4} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$$



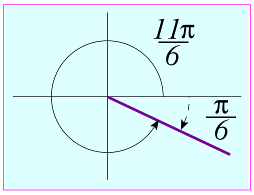
2 Find the exact value of $cot\,\dfrac{11\pi}{6}$.

Solution:

As $cot\,\theta = \dfrac{1}{tan\,\theta}$ we first solve $tan\,\dfrac{11\pi}{6}$

$$tan\,\frac{11\pi}{6} = -\,\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

$$cot\,\frac{11\pi}{6} = \frac{1}{-\frac{1}{\sqrt{3}}} = -\sqrt{3}$$



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41.5 Pythagorean Identities

From C1/C2 we established the Pythagorean Identity:

$$\cos^2\theta + \sin^2\theta \equiv 1$$

Two other versions can be derived from this identity.

Version 1

Divide the above by $\sin^2\theta$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

Version 2

Divide the above by $\cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

41.5.1 Example:

1 Solve $3\sec^2\theta - 5\tan\theta - 4 = 0$ for θ between $0 \leqslant \theta \leqslant 360°$

Solution:

$$3\sec^2\theta - 5\tan\theta - 4 = 0$$

$$3\left(1 + \tan^2\theta\right) - 5\tan\theta - 4 = 0$$

$$3 + 3\tan^2\theta - 5\tan\theta - 4 = 0$$

$$3\tan^2\theta - 5\tan\theta - 1 = 0$$

$$\tan\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 4 \times 3 \times 1}}{6}$$

$$= \frac{5 \pm \sqrt{13}}{6}$$

$$\therefore \tan\theta = 1{\cdot}847 \quad \text{or} \quad \tan\theta = -0{\cdot}180$$

$$\tan\theta = 1{\cdot}847 \quad \Rightarrow \quad \theta = 61{\cdot}6°,\ 241{\cdot}6°$$

$$\tan\theta = -0{\cdot}180 \quad \Rightarrow \quad \theta = 169{\cdot}8°,\ 349{\cdot}8°$$

2 Show that:

$$\frac{\sec^2\theta - 1}{\sec^2\theta} \equiv \sin^2\theta$$

Solution:

Using the LHS:

$$\frac{\sec^2\theta - 1}{\sec^2\theta} \equiv \frac{1 + \tan^2\theta - 1}{\sec^2\theta} \equiv \frac{\tan^2\theta}{\sec^2\theta}$$

$$\equiv \tan^2\theta\cos^2\theta$$

$$\equiv \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta$$

$$\equiv \sin^2\theta$$

$$\equiv RHS$$

41.6 Compound Angle (Addition) Formulae

The expansion of expressions of the form of $\sin(A \pm B)$, $\cos(A \pm B)$, & $\tan(A \pm B)$ are completed using the following **Compound Angle** or **Addition** identities.

The proof of these first four are not required for the exam, but they should be learnt.

$$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

From the four identities above, the identities for $\tan(A \pm B)$ can be derived and could be asked for in the exam.

For : $\tan(A + B)$

$$\tan(A + B) \equiv \frac{\sin(A + B)}{\cos(A + B)}$$
$$\equiv \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$
$$\equiv \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$
$$\equiv \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}}$$
$$\equiv \frac{\tan A + Tan B}{1 - \tan A \tan B}$$

Similarly for : $\tan(A - B)$

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan(A + B) \equiv \frac{\tan A + Tan B}{1 - \tan A \tan B}$$
$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

41.6.1 Example:

1 Evaluate *sin 75°*, (non calculator method).

Solution:

The solution to all these type of problems is to split the angle up into the sum or difference of two angles where the trig value is known for various standard angles like 30°, 45°, 60°, 90°, or 180°

$$\sin 75° = \sin(30° + 45°)$$

$$= \sin 30° \cos 45° + \cos 30° \sin 45°$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}\sqrt{3}}{4}$$

$$= \frac{\sqrt{2}(1 + \sqrt{3})}{4}$$

2 Evaluate *cos 105°*, (non calculator method).

Solution:

$$\cos 105° = \cos(60° + 45°)$$

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(105) = \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}\sqrt{3}}{4}$$

$$\cos 105° = \frac{\sqrt{2}(1 - \sqrt{3})}{4} = -0{\cdot}259$$

Note that a cosine in the second quadrant will be negative, so the answer is consistent.

3 Evaluate *tan (−15°)*, (non calculator method).

Solution:

$$\tan(-15°) = \tan(45° - 60°)$$

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + tab\, A \tan B}$$

$$\tan(-15°) = \frac{\tan 45° - \tan 60°}{1 + \tan 45° \tan 60°}$$

$$= \frac{1 - \sqrt{3}}{1 + 1 \times \sqrt{3}}$$

$$= \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

4 Evaluate $\cos(A + B)$ & $\tan(A - B)$ given that A is obtuse and $\sin A = \frac{3}{5}$, B is acute and $\sin B = \frac{12}{13}$.

Solution:

First, find the values for $\cos A$, $\cos B$, $\tan A$, & $\tan B$.

A is obtuse which means quadrant 2, therefore *sin* is +ve, and both *tan* & *cos* are −ve.

$$\sin A = \frac{3}{5} \quad \text{(recognise this a 3, 4, 5 right angled } \Delta)$$

$$\therefore \tan A = -\frac{3}{4} \quad \text{and} \quad \cos A = -\frac{4}{5}$$

B is acute which means quadrant 1, therefore *sin, tan* & *cos* are +ve.

$$\sin B = \frac{12}{13} \quad \text{(recognise this a 5, 12, 13 right angled } \Delta)$$

$$\therefore \tan B = \frac{12}{5} \quad \text{and} \quad \cos A = \frac{5}{13}$$

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(A + B) = -\frac{4}{5}\cdot\frac{5}{13} - \frac{3}{5}\cdot\frac{12}{13} = -\frac{56}{65}$$

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\equiv \frac{-\frac{3}{4} - \frac{12}{5}}{1 + \left(-\frac{3}{4}\right)\frac{12}{5}} = \frac{63}{16}$$

5 Prove that:

$$\frac{\sin(A - B)}{\cos A \cos B} + \frac{\sin(B - C)}{\cos B \cos C} + \frac{\sin(C - A)}{\cos C \cos A} \equiv 0$$

Solution:

Using the LHS

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A - \cos C \sin A}{\cos C \cos A}$$

$$= \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A}{\cos C \cos A} - \frac{\cos C \sin A}{\cos C \cos A}$$

$$= \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} + \frac{\sin B}{\cos B} - \frac{\sin C}{\cos C} + \frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}$$

$$= \tan A - \tan B + \tan B - \tan C + \tan C - \tan A$$

$$= 0$$

$$= RHS$$

6 Show that:

$$\cos\left(\frac{\pi}{2} - x\right) \equiv \sin x$$

Solution:

Using the LHS

$$\cos\left(\frac{\pi}{2} - x\right) \equiv \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$

$$\equiv (0)\cos x + (1)\sin x$$

$$\equiv \sin x$$

$$\equiv RHS$$

7 Solve $2\cos\theta = \sin(\theta + 30°)$ for $0 \leqslant \theta \leqslant 360°$

Solution:

$$2\cos\theta = \sin(\theta + 30°)$$

$$2\cos\theta = \sin\theta\cos 30° + \cos\theta\sin 30°$$

$$2\cos\theta = \sin\theta \times \frac{\sqrt{3}}{2} + \cos\theta \times \frac{1}{2}$$

$$2\cos\theta - \frac{1}{2}\cos\theta = \frac{\sqrt{3}}{2}\sin\theta$$

$$\frac{3}{2}\cos\theta = \frac{\sqrt{3}}{2}\sin\theta$$

$$3\cos\theta = \sqrt{3}\sin\theta$$

$$\frac{3}{\sqrt{3}} = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\theta = 60°, \ 240°$$

41.7 Double Angle Formulae

The **Double Angle** formulae are just special cases of the compound angle formulae where $A = B$. Recall also that $\cos^2\theta + \sin^2\theta \equiv 1$. This gives rise to the following:

$$\sin 2A \equiv 2\sin A\cos A \qquad \{A = B \text{ in } \sin(A + B)\}$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \qquad \{A = B \text{ in } \cos(A + B)\}$$

$$\cos 2A \equiv 2\cos^2 A - 1 \qquad \{\sin^2 A = 1 - \cos^2 A\}$$

$$\cos 2A \equiv 1 - 2\sin^2 A \qquad \{\cos^2 A = 1 - \sin^2 A\}$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A} \qquad \{A = B \text{ in } \tan(A + B)\}$$

$$\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A) \qquad \{\text{Re-arranging}\}$$

$$\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$$

$$\tan^2 A \equiv \frac{1 - \cos 2A}{1 + \cos 2A} \qquad \{\text{see below}\}$$

Notice how the double angle formulae, in the form of:

$$\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A) \qquad \sin^2 A \equiv \frac{1}{2}(1 - \cos 2A) \qquad \tan^2 A \equiv \frac{1 - \cos 2A}{1 + \cos 2A}$$

act to reduce the power of $\cos^2 A$, $\sin^2 A$ & $\tan^2 A$. Think of these as the power reduction formulae.

41.7.1 Example:

1 Show that:

$$tan^2\theta \equiv \frac{1 - cos\,2\theta}{1 + cos\,2\theta}$$

Solution:

Using the LHS:

$$tan^2\theta \equiv \frac{sin^2\theta}{cos^2\theta}$$

$$\equiv \frac{\frac{1}{2}(1 - cos\,2\theta)}{\frac{1}{2}(1 + cos\,2\theta)}$$

$$\equiv \frac{1 - cos\,2\theta}{1 + cos\,2\theta}$$

$$\equiv RHS$$

2 Solve $1 - 2sin\,\theta - 4cos\,2\theta = 0$ for θ between $0 \leqslant \theta \leqslant 360°$

Solution:

$$1 - 2sin\,\theta - 4cos\,2\theta = 0$$

$$1 - 2sin\,\theta - 4(1 - 2sin^2\theta) = 0$$

$$1 - 2sin\,\theta - 4 + 8sin^2\theta = 0$$

$$8sin^2\theta - 2sin\,\theta - 3 = 0$$

$$(4sin\,\theta - 3)(2sin\,\theta + 1) = 0$$

$$\therefore sin\,\theta = \frac{3}{4} \quad or \quad sin\,\theta = -\frac{1}{2}$$

3 Simplify:

$$\frac{sin\,x}{1 + cos\,x}$$

Solution:

Now $\quad sin\,A \equiv 2sin\,½A\,cos\,½A \quad \& \quad cos\,A \equiv 2cos^2\,½A - 1$

$$\frac{sin\,x}{1 + cos\,x} = \frac{2sin\,½x\,cos\,½x}{1 + 2cos^2\,½x - 1}$$

$$= \frac{2sin\,½x\,cos\,½x}{2cos^2\,½x}$$

$$= \frac{sin\,½x}{cos\,½x}$$

$$= tan\,½x$$

4 Express $cos^4 x$ in terms of cosines of multiples of x.

Solution:

$$cos^4 A \equiv \left(cos^2 A\right)^2$$
$$\equiv \left(\frac{1}{2}(1 + cos\, 2A)\right)^2$$
$$\equiv \frac{1}{4}(1 + cos\, 2A)^2$$
$$\equiv \frac{1}{4}\left(1 + 2cos\, 2A + cos^2 2A\right)$$
$$\equiv \frac{1}{4}\left(1 + 2cos\, 2A + \frac{1}{2}(1 + cos\, 4A)\right)$$
$$\equiv \frac{1}{4}\left(1 + 2cos\, 2A + \frac{1}{2} + \frac{1}{2}cos\, 4A\right)$$
$$\equiv \frac{1}{8}(2 + 4cos\, 2A + 1 + cos\, 4A)$$
$$\equiv \frac{1}{8}(3 + 4cos\, 2A + cos\, 4A)$$

5 If $tan\,\theta = \frac{3}{4}$ and θ is acute, find the values of $tan\, 2\theta$, $tan\, 4\theta$, $tan\,\frac{\theta}{2}$

Solution:

To solve use $tan\, 2A \equiv \dfrac{2tan\, A}{1 - tan^2 A}$ with $A = \theta,\ A = 2\theta,\ A = \frac{\theta}{2}$

$$A = \theta, \qquad tan\, 2\theta \equiv \frac{2tan\,\theta}{1 - tan^2\theta} = \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \frac{24}{7}$$

$$A = 2\theta, \qquad tan\, 4\theta \equiv \frac{2tan\, 2\theta}{1 - tan^2 2\theta} = \frac{2\left(\frac{24}{7}\right)}{1 - \left(\frac{24}{7}\right)^2} = -\frac{336}{527}$$

$$A = \frac{\theta}{2}, \qquad tan\,\theta \equiv \frac{2tan\,\frac{\theta}{2}}{1 - tan^2\frac{\theta}{2}} = \frac{3}{4} \qquad \text{(given)}$$

$$4 \times 2tan\,\frac{\theta}{2} = 3\left(1 - tan^2\frac{\theta}{2}\right)$$

$$8\, tan\,\frac{\theta}{2} = 3 - 3tan^2\frac{\theta}{2}$$

$$3tan^2\frac{\theta}{2} + 8\, tan\,\frac{\theta}{2} - 3 = 0$$

$$\left(3tan\,\frac{\theta}{2} - 1\right)\left(tan\,\frac{\theta}{2} + 3\right) = 0$$

$$\therefore \qquad tan\,\frac{\theta}{2} = \frac{1}{3} \quad or \quad tan\,\frac{\theta}{2} = -3$$

Now θ is acute, hence $\frac{\theta}{2}$ is acute $\quad \Rightarrow\ tan\,\frac{\theta}{2}$ is +ve

$$\therefore \qquad tan\,\frac{\theta}{2} = \frac{1}{3}$$

6 Eliminate θ from the equations $x \equiv \cos 2\theta$, $y \equiv \sec\theta$.

Solution:

Using $\cos 2A \equiv 2\cos^2 A - 1$

$$x \equiv \cos 2\theta \qquad\qquad y \equiv \sec\theta$$

$$x \equiv 2\cos^2\theta - 1 \qquad\qquad y \equiv \frac{1}{\cos\theta}$$

$$\frac{1}{y} \equiv \cos\theta$$

$$\therefore\ \cos^2\theta \equiv \left(\frac{1}{y}\right)^2$$

$$\therefore\ x \equiv 2\left(\frac{1}{y}\right)^2 - 1$$

$$y^2 x \equiv 2 - y^2$$

$$y^2 x + y^2 \equiv 2$$

$$y^2(x + 1) \equiv 2$$

7 Prove that:

$$\tan\theta + \cot\theta \equiv \frac{1}{\sin\theta\cos\theta}$$

Solution:

Using the LHS:

$$LHS \equiv \tan\theta + \cot\theta$$

$$\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\equiv \frac{\sin\theta\sin\theta + \cos\theta\cos\theta}{\sin\theta\cos\theta}$$

$$\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$\equiv \frac{1}{\sin\theta\cos\theta}$$

$$\equiv RHS$$

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41.8 Triple Angle Formulae

This is just an extension of the compound angle identity, replacing $A+B$ with $2A+A$, which gives us:

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

$$\tan 3A \equiv \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

The same technique can be used to find other double combinations such as:

$$\cos 6A \equiv \cos^2 3A - \sin^2 3A$$

41.8.1 Example:

1 Prove that:

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

Solution:

Using the LHS:

$$\begin{aligned}
\sin 3A &\equiv \sin(2A + A) \\
&\equiv \sin 2A \cos A + \cos 2A \sin A \\
&\equiv (2\sin A \cos A)\cos A + (1 - 2\sin^2 A)\sin A \\
&\equiv 2\sin A \cos^2 A + \sin A - 2\sin^3 A \\
&\equiv 2\sin A\,(1 - \sin^2 A) + \sin A - 2\sin^3 A \\
&\equiv 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A \\
&\equiv 3\sin A - 4\sin^3 A \\
&\equiv RHS
\end{aligned}$$

2 Prove that:

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

Solution:

Using the LHS:

$$\begin{aligned}
\cos 3A &\equiv \cos(2A + A) \\
&\equiv \cos 2A \cos A + \sin 2A \sin A \\
&\equiv (2\cos^2 A - 1)\cos A + (2\sin A \cos A)\sin A \\
&\equiv 2\cos^3 A - \cos A - 2\sin^2 A \cos A \\
&\equiv 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A \\
&\equiv 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A \\
&\equiv 4\cos^3 A - 3\cos A \\
&\equiv RHS
\end{aligned}$$

41.9 Half Angle Formulae

This is an extension of the double angle identity, replacing A with $\frac{A}{2}$.

This is easily derived:

$$\cos 2A \equiv 2\cos^2 A - 1$$

$$\cos A \equiv 2\cos^2 \frac{A}{2} - 1 \qquad \text{substitute } A = \frac{A}{2}$$

$$\cos^2 \frac{A}{2} \equiv \frac{1}{2}(1 + \cos A)$$

Similarly for $\sin 2A$.

$$\sin^2 \frac{A}{2} \equiv \frac{1}{2}(1 - \cos A)$$

$$\cos^2 \frac{A}{2} \equiv \frac{1}{2}(1 + \cos A)$$

$$\tan \frac{A}{2} \equiv \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A}$$

41.9.1 Example:

1

41.10 Factor Formulae

Using the **Factor formulae** any sum or difference of sines or cosines can be expressed as a product of sines and cosines. Called the factor formulae because factorising an expression means converting it into a product.

The factor formulae are found easily enough: take two compound angle formulae, for either the sine or cosines, and add or subtract the identities.

$$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B \qquad (1)$$

$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B \qquad (2)$$

Add identities (1) & (2)

$$\sin(A + B) + \sin(A - B) \equiv 2 \sin A \cos B \qquad (1 + 2)$$

Let: $A + B = P \qquad A - B = Q$

$\therefore \qquad A = \dfrac{P + Q}{2} \qquad B = \dfrac{P - Q}{2}$

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P + Q}{2}\right)\cos\left(\frac{P - Q}{2}\right)$$

Similar results can be obtained for $(\sin P - \sin Q)$ and $(\cos P \pm \cos Q)$.

Sum to Product rules:

$$\sin A + \sin B = 2 \sin\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)$$

Or $\qquad \cos A - \cos B = 2 \sin\left(\dfrac{A + B}{2}\right) \sin\left(\dfrac{B - A}{2}\right)$ *Note the gotcha in the signs*

Alternative format:

An alternative format in terms of A & B is as follows:

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B$$

$$\sin(A + B) - \sin(A - B) = 2\cos A \sin B$$

$$\cos(A + B) + \cos(A - B) = 2\cos A \cos B$$

$$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$$

Product to Sum rules:

These can be re-arranged to give a product to sum rule, which is useful for integration.

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2\cos A \sin B = \sin(A + B) - \sin(A - B)$$

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$-2\sin A \sin B = \cos(A + B) - \cos(A - B)$$

41.10.1 Example:

1 Show that:

$$\tan 2\theta \equiv \frac{\sin\theta + \sin 3\theta}{\cos\theta + \cos 3\theta}$$

Solution:

Using the RHS:

$$\frac{\sin\theta + \sin 3\theta}{\cos\theta + \cos 3\theta} \equiv \frac{2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)}{2\cos\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)}$$

$$\equiv \frac{\sin\left(\frac{4\theta}{2}\right)}{\cos\left(\frac{4\theta}{2}\right)}$$

$$\equiv \frac{\sin(2\theta)}{\cos(2\theta)}$$

$$\equiv \tan 2\theta$$

41.11 Topical Tips on Proving Identities

There are a number of guidelines you can use in order to prove identities. There are four basic methods:

- Start with the LHS and work towards the RHS expression.
- Start with the RHS and work towards the LHS expression.
- Subtract one side from the other and set the expression to zero
- Divide one side by the other and make the expression equal to one.
- Some general advice:
  - As a guide start with the most complicated side first
  - Note the fuctions that are in the expression you are aiming towards and work towards converting to those functions
  - Recognise opportunities to use the basic identities
  - Pairings of sines & cosines; secants & tangents; and cosecants & cotangents, work well together
  - Proving identities is not the same as solving equations. You cannot add or subtract quantities to both sides or cross multiply as you cannot assume that the given identity is, in fact, equal.

41.12 Trig Identity Digest

41.12.1 Trig Identities

$$\sin\theta \equiv \cos\left(\frac{1}{2}\pi - \theta\right) \qquad \sin x = \cos(90° - x)$$

$$\cos\theta \equiv \sin\left(\frac{1}{2}\pi - \theta\right) \qquad \cos x = \sin(90° - x)$$

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

41.12.2 Pythagorean Identities

$$\cos^2\theta + \sin^2\theta \equiv 1 \qquad (1)$$

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta \qquad \text{(Division of (1) by } \sin^2\theta)$$

$$1 + \tan^2\theta \equiv \sec^2\theta \qquad \text{(Division of (1) by } \cos^2\theta)$$

41.12.3 Compound Angle (Addition) Identities

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

41.12.4 Double Angle Identities

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A$$

$$\equiv 2\cos^2 A - 1 \qquad (\sin^2\theta = 1 - \cos^2\theta)$$

$$\equiv 1 - \sin^2 A \qquad (\cos^2\theta = 1 - \sin^2\theta)$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

41.12.5 Triple Angle Identities

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

$$\tan 3A \equiv \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

41.12.6 Half Angle Identities

$$\cos^2\frac{A}{2} \equiv \frac{1}{2}(1 + \cos A)$$

$$\sin^2\frac{A}{2} \equiv \frac{1}{2}(1 + \cos A)$$

41.12.7 Factor formulæ:

Sum to Product rules:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Or $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$ *Note the gotcha in the signs*

Alternative format:

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$$

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$$

Product to Sum rules:

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$$

41.12.8 Small t Identities

If $t = \tan \tfrac{1}{2}\theta$

$$\sin\theta \equiv \frac{2t}{1+t^2}$$

$$\cos\theta \equiv \frac{1-t^2}{1+t^2}$$

$$\tan\theta \equiv \frac{2t}{1-t^2}$$

42 • C3 • Trig: Inverse Functions

42.1 Inverse Trig Functions Intro

The basic **Inverse Trig Functions** are $sin^{-1}x$, $cos^{-1}x$, and $tan^{-1}x$.

Now $sin^{-1}x$ reads as "the angle whose *sin* is…"

similarly $cos^{-1}x$ reads as "the angle whose *cos* is…"

and $tan^{-1}x$ reads as "the angle whose *tan* is…"

> For the avoidance of doubt, the reciprocal of a trig function is written, for example, as $(\sin x)^{-1}$.

Hence,

If $\sin\theta = 0{\cdot}5$

$$\theta = sin^{-1}(0{\cdot}5)$$

$$\theta = 30°$$

i.e. the angle whose *sin* is 0·5 is 30° Remember that $sin^{-1}x$ is an angle.

An alternative way of writing $\theta = sin^{-1}x$ is $\theta = \arcsin x$, so we can say that:

$$\sin\theta = x \quad\Rightarrow\quad \theta = \arcsin x$$

$$\cos\theta = x \quad\Rightarrow\quad \theta = \arccos x$$

$$\tan\theta = x \quad\Rightarrow\quad \theta = \arctan x$$

For a inverse function to exist, recall that the function and its inverse must have a one to one relationship or mapping. The functions of $\sin x$, $\cos x$, and $\tan x$ are many to one mappings, so any inverse mapping will be many to one.

However, if we restrict the domain, then we can create a one to one relationship and the two curves will be a reflection of each other about the line $y = x$.

There are, of course, an infinite number of solutions to a trig function, but restricting the domain gives only one solution called the **principal value** which is the one given on a calculator.

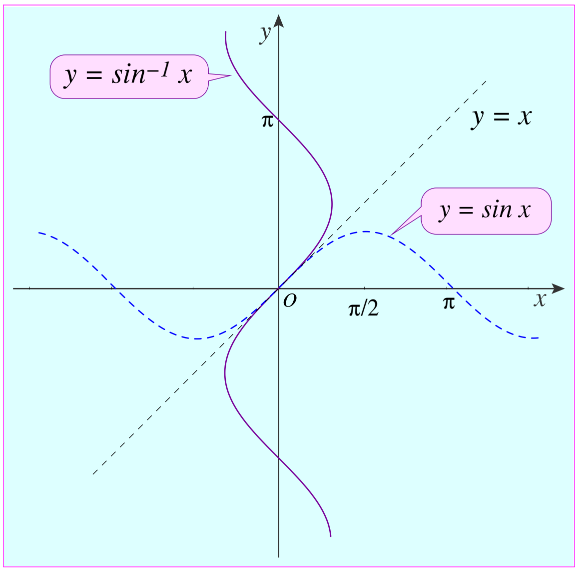
Restrictions imposed on the main trig functions are:

Funtion	*Domain* °	*Domain (radians)*
$y = \sin\theta$	$-90 \leqslant \theta \leqslant 90$	$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$
$y = \cos\theta$	$0 \leqslant \theta \leqslant 180$	$0 \leqslant \theta \leqslant \pi$
$y = \tan\theta$	$-90 \leqslant \theta \leqslant 90$	$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

42.2 Inverse Sine Function

The reflection of $y = \sin x$ in the line $y = x$

give the inverse which is a one to many relationship or mapping and is therefore not a function.



Restrict the domain of $y = \sin x$ to:

$$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

and the range becomes:

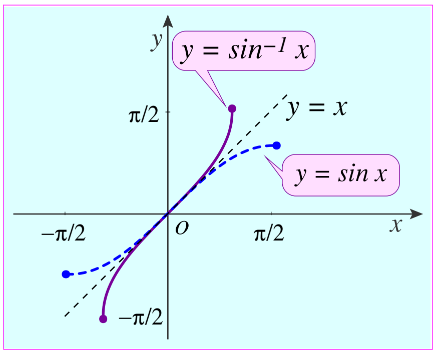
$$-1 \leqslant \sin x \leqslant 1$$

The inverse function is now created, with a domain of

$$-1 \leqslant x \leqslant 1$$

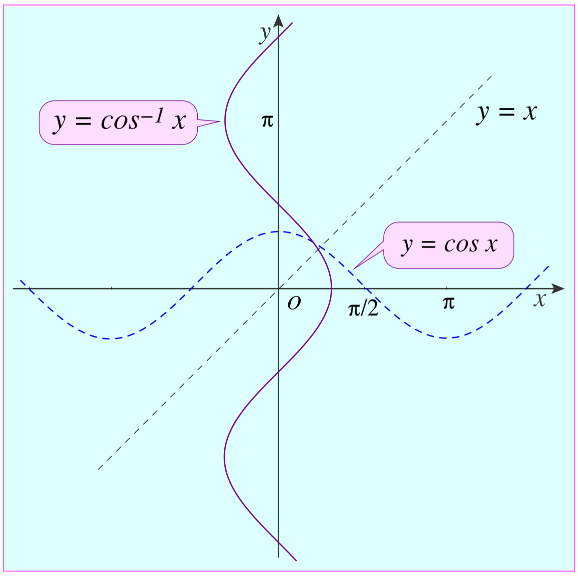
and a range of

$$-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$$



42.3 Inverse Cosine Function

The reflection of $y = \cos x$ in the line $y = x$

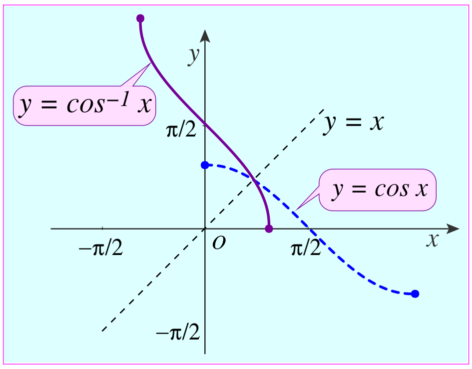


Restrict the domain of $y = \cos x$ to:

$$0 \leqslant x \leqslant \pi$$

and the range becomes:

$$-1 \leqslant \cos x \leqslant 1$$



The inverse function is now created, with a domain of

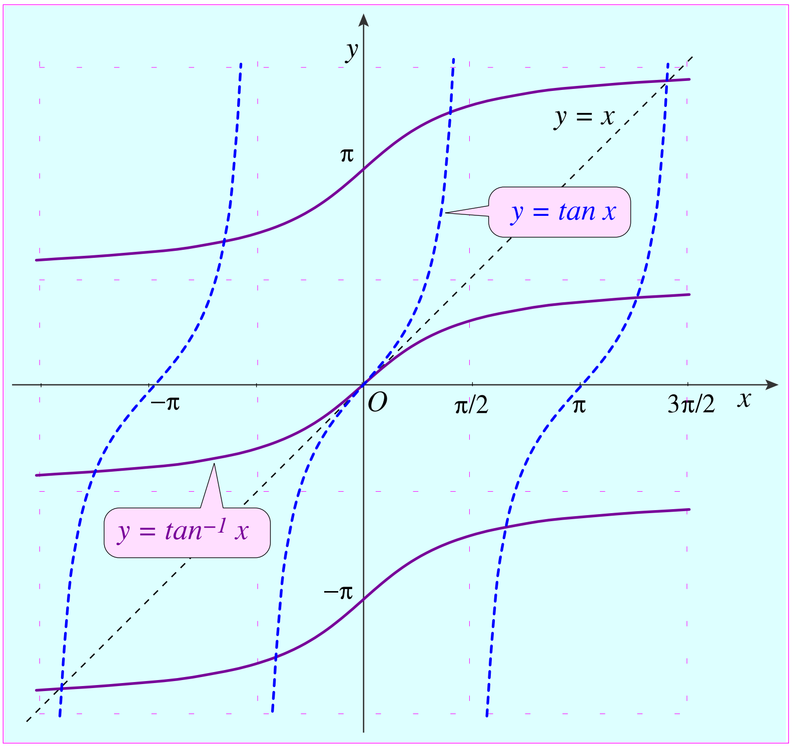
$$-1 \leqslant x \leqslant 1$$

and a range of

$$-\frac{\pi}{2} \leqslant \cos^{-1}x \leqslant \frac{\pi}{2}$$

42.4 Inverse Tangent Function

The reflection of $y = \tan x$ in the line $y = x$



Restrict the domain of $y = \tan x$ to:

$$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

and the range becomes:

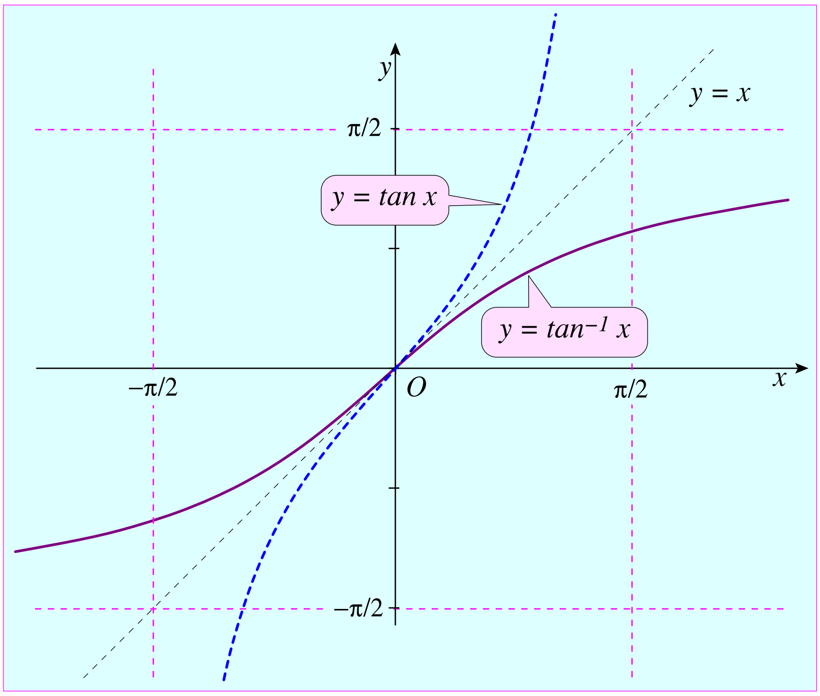
$$\tan x \in \mathbb{R}$$

The inverse function is now created, with a domain of

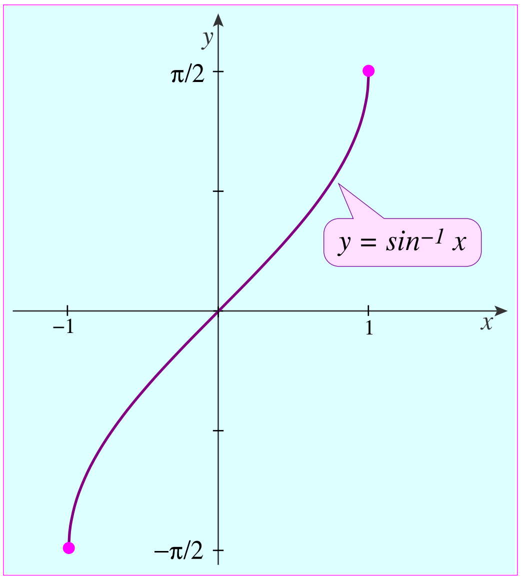
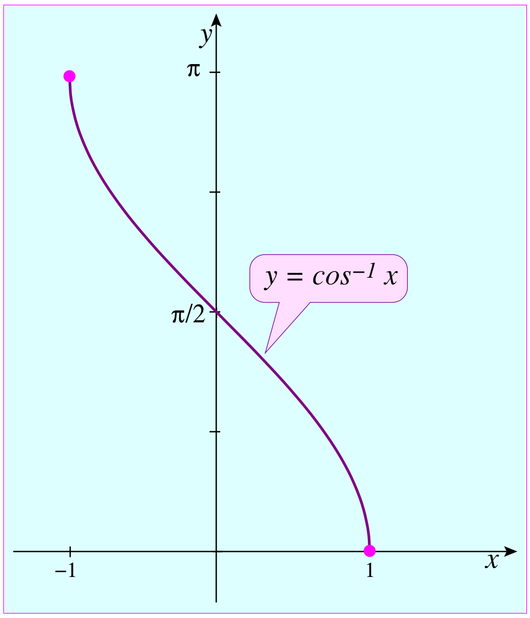
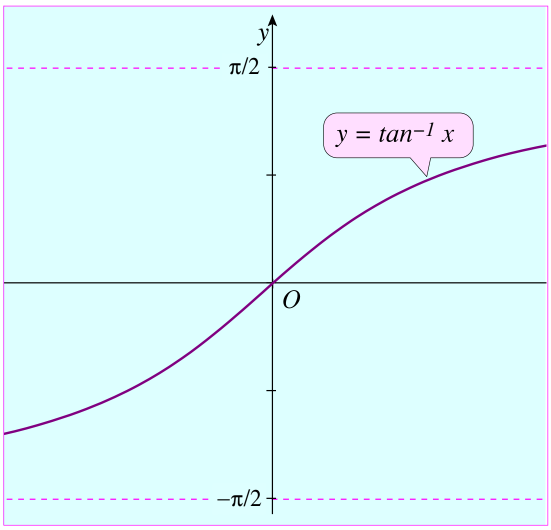
$$x \in \mathbb{R}$$

and a range of

$$-\frac{\pi}{2} \leqslant \tan^{-1} x \leqslant \frac{\pi}{2}$$



42.5 Inverse Trig Function Summary Graphs

Function	Properties	Illustration
$y = sin^{-1} x$ $y = arcsin\,x$	**Inverse Sine Function:** Odd function Restricted Domain: $-1 \leqslant x \leqslant 1$ Range: $-\frac{\pi}{2} \leqslant sin^{-1}x \leqslant \frac{\pi}{2}$ Intercept: (0, 0) Symmetric about the origin – has rotational symmetry, order 2. Increasing function	
$y = cos^{-1} x$ $y = arccos\,x$	**Inverse Cosine Function:** Restricted Domain: $-1 \leqslant x \leqslant 1$ Range: $0 \leqslant cos^{-1}x \leqslant \pi$ y-intercept $\left(0, \frac{\pi}{2}\right)$ Decreasing function	
$y = tan^{-1} x$ $y = arctan\,x$	**Inverse Tangent Function:** Odd function Domain: $x \in \mathbb{R}$ Range: $-\frac{\pi}{2} \leqslant tan^{-1}x \leqslant \frac{\pi}{2}$ Intercept (0, 0) Horizontal asymptotes: $y = \pm\frac{\pi}{2}$ Symmetric about the origin – has rotational symmetry order 2. Increasing function	

43 • C3 • Trig: Harmonic Form

43.1 Form of $a \cos x + b \sin x$

Using the compound angle identity $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$, then any function of the form $a \cos x + b \sin x$ can be written as $R \sin(x + \alpha)$ where $R > 0$ and angle α is acute.

We also find that $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$.

$$a \sin x + b \cos x = R \sin(x + \alpha)$$

This new form of function is useful in solving equations, especially when finding max & min values, as well as sketching graphs of the form $y = a \sin x + b \cos x$. This is often called the harmonic form.

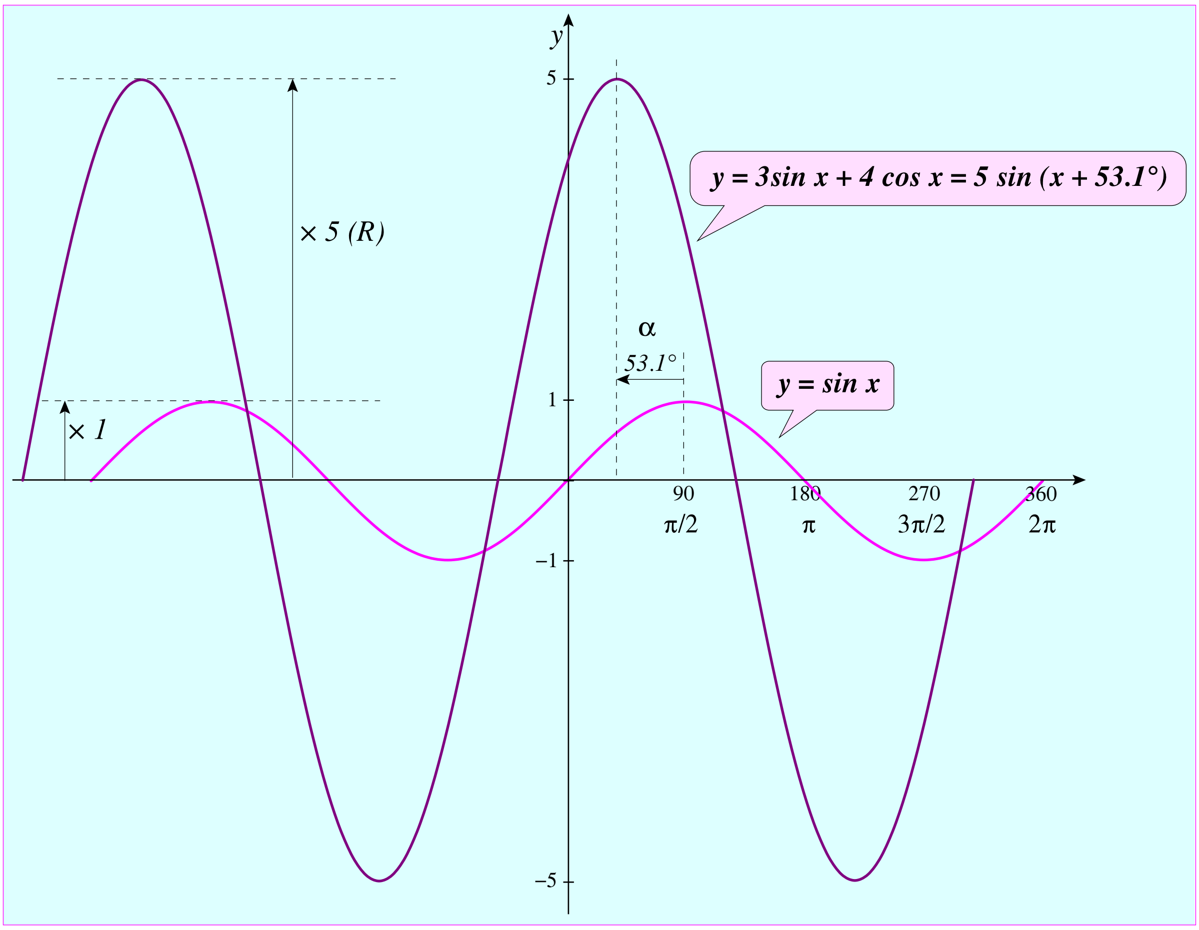
Plotting an equation of the form $a \cos x + b \sin x$ gives a sinusoidal wave form, which appears as a translation of $\sin x$, which in this case, is translated in the negative x direction by a factor of α and stretched in the y direction by the factor R.

Example:

$$3 \sin x + 4 \cos x \equiv 5 \sin(x + 53.1°)$$

Compare with:

$$a \sin x + b \cos x = R \sin(x + \alpha)$$



a sin x + b cos x

43.2 Proving the Identity

Show that the following are true:

- $a \sin x + b \cos x \equiv R \sin (x + \alpha)$
- $R = \sqrt{a^2 + b^2}$
- $\alpha = \tan^{-1} \frac{b}{a}$

Take the RHS and use the Compound Angle Identity to expand expression

$$\begin{aligned} R \sin (x + \alpha) &= R(\sin x \cos \alpha + \cos x \sin \alpha) \\ &= R \sin x \cos \alpha + R \cos x \sin \alpha \\ &= R \cos \alpha \sin x + R \sin \alpha \cos x \end{aligned}$$

Since R and α are both constants, therefore, $R \cos \alpha$ & $R \sin \alpha$ are both constants.

Hence we can say: $R \sin (x + \alpha) = a \sin x + b \cos x$

Equate the coefficients on the RHS:

where $a = R \cos \alpha \qquad (1)$

& $b = R \sin \alpha \qquad (2)$

Divide (1) & (2) $\frac{R \sin \alpha}{R \cos \alpha} = \frac{b}{a}$

$\therefore \quad \tan \alpha = \frac{b}{a}$

$\alpha = \tan^{-1} \frac{b}{a}$

Take (1) & (2) and square and add:

$$R^2 \cos^2 \alpha = a^2$$

$$R^2 \sin^2 \alpha = b^2$$

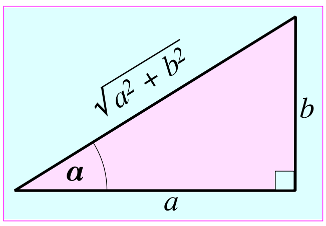
$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = a^2 + b^2$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = a^2 + b^2$$

but $(\cos^2 \alpha + \sin^2 \alpha) = 1$

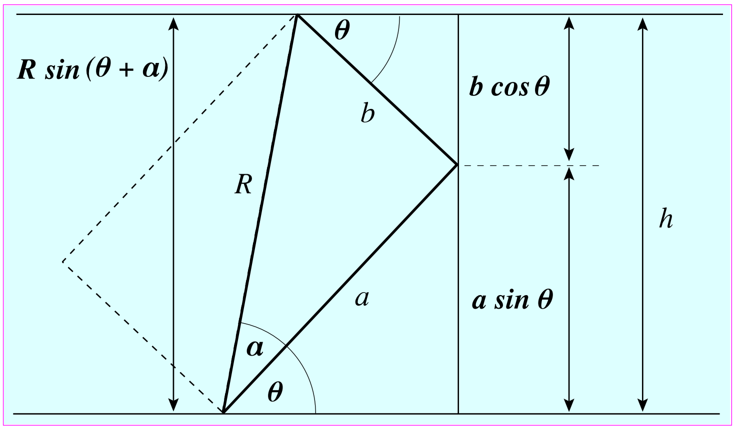
$\therefore \quad R^2 = a^2 + b^2$

$\therefore \quad R = \sqrt{a^2 + b^2}$



43.3 Geometric View of the Harmonic Form

Consider the diagram below:



Geometric View of the Harmonic Form

$$h = a \sin\theta + b\cos\theta$$

$$h = R\sin(\theta + \alpha)$$

$$\therefore \quad a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$$

Sometimes dressed up as a door through a hole problem:-)

43.4 Choosing the Correct Form

The key is to choose a method that ensures α is acute.

$R\sin(x + \alpha) \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$ use for $a\sin x + b\cos x$

$R\sin(x - \alpha) \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$ use for $a\sin x - b\cos x$

$R\cos(x + \alpha) \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$ use for $a\cos x - b\sin x$

$R\cos(x - \alpha) \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$ use for $a\cos x + b\sin x$

$$a\sin x + b\cos x \equiv R\sin(x + \alpha)$$

$$a\sin x - b\cos x \equiv R\sin(x - \alpha)$$

$$a\cos x - b\sin x \equiv R\cos(x + \alpha)$$

$$a\cos x + b\sin x \equiv R\cos(x - \alpha)$$

Note that $a\sin x + b\cos x$ has two solutions, as $a\sin x + b\cos x$ and $a\cos x + b\sin x$.

$a\sin x + b\cos x \equiv R\sin(x + \alpha) \quad \Rightarrow R\cos\alpha = a \quad R\sin\alpha = b \}$

$a\sin x - b\cos x \equiv R\sin(x - \alpha) \quad \Rightarrow R\cos\alpha = a \quad -R\sin\alpha = -b \} \quad \tan\alpha = \dfrac{b}{a}$

$a\cos x - b\sin x \equiv R\cos(x + \alpha) \quad \Rightarrow R\cos\alpha = a \quad -R\sin\alpha = -b \}$

$a\cos x + b\sin x \equiv R\cos(x - \alpha) \quad \Rightarrow R\cos\alpha = a \quad R\sin\alpha = b \}$

Note that $\tan\alpha$ is positive in each case.

43.5 Worked Examples

43.5.1 Example:

1 Express $cos\,\theta - sin\,\theta$ in the $R\,cos\,(\theta \pm \alpha)$ form.

Solution:

$$cos\,\theta - sin\,\theta \equiv R\,cos\,(\theta + \alpha)$$

$$cos\,\theta - sin\,\theta \equiv R\,(cos\,\theta\,cos\,\alpha - sin\,\theta\,sin\,\alpha)$$

$$cos\,\theta - sin\,\theta \equiv R\,cos\,\theta\,cos\,\alpha - R\,sin\,\theta\,sin\,\alpha$$

Equate the coefficients:

$$1 = R\,cos\,\alpha$$

$$1 = R\,sin\,\alpha$$

$$tan\,\alpha = \frac{R\,sin\,\alpha}{R\,cos\,\alpha} = \frac{1}{1} = 1$$

$$\alpha = 45°$$

From pythag, hypotenuse $= \sqrt{2}$

$$\therefore \quad cos\,\theta - sin\,\theta \equiv \sqrt{2}\,cos\,(\theta + 45°)$$

2 Express $5\,cos\,\theta + 12\,sin\,\theta$ in the $R\,cos\,(\theta \pm \alpha)$ form.

Solution:

$$5\,cos\,\theta + 12\,sin\,\theta \equiv R\,cos\,(\theta - \alpha)$$

$$5\,cos\,\theta + 12\,sin\,\theta \equiv R\,cos\,\theta\,cos\,\alpha + R\,sin\,\theta\,sin\,\alpha$$

Equate the coefficients:

$$5 = R\,cos\,\alpha$$

$$12 = R\,sin\,\alpha$$

$$tan\,\alpha = \frac{R\,sin\,\alpha}{R\,cos\,\alpha} = \frac{12}{5}$$

$$\alpha = 67{\cdot}4°$$

From pythag, hypotenuse $= \sqrt{12^2 + 5^2} = 13$

$$\therefore \quad 5\,cos\,\theta + 12\,sin\,\theta \equiv 13\,cos\,(\theta - 67{\cdot}4°)$$

3 Express $5\,sin\,\theta - 8\,cos\,\theta$ in the $R\,sin\,(\theta \pm \alpha)$ form.

Solution:

$$5\,sin\,\theta - 8\,cos\,\theta \equiv R\,sin\,(\theta - \alpha)$$

$$\equiv R\,sin\,\theta\,cos\,\alpha - R\,cos\,\theta\,sin\,\alpha$$

Equate the coefficients: $\quad 5 = R\,cos\,\alpha$

$$8 = R\,sin\,\alpha$$

$$tan\,\alpha = \frac{R\,sin\,\alpha}{R\,cos\,\alpha} = \frac{8}{5}$$

$$\alpha = 58°$$

From pythag, hypotenuse $= \sqrt{8^2 + 5^2} = \sqrt{89}$

$$\therefore \quad 5\,sin\,\theta - 8\,cos\,\theta \equiv \sqrt{89}\,sin\,(\theta + 58°)$$

4 Solve $\cos\theta - 7\sin\theta = 2$ (for 0° to 360°)

Solution:

$$\cos\theta - 7\sin\theta \equiv R\cos(\theta + \alpha)$$

$$\cos\theta - 7\sin\theta \equiv R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

$$\cos\theta - 7\sin\theta \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

Equate the coefficients: $1 = R\cos\alpha$

$$7 = R\sin\alpha$$

$$\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = \frac{7}{1} = 7$$

$$\alpha = 81{\cdot}9°$$

From pythag, hypotenuse $= \sqrt{50}$

$$\therefore \quad \cos\theta - 7\sin\theta \equiv \sqrt{50}\cos(\theta + 81{\cdot}9°)$$

But $\cos\theta - 7\sin\theta = 2$

$$\therefore \quad \sqrt{50}\cos(\theta + 81{\cdot}9) = 2$$

$$\theta + 81{\cdot}9 = \cos^{-1}\frac{2}{\sqrt{50}}$$

$$\theta + 81{\cdot}9 = 73{\cdot}57,\ 286{\cdot}43,\ 433{\cdot}87$$

$$\theta = -8{\cdot}33,\ 204{\cdot}53,\ 351{\cdot}67$$

Discount the first solution of −8.2 as this is outside the required range.

5 Express $5\sin\theta + 12\cos\theta$ in the $R\sin(\theta + \alpha)$ form, and show that $5\sin\theta + 12\cos\theta + 7 \leqslant 20$

Solution:

$$5\sin\theta + 12\cos\theta \equiv R\sin(\theta + \alpha)$$

$$5\sin\theta + 12\cos\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

Equate the coefficients: $5 = R\cos\alpha$

$$12 = R\sin\alpha$$

$$\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = \frac{12}{5}$$

$$\alpha = 67{\cdot}4°$$

From pythag, hypotenuse $= \sqrt{12^2 + 5^2} = 13$

$$\therefore \quad 5\sin\theta + 12\cos\theta \equiv 13\sin(\theta + 67{\cdot}4°)$$

$$-1 \leqslant \sin(\theta + 67{\cdot}4) \leqslant 1$$

$$-13 \leqslant 13\sin(\theta + 67{\cdot}4) \leqslant 13$$

$$-13 \leqslant (5\sin\theta + 12\cos\theta) \leqslant 13$$

$$-13 + 7 \leqslant (5\sin\theta + 12\cos\theta + 7) \leqslant 13 + 7$$

$$-6 \leqslant (5\sin\theta + 12\cos\theta + 7) \leqslant 20$$

Hence: $(5\sin\theta + 12\cos\theta + 7) \leqslant 20$

6 Find the minimum & maximum values of $\cos\theta - 7\sin\theta$ and the corresponding values of θ

Solution:

From a previous example above:

$$\cos\theta - 7\sin\theta \equiv \sqrt{50}\cos(\theta + 81{\cdot}9°)$$

$$-1 \leqslant \cos(\theta + 81{\cdot}9) \leqslant 1$$

$$-\sqrt{50} \leqslant \sqrt{50}\cos(\theta + 81{\cdot}9) \leqslant \sqrt{50}$$

$$\therefore \quad -\sqrt{50} \leqslant (\cos\theta - 7\sin\theta) \leqslant \sqrt{50}$$

$\therefore$ Min value of $(\cos\theta - 7\sin\theta)$ is $-\sqrt{50}$

Min value occurs when: $\cos(\theta + 81{\cdot}9) = -1$

$$\theta + 81{\cdot}9 = 180°$$

$$\theta = 180° - 81{\cdot}9$$

$$\theta = 98{\cdot}1°$$

Min value of $-\sqrt{50}$ occurs when $\theta = 98{\cdot}1°$

$\therefore$ Max value of $(\cos\theta - 7\sin\theta)$ is $\sqrt{50}$

Min value occurs when: $\cos(\theta + 81{\cdot}9) = 1$

$$\theta + 81{\cdot}9 = 0°$$

$$\theta = -81{\cdot}9°,\ 278{\cdot}1$$

$$\theta = 278{\cdot}1°$$

Max value of $\sqrt{50}$ occurs when $\theta = 278{\cdot}1°$

7 Find the minimum & maximum values of $2\sin\theta + 7\cos\theta$ and the corresponding values of θ

Solution:

Find that: $2\sin\theta + 7\cos\theta \equiv \sqrt{53}\cos(\theta - 15{\cdot}9°)$

$$-1 \leqslant \cos(\theta - 15{\cdot}9°) \leqslant 1$$

$$-\sqrt{53} \leqslant \sqrt{53}\cos(\theta - 15{\cdot}9°) \leqslant \sqrt{53}$$

$$\therefore \quad -\sqrt{53} \leqslant (2\sin\theta + 7\cos\theta) \leqslant \sqrt{53}$$

$\therefore$ Min value of $(2\sin\theta + 7\cos\theta)$ is $-\sqrt{53}$

Min value occurs when: $\cos(\theta - 15{\cdot}9°) = -1$

$$\theta - 15{\cdot}9 = 180°$$

$$\theta = 180° + 15{\cdot}9$$

$$\theta = 195{\cdot}9°$$

$\therefore$ Min value of $-\sqrt{53}$ occurs when $\theta = 195{\cdot}9°$

Max value of $2\sin\theta + 7\cos\theta$ is $\sqrt{53}$

Min value occurs when: $\cos(\theta - 15{\cdot}9°) = 1$

$$\theta - 15{\cdot}9 = 0°$$

$$\theta = 15{\cdot}9°$$

Max value of $\sqrt{53}$ occurs when $\theta = 15{\cdot}9°$

8 Express $3\cos\theta - 2\sin\theta$ in the $R\cos(\theta \pm \alpha)$ form.

Solution:

$$3\cos\theta - 2\sin\theta \equiv R\cos(\theta + \alpha)$$

for $a\cos x - b\sin x$ use $\quad R\cos(x + \alpha) \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$

$$3\cos\theta - 2\sin\theta \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

Equate the coefficients:

$$3 = R\cos\alpha$$

$$2 = R\sin\alpha$$

$$\tan\alpha = \frac{b}{a} = \frac{2}{3}$$

$$\alpha = 33{\cdot}7°$$

From pythag, hypotenuse $= \sqrt{3^2 + 2^2} = \sqrt{13}$

$$\therefore \quad 3\cos\theta - 2\sin\theta \equiv \sqrt{13}\cos(\theta - 33{\cdot}7°)$$

Note: The tangent above is not negative.

43.6 Harmonic Form Digest

$$a \sin x \pm b \cos x \equiv R \sin (x \pm \alpha)$$

$$a \cos x \pm b \sin x \equiv R \cos (x \mp \alpha) \qquad \text{(watch signs)}$$

$$R = \sqrt{a^2 + b^2} \qquad R \cos \alpha = a \qquad R \sin \alpha = b$$

$$\tan \alpha = \frac{b}{a} \qquad 0 < a < \frac{\pi}{2}$$

Recall

$$\sin (A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$R \sin (x + \alpha) \equiv R \sin x \cos \alpha + R \cos x \sin \alpha$ use for $a \sin x + b \cos x$

$R \sin (x - \alpha) \equiv R \sin x \cos \alpha - R \cos x \sin \alpha$ use for $a \sin x - b \cos x$

$R \cos (x + \alpha) \equiv R \cos x \cos \alpha - R \sin x \sin \alpha$ use for $a \cos x - b \sin x$

$R \cos (x - \alpha) \equiv R \cos x \cos \alpha + R \sin x \sin \alpha$ use for $a \cos x + b \sin x$

44 • C3 • Relation between dy/dx and dx/dy

44.1 Relation between dy/dx and dx/dy

It is very tempting to think that $\frac{dy}{dx}$ is a fraction, and treat it as such.

Strictly speaking $\frac{dy}{dx}$ is a function, and should not be confused with a fraction, although in practise it often appears to behave like one.

Deriving the link between $\frac{dy}{dx}$ and $\frac{dx}{dy}$ is as follows:

From C1 recall that a derivative of a function is defined as:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Now $\frac{\delta y}{\delta x}$ **is** a fraction, hence:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\frac{\delta x}{\delta y}}$$

As $\delta x \to 0$ then $\delta y \to 0$

$$\frac{dy}{dx} = \frac{1}{\lim_{\delta y \to 0} \frac{\delta x}{\delta y}} = \frac{1}{\frac{dx}{dy}}$$

Hence $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

or $$\frac{dy}{dx} \times \frac{dx}{dy} = 1$$

E.g. Consider:

$$y = ax + b \qquad \Rightarrow \qquad \frac{dy}{dx} = a$$

Rearrange to make x the subject:

$$x = \frac{y - b}{a}$$

$$x = \frac{y}{a} - \frac{b}{a} \qquad \Rightarrow \qquad \frac{dx}{dy} = \frac{1}{a}$$

$$\therefore \quad \frac{dy}{dx} \times \frac{dx}{dy} = a \times \frac{1}{a}$$

$$= 1$$

44.2 Finding the Differential of $x = g(y)$

Don't assume that every differential has to start with a $\frac{dy}{dx}$

44.2.1 Example:

1 Find the gradient of $x = y^3 + 6y$ at the point (7, 1).

Note that a gradient is given as $\frac{dy}{dx}$.

Solution:

$$\frac{dx}{dy} = 3y^2 + 6$$

At $y = 1$ $\quad \frac{dx}{dy} = 3 \times 1 + 6 = 9$

Recall: $\quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{9}$

2 Find the differential of $y = \ln x$

Solution:

$$y = \ln x$$

$$e^y = x$$

$$x = e^y$$

$$\frac{dx}{dy} = e^y$$

But $\quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d[\ln x]}{dx} = \frac{1}{x}$$

44.3 Finding the Differential of an Inverse Function

This relationship $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ can be used to find the differential of an inverse function. Recall that for a one to one function there is an inverse relationship. We can treat either x or y as the dependent variable.
We can, therefore, write:

$$y = f(x) \qquad \Rightarrow \qquad x = f^{-1}(y)$$

The differential of the function and its inverse is linked by the relationship:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

The advantage of this relationship is that you don't need to know the exact inverse function.

44.3.1 Example:

1 Find the gradient of the inverse function at the point (2, 1), where the function is defined as $f(x) = x^4 + 3x^2 - 2$.

Solution:

$$y = f(x) \qquad \Rightarrow \qquad x = f^{-1}(y)$$

$$y = x^4 + 3x^2 - 2$$

$$x = y^4 + 3y^2 - 2$$

Rearranging to make y the subject is not required since:

$$\frac{dx}{dy} = 4y^3 + 6y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4y^3 + 6y}$$

$$\frac{dy}{dx} = \frac{1}{4 + 6} = \frac{1}{10}$$

2

45 • C3 • Differentiation: The Chain Rule

45.1 Composite Functions Revised

Recall that a composite function, otherwise known as a 'function of a function', is formed by applying one function, then immediately applying another function to the result of the first function.

In simple terms:

$$\text{Input } x \xrightarrow{f} \text{output } f(x)$$

$$\text{Input } f(x) \xrightarrow{g} \text{output } g\left[f(x)\right] \text{ or } gf(x)$$

In other words, apply f to x first, then g to $f(x)$. You read the result, $gf(x)$, from right to left.

E.g.

If $y = (x + 3)^3$ then y is said to be a function of x.

If we make $u = x + 3$ then $y = u^3$ and so y is a function of u and u is a function of x. In function notation we would write:

$$F(x) = gf(x)$$

where $g(u) = u^3$ and $f(x) = x + 3$

Reading a function of a function:

$(x^2 - 4)^3$ is a cubic function g, of a quadratic function f

$\sqrt{(1 - x)^3}$ is a square root function g, of a cubic function f

45.2 Intro to the Chain Rule

We have seen from earlier modules that in order to differentiate a polynomial such as $(2x + 3)^3$ we can use the Binomial theorem to expand the brackets and differentiate each term individually. However, a problem arises if we want to differentiate something like $(2x + 3)^{42}$. Using the Binomial theorem would be tedious to say the least, but as always in mathematics, there is generally an easier way.

The answer to this and many other problems involving composite functions is the **chain rule**, which is given as:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where y is a function of u and u is a function of x.

In function terminology we write

$$F'(x) = g'(f(x)) \times f'(x)$$

where $F(x) = gf(x)$ and $F(x) = g(u)$ and $u = f(x)$

In other words, if $g(u)$ is the outside function and $f(x)$ is the inside function we differentiate the outside function and multiply by the differential of the inside function.

45.3 Applying the Chain Rule

Typical examples of composite functions that can be differentiated by the chain rule are:

$$(x^2 - 4)^3 \qquad \sqrt{(1 - x^3)} \qquad e^{2x+5} \qquad ln\,(3x^2 - 2) \qquad \frac{1}{(x^2 - 4)^3}$$

45.3.1 Example:

1 Find $\frac{dy}{dx}$ when $y = (x^2 - 4)^5$

Solution:

$$y = (x^2 - 4)^5$$

$$\Rightarrow \quad u = x^2 - 4 \qquad \Rightarrow \qquad y = u^5$$

$$\therefore \quad \frac{du}{dx} = 2x \qquad\qquad \frac{dy}{du} = 5u^4$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times 2x$$

$$\therefore \quad \frac{dy}{dx} = 10x\,(x^2 - 4)^4$$

Alternative Solution:

$$\frac{dy}{dx} = \frac{d}{du}(u^5) \times \frac{d}{dx}(x^2 - 4) = 5\,(x^2 - 4)^4 \times 2x \qquad\qquad etc$$

2 Find $\frac{dy}{dx}$ when $y = \sqrt{(1 - x^3)}$

Solution:

$$y = \sqrt{(1 - x^3)} \qquad \Rightarrow \qquad y = (1 - x^3)^{\frac{1}{2}}$$

$$\Rightarrow \quad u = 1 - x^3 \qquad \Rightarrow \qquad y = u^{\frac{1}{2}}$$

$$\therefore \quad \frac{du}{dx} = -3x^2 \qquad\qquad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times (-3x^2)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{3}{2}x^2(1 - x^3)^{-\frac{1}{2}}$$

$$= -\frac{3x^2}{2\sqrt{(1 - x^3)}}$$

Alternative Solution:

$$\frac{dy}{dx} = \frac{d}{du}\left(u^{\frac{1}{2}}\right) \times \frac{d}{dx}(1 - x^3) = \frac{1}{2}(1 - x^3)^{-\frac{1}{2}} \times (-3x^2) \qquad\qquad etc$$

3 Find $\frac{dy}{dx}$ when $y = \ln(3x^2 - 2)$

Solution:

$$y = \ln(3x^2 - 2)$$

$$\Rightarrow \quad u = 3x^2 - 2 \quad \Rightarrow \quad y = \ln u$$

$$\therefore \frac{du}{dx} = 6x \qquad \frac{dy}{du} = \frac{1}{u}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 6x$$

$$\therefore \frac{dy}{dx} = \frac{6x}{3x^2 - 2}$$

4 Find $\frac{dy}{dx}$ when $y = e^{2x+5}$

Solution:

$$y = e^{2x+5}$$

$$\Rightarrow \quad u = 2x + 5 \quad \Rightarrow \quad y = e^u$$

$$\therefore \frac{du}{dx} = 2 \qquad \frac{dy}{du} = e^u$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2$$

$$\therefore \frac{dy}{dx} = 2e^{2x+5}$$

5 Take the parametric curve defined by $x = 2t^2$ & $y = 4t$. Point P has the co-ordinates, $(2p^2, 4p)$. Find the gradient at point P:

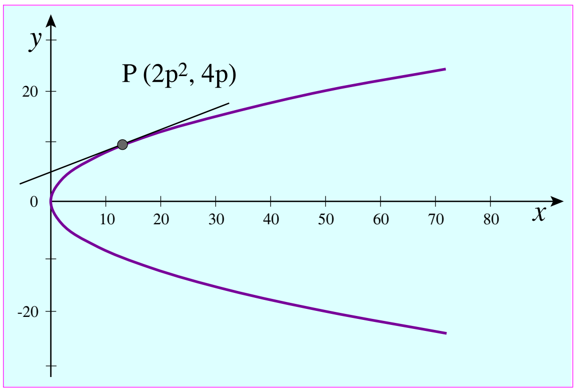
Solution:

Draw a sketch!!!!!!

$$t = \frac{y}{4} \quad \Rightarrow$$

$$x = 2\left(\frac{y}{4}\right)^2 \quad \Rightarrow$$

$$y^2 = 8x$$



$$x = 2t^2 \qquad y = 4t$$

$$\therefore \frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 4$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 4 \times \frac{1}{4t} = \frac{1}{t}$$

At point $P(2p^2, 4p)$; $y = 4p \quad \Rightarrow \quad \therefore 4p = 4t \quad \Rightarrow \quad p = t$

The gradient at point $P = \frac{1}{p}$

45.4 Using the Chain Rule Directly

After some practise, it is possible to use the chain rule with out formally writing down each stage. We can express this by writing the rule as:

If $\quad y = u^n$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{d\,[u^n]}{du} \times \frac{du}{dx}$$

$$= n\,u^{n-1}\left[\frac{du}{dx}\right]$$

45.4.1 Example:

$$y = (6x + 8)^4 \quad \Rightarrow \quad \frac{dy}{dx} = 4\,(6x + 8)^3 \times 6$$

$$y = (ax + b)^n \quad \Rightarrow \quad \frac{dy}{dx} = n\,(ax + b)^3 \times a$$

$$y = ln\,(4x - 1) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{4x - 1} \times 4 = \frac{4}{4x - 1}$$

$$y = \sqrt{4e^{3x} + 2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left[4e^{3x} + 2\right]^{-\frac{1}{2}} \times 12e^{3x} = \frac{6e^{3x}}{\sqrt{4e^{3x} + 2}}$$

$$y = \frac{1}{x^4 + 2} \quad \Rightarrow \quad \frac{dy}{dx} = -1\left[x^4 + 2\right]^{-2} \times 4x^3 = -\frac{4x^3}{(x^4 + 2)^2}$$

$$y = ln\,kx \quad \Rightarrow \quad \frac{d}{dx} = \frac{1}{kx} \times k = \frac{1}{x}$$

$$y = ln\,(ax + b) \quad \Rightarrow \quad \frac{d}{dx} = \frac{1}{ax + b} \times a = \frac{a}{ax + b}$$

45.5 Related Rates of Change

See also 50 • C3 • Differentiation: Rates of Change

The Chain Rule is a powerful way of connecting the rates of change of two dependent variables.

Consider a sphere, in which the volume is increasing at a given rate. Since the volume of the sphere is connected to the radius, how can the rate of increase in the radius be calculated?

If we are given the rate of increase in the volume, we have a value for $\frac{dV}{dt}$. The volume is connected to the radius via the function:

$$V = \frac{4}{3}\pi r^3 \quad \text{and hence} \quad \frac{dV}{dr} = 4\pi r^2$$

The required rate of increase in the radius is given by $\frac{dr}{dt}$. We can connect all these related rates of change using the chain rule such that:

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$$

If the volume is increasing at $980\ cm^3\ per\ hour$ and $r = 7\ cm$ at time t, then:

$$980 = 4\pi \times 49 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{980}{196\pi} = \frac{5}{\pi} = 1{\cdot}59\ cm/hour$$

45.5.1 Example:

1 Let A be the surface area of a spherical balloon. What is the rate of increase in the surface area of the balloon when the radius r is 6 cm, and the radius is increasing at 0·08 cm/sec?

Solution:

We want to find $\frac{dA}{dt}$, and we know that $A = 4\pi r^2$

$$\frac{dA}{dr} = 8\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 8\pi \times 6 \times 0{\cdot}08$$

$$\frac{dA}{dt} = 3{\cdot}84\pi\ \text{cm}^2/sec$$

2 An ice cube of side, $6x$ cm, melts at a constant rate of $0{\cdot}8\ \text{cm}^3/\text{min}$.
Find the rate at which x and the surface area A changes with time, when $x = 2$

Solution:

We want to find $\frac{dx}{dt}$ & $\frac{dA}{dt}$

The volume of the cube is $V = (6x)^3 \quad \Rightarrow 216x^3$

The surface area of the cube is $A = 6(6x)^2 \quad \Rightarrow 216x^2$

Now: $$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = -0{\cdot}8$$ (Negative as it is a decreasing value)

$$-0{\cdot}8 = 3 \times 216x^2 \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{0{\cdot}8}{3 \times 216 \times 4} = -0{\cdot}000308\ \text{cm/min}$$

Also: $$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$\frac{dA}{dt} = 2 \times 216x \times \left(-\frac{0{\cdot}8}{3 \times 216 \times 4}\right)$$

$$\frac{dA}{dt} = 2 \times 216 \times 2 \times \left(-\frac{0{\cdot}8}{3 \times 216 \times 4}\right)$$

$$\frac{dA}{dt} = \left(-\frac{0{\cdot}8}{3}\right) = -0{\cdot}267\ \text{cm}^2/\text{min}$$

3

45.6 Deriving the Chain rule

Start with a composite function of $y = gf(x)$ where $y = g(u)$ and $u = f(x)$. An increase in x by a small amount δx means a corresponding increase in u, by a small amount δu and hence y by δy.

$$\text{Now} \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Since δy, δu, δx can be handled algebraically, we have:

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

As $\delta x \to 0, \ \delta u \to 0$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta u \to 0} \left(\frac{\delta y}{\delta u} \right) \times \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} \right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

45.7 Chain Rule Digest

Used to differentiate composite functions.

If y is a function of u and u is a function of x then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

In function terminology:

$$F'(x) = g'(f(x)) \times f'(x)$$

where $F(x) = gf(x)$ and $F(x) = g(u)$ and $u = f(x)$

$$\frac{dy}{dx} = \frac{d[g(u)]}{du} \times \frac{du}{dx}$$

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$$

$$\frac{d}{dx}k[f(x)]^n = kn[f(x)]^{n-1} f'(x)$$

$$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$$

$$\frac{d}{dx}\left[e^{f(x)}\right] = \frac{d[f(x)]}{du} \times e^{f(x)} \quad \Rightarrow \quad = f'(x)e^{f(x)}$$

$$\frac{d}{dx}\left[ke^{f(x)}\right] = k\frac{d[f(x)]}{du} \times e^{f(x)} \quad \Rightarrow \quad = kf'(x)e^{f(x)}$$

$$\frac{d}{dx}\left[e^{x}\right] = e^{x}$$

$$\frac{d}{dx}\left[e^{kx}\right] = ke^{kx}$$

$$\frac{d}{dx}\left[e^{ax+b}\right] = ae^{ax+b}$$

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left[k\ln f(x)\right] = k\frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[\ln kx\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[\ln(ax+b)\right] = \frac{a}{ax+b}$$

46 • C3 • Differentiation: Product Rule

46.1 Differentiation: Product Rule

Assume y to be a function of x such that $y = f(x)$. Then consider y to be a product of two functions u and v, which themselves are also functions of x.

We now have:

$$y = uv \qquad \text{where } u \text{ and } v \text{ are functions of } x$$

In this situation, where y is a product of two functions we use the **Product Rule**, thus:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $\frac{du}{dx}$ is u differentiated w.r.t x

& $\frac{dv}{dx}$ is v differentiated w.r.t x

In function notation the rule is $\quad y = f(x)\,g(x) \qquad \frac{dy}{dx} = f(x)\,g'(x) + f'(x)\,g(x)$

or $\quad (uv)' = uv' + vu'$

Note: other text books sometimes have the product rule laid out slightly differently. Use whatever you find comfortable learning. e.g.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

46.2 Deriving the Product Rule

Starting with $y = uv$ and increasing x by a small amount δx, with corresponding increases in y, u and v, we have:

$$y + \delta y = (u + \delta u)(v + \delta v)$$

Substituting $y = uv$ $\qquad uv + \delta y = uv + u\delta v + v\delta u + \delta u\delta v$

Subtracting uv from both sides $\qquad \delta y = u\delta v + v\delta u + \delta u\delta v$

Divide by δx $\qquad \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}\delta u$

As $\delta x \to 0 \qquad \frac{\delta y}{\delta x} \to \frac{dy}{dx}, \qquad \frac{\delta u}{\delta x} \to \frac{du}{dx}, \qquad \frac{\delta v}{\delta x} \to \frac{dv}{dx}, \qquad \delta u \to 0$

More formerly $\qquad \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{dy}{dx}, \qquad \lim_{\delta x \to 0}\frac{\delta u}{\delta x} = \frac{du}{dx} \qquad \lim_{\delta x \to 0}\frac{\delta v}{\delta x} = \frac{dv}{dx} \qquad$ and $\lim_{\delta x \to 0}\delta u = 0$

$$\therefore \quad \frac{dy}{dx} = u\lim_{\delta x \to 0}\frac{\delta v}{\delta x} + v\lim_{\delta x \to 0}\frac{\delta u}{\delta x} + \lim_{\delta x \to 0}\frac{\delta v}{\delta x}\lim_{\delta x \to 0}\delta u$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

46.3 Product Rule: Worked Examples

46.3.1 Example:

1 Differentiate $y = 2x(x-1)^4$ and find the stationary points.

Solution:

$$u = 2x \qquad v = (x-1)^4$$

$$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = 4(x-1)^3$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = 2x \times 4(x-1)^3 + (x-1)^4 \times 2$$

$$= 8x(x-1)^3 + 2(x-1)^4$$

$$= 2(x-1)^3[4x + (x-1)]$$

$$= 2(x-1)^3(5x-1)$$

Stationary points when $\frac{dy}{dx} = 0$

$$2(x-1)^3(5x-1) = 0$$

$$x = 1 \quad \textit{and} \quad x = \frac{1}{5}$$

2 Differentiate $y = x^2(x^2+7)^2$

Solution:

$$u = x^2 \qquad v = (x^2+7)^2$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 2(x^2+7) \times 2x \quad \Rightarrow 4x(x^2+7) \quad \text{(chain rule)}$$

$$\therefore \quad \frac{dy}{dx} = x^2 \times 4x(x^2+7) + (x^2+7)^2 \times 2x$$

$$= 4x^3(x^2+7) + 2x(x^2+7)^2$$

$$= 2x(x^2+7)[2x^2 + (x^2+7)]$$

$$= 2x(x^2+7)(3x^2+7)$$

3 Differentiate $y = xe^x$

Solution:

$$u = x \qquad v = e^x$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = e^x$$

$$\therefore \quad \frac{dy}{dx} = x \times e^x + e^x \times 1$$

$$= e^x(x+1)$$

4 Differentiate $y = (x^2 + 4)(x^5 + 7)^4$

Solution:

$u = (x^2 + 4) \qquad v = (x^5 + 7)^4$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 4(x^5 + 7)^3 \times 5x^4 \quad \Rightarrow 20x^4(x^5 + 7)^3$

$$\therefore \quad \frac{dy}{dx} = (x^2 + 4) \times 20x^4(x^5 + 7)^3 + (x^5 + 7)^4 \times 2x$$

$$= 20x^4(x^5 + 7)^3(x^2 + 4) + 2x(x^5 + 7)^4$$

$$= 2x(x^5 + 7)^3\left[10x^3(x^2 + 4) + (x^5 + 7)\right]$$

$$= 2x(x^5 + 7)^3(11x^5 + 40x^3 + 7)$$

5 Differentiate $y = (x + 4)(x^2 - 1)^{\frac{1}{2}}$

Solution:

$u = (x + 4) \qquad v = (x^2 - 1)^{\frac{1}{2}}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \times 2x \quad \Rightarrow x(x^2 - 1)^{-\frac{1}{2}}$

$$\therefore \quad \frac{dy}{dx} = (x + 4) \times x(x^2 - 1)^{-\frac{1}{2}} + (x^2 - 1)^{\frac{1}{2}} \times 1$$

$$= x(x + 4)(x^2 - 1)^{-\frac{1}{2}} + (x^2 - 1)^{\frac{1}{2}}$$

$$= (x^2 - 1)^{-\frac{1}{2}}\left[x(x + 4) + (x^2 - 1)^1\right]$$

$$= (x^2 - 1)^{-\frac{1}{2}}(2x^2 + 4x - 1)$$

$$= \frac{(2x^2 + 4x - 1)}{(x^2 - 1)^{\frac{1}{2}}} = \frac{2x^2 + 4x - 1}{\sqrt{x^2 - 1}}$$

6 Differentiate $y = x^4 \sin x$

Solution:

$u = x^4 \qquad v = \sin x$

$\frac{du}{dx} = 4x^3 \qquad \frac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = x^4 \times \cos x + \sin x \times 4x^3$$

$$= x^4 \cos x + 4x^3 \sin x$$

$$= x^3(x \cos x + 4 \sin x)$$

7 Differentiate $y = \cos x \sin x$

Solution:

$$u = \cos x \qquad v = \sin x$$

$$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \cos x \times \cos x + \sin x \times (-\sin x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

8 Differentiate $y = a\,e^{bx} \sin ax$

Solution:

$$u = a\,e^{bx} \qquad v = \sin ax$$

$$\frac{du}{dx} = ab\,e^{bx} \qquad \frac{dv}{dx} = a\cos ax$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = a\,e^{bx} \times a\cos ax + \sin ax \times ab\,e^{bx}$$

$$= a^2 e^{bx} \cos ax + ab\,e^{bx} \sin ax$$

$$= a\,e^{bx}(a\cos ax + b\sin ax)$$

46.4 Topical Tips

Leave answers in factorised form. It is then easier to find the stationary points on a curve.

47 • C3 • Differentiation: Quotient Rule

47.1 Differentiation: Quotient Rule

Assume y to be a function of x such that $y = f(x)$. Then consider y to be a quotient of two functions u and v, which themselves are also functions of x. We now have:

$$y = \frac{u}{v} \qquad \text{where } u \text{ and } v \text{ are functions of } x$$

In this situation, where y is a quotient of two functions we use the **Quotient Rule**, thus:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Alternative forms of the equation as given in the exam formulae book:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

47.2 Quotient Rule Derivation

Starting with $y = \frac{u}{v}$ and the product rule we have:

$$\begin{aligned}
y &= \frac{u}{v} \\
u &= yv \\
\frac{du}{dx} &= y\frac{dv}{dx} + v\frac{dy}{dx} \\
v\frac{dy}{dx} &= \frac{du}{dx} - y\frac{dv}{dx} \\
v\frac{dy}{dx} &= \frac{du}{dx} - \frac{u}{v}\frac{dv}{dx} \\
\frac{dy}{dx} &= \frac{1}{v}\left[\frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}\right] \\
&= \frac{1}{v}\left[\frac{v}{v}\frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}\right] \\
&= \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right] \\
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}
\end{aligned}$$

47.3 Quotient Rule: Worked Examples

47.3.1 Example:

1 Differentiate $y = \dfrac{x}{x+1}$

Solution:

$$u = x \qquad v = x + 1 \qquad \text{Recall: } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+1)\times 1 - x \times 1}{(x+1)^2}$$

$$= \frac{x + 1 - x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

2 Differentiate $y = \dfrac{x+2}{x^2+3}$

Solution:

$$u = x + 2 \qquad v = x^2 + 3$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2+3)\times 1 - (x+2)\times 2x}{(x^2+3)^2}$$

$$= \frac{(x^2+3) - 2x(x+2)}{(x^2+3)^2}$$

$$= \frac{x^2 + 3 - 2x^2 - 4x}{(x^2+3)^2}$$

$$= \frac{3 - x^2 - 4x}{(x^2+3)^2}$$

3 Differentiate $y = \dfrac{3x}{e^{4x}}$

Solution:

$$u = 3x \qquad v = e^{4x}$$

$$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = 4e^{4x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^{4x}\times 3 - 3x \times 4e^{4x}}{(e^{4x})^2}$$

$$= \frac{3e^{4x} - 12x\,e^{4x}}{(e^{4x})^2} \quad \Rightarrow \quad \frac{3e^{4x}(1 - 4x)}{(e^{4x})^2}$$

$$= \frac{3(1-4x)}{e^{4x}}$$

4 Differentiate $y = \sqrt{\dfrac{x+1}{x^2+1}}$

Solution:

$$y = \frac{(x+1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}}$$

$$u = (x+1)^{\frac{1}{2}} \qquad v = (x^2+1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} \qquad \frac{dv}{dx} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x \qquad \Rightarrow x(x^2+1)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2+1)^{\frac{1}{2}} \times \frac{1}{2}(x+1)^{-\frac{1}{2}} - (x+1)^{\frac{1}{2}} \times x(x^2+1)^{-\frac{1}{2}}}{\left[(x^2+1)^{\frac{1}{2}}\right]^2}$$

$$= \frac{\frac{1}{2}(x^2+1)^{\frac{1}{2}}(x+1)^{-\frac{1}{2}} - x(x+1)^{\frac{1}{2}}(x^2+1)^{-\frac{1}{2}}}{(x^2+1)}$$

$$= \frac{\frac{1}{2}(x+1)^{-\frac{1}{2}}(x^2+1)^{-\frac{1}{2}}\left[(x^2+1) - 2x(x+1)\right]}{(x^2+1)}$$

$$= \frac{(x^2+1) - 2x^2 - 2x}{2(x^2+1)^{\frac{1}{2}}(x+1)^{\frac{1}{2}}(x^2+1)}$$

$$= \frac{1 - 2x - x^2}{2(x^2+1)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}$$

5 Differentiate $y = \dfrac{\ln x}{x+1}$ and find the gradient at $x = e$

Solution:

$$u = \ln x \qquad v = x + 1 \qquad \text{Recall:} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+1) \times \frac{1}{x} - \ln x \times 1}{(x+1)^2}$$

$$= \frac{x + 1 - x\ln x}{x(x+1)^2}$$

$$= \frac{e + 1 - e}{e(e+1)^2}$$

$$= \frac{1}{e(e+1)^2}$$

47.4 Topical Tips

Some quotients can be simplified such that the product or the chain rule can be used which are probably easier to handle.

E.g.

$$y = \frac{5}{(3x + 2)^2} \Rightarrow 5(3x + 2)^{-2} \quad \Rightarrow \quad \frac{dy}{dx} = -30(3x + 2)^{-3} \quad \text{chain rule}$$

$$y = \frac{1 - x}{1 + x} \Rightarrow (1 - x)(1 + x)^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2}{(1 + x)^2} \quad \text{product rule}$$

Note how the quotient rule is given in the formulae book:

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

Compare with our derivation:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

48 • C3 • Differentiation: Exponential Functions

48.1 Differentiation of e^x

Recall from Exponential & Log Functions that the value of e is chosen such that the gradient function of $y = e^x$ is the same as the original function and that when $x = 0$ the gradient of $y = e^x$ is 1.

Hence:

$$y = e^x \qquad \frac{dy}{dx} = e^x$$

$$y = e^{kx} \qquad \frac{dy}{dx} = ke^{kx}$$

$$y = e^{f(x)} \qquad \frac{dy}{dx} = f'(x)\,e^{f(x)}$$

48.1.1 Example:

1 Differentiate $y = 5e^{3x} + 2e^{-4x}$

Solution:

$$\frac{dy}{dx} = 5 \times 3e^{3x} + 2 \times (-4)\,e^{-4x}$$

$$\therefore \quad \frac{dy}{dx} = 15e^{3x} - 8e^{-4x}$$

2 Differentiate $y = \frac{1}{3}e^{9x}$ and find the equation of the tangent at $x = 0$

Solution:

$$\frac{dy}{dx} = \frac{1}{3} \times 9e^{9x} = 3e^{9x}$$

$$\therefore \quad \text{When } x = 0, \quad \frac{dy}{dx} = 3$$

$$\text{and} \qquad y = \frac{1}{3}e^0 = \frac{1}{3}$$

Now equation of a straight line is $y = mx + c$

$\therefore$ Equation of the tangent is $\quad y = 3x + \frac{1}{3}$

3 Differentiate $y = e^{x^3}$

Solution:

$u = x^3 \qquad \frac{du}{dx} = 3x^2$

$y = e^u \qquad \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = 3x^2 \times e^u = 3x^2\, e^u$$

$$\therefore \quad \frac{dy}{dx} = 3x^2\, e^{x^3}$$

4 Differentiate $y = e^{(x-1)^2}$

Solution:

$t = x - 1 \qquad \frac{dt}{dx} = 1$

$u = (t)^2 \qquad \frac{du}{dt} = 2t$

$y = e^u \qquad \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = e^u \times 2t \times 1 = 2t\, e^u$$

$$\therefore \quad \frac{dy}{dx} = 2(x - 1)\, e^{(x-1)^2}$$

5

49 • C3 • Differentiation: Log Functions

49.1 Differentiation of ln x

Recall that $ln\,x$ is the reciprocal function of e^x and that $y = e^x$ is a reflection of $y = ln\,x$ in the line $y = x$

$$y = ln\,x \qquad \frac{dy}{dx} = \frac{1}{x}$$

$$y = ln\,f(x) \qquad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Note that if: $y = ln\,kx \quad \Rightarrow \quad y = ln\,k + lnx$

$$\therefore \quad \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

This can be shown thus:

Recall that: $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} \qquad (1)$

If $\quad y = ln\,x \quad$ then $\quad x = e^y$

Differntiate w.r.t to y $\quad \dfrac{dx}{dy} = e^y$

Hence $\quad \dfrac{dx}{dy} = x$

From (1) $\quad \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{x}$

49.2 Worked Examples

49.2.1 Example:

1 Differentiate $y = ln\,x^2$

Solution:

$$u = x^2 \qquad \frac{du}{dx} = 2x$$

$$y = ln\,u \qquad \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 2x \times \frac{1}{u} = \frac{2x}{u}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$$

2 Differentiate $y = \ln\left(x^2\sqrt{2x^3 + 3}\right)$

Solution:

Let $z = x^2(2x^3 + 3)^{\frac{1}{2}}$ and $z = uv$

Where $u = x^2$ and $v = (2x^3 + 3)^{\frac{1}{2}}$

$\therefore \quad \dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = \dfrac{1}{2}(2x^3 + 3)^{-\frac{1}{2}} \times 6x^2 \Rightarrow 3x^2(2x^3 + 3)^{-\frac{1}{2}}$

$$\therefore \quad \frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^2 \times 3x^2(2x^3 + 3)^{-\frac{1}{2}} + (2x^3 + 3)^{\frac{1}{2}} \times 2x$$

$$= 3x^4(2x^3 + 3)^{-\frac{1}{2}} + 2x(2x^3 + 3)^{\frac{1}{2}}$$

but $$\frac{dy}{dx} = \frac{3x^4(2x^3 + 3)^{-\frac{1}{2}} + 2x(2x^3 + 3)^{\frac{1}{2}}}{x^2(2x^3 + 3)^{\frac{1}{2}}} = \frac{3x^4(2x^3 + 3)^{-\frac{1}{2}}}{x^2(2x^3 + 3)^{\frac{1}{2}}} + \frac{2x(2x^3 + 3)^{\frac{1}{2}}}{x^2(2x^3 + 3)^{\frac{1}{2}}}$$

$$= \frac{3x^2}{(2x^3 + 3)} + \frac{2}{x}$$

3 Differentiate $y = e^{x \ln 2}$

Solution:

$u = x \ln 2 \qquad \dfrac{du}{dx} = \ln 2$

$y = e^u \qquad \dfrac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \ln 2 \times e^u = \ln(2)\, e^u$$

$$\therefore \quad \frac{dy}{dx} = e^{x \ln 2} \ln(2)$$

4 Differentiate $y = e^{(x-1)^2}$

Solution:

$t = x - 1 \qquad \dfrac{dt}{dx} = 1$

$u = (t)^2 \qquad \dfrac{du}{dt} = 2t$

$y = e^u \qquad \dfrac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = e^u \times 2t \times 1 = 2t\, e^u$$

$$\therefore \quad \frac{dy}{dx} = 2(x - 1)\, e^{(x-1)^2}$$

50 • C3 • Differentiation: Rates of Change

50.1 Connected Rates of Change

Differentiation is all about rates of change. In other words, how much does y change with respect to x. Thinking back to the definition of a straight line, the gradient of a line is given by the change in y co-ordinates divided by the change in x co-ordinates. So it should come as no surprise that differentiation also gives the gradient of a curve at any given point.

Perhaps the most obvious example of rates of change is that of changing distance with time which we call speed. This can be taken further, and if the rate of change of speed with respect to time is measured we get acceleration.

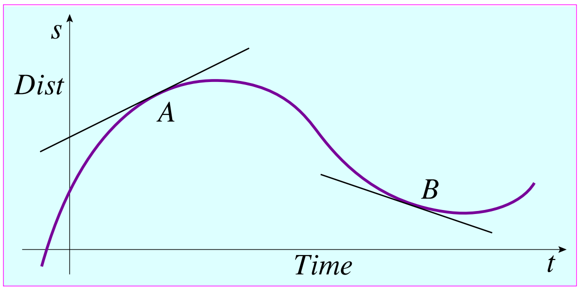
In terms of differentiation this can be written as:

$$\frac{ds}{dt} = v \qquad \text{where } s = \text{distance, } t = \text{time \& } v = \text{velocity}$$

$$\frac{dv}{dt} = a \qquad \text{where } s = \text{distance, } t = \text{time \& } a = \text{acceleration}$$

$$\frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt} \cdot \frac{ds}{dt} = \frac{d^2s}{dt^2}$$

The gradient at A is the rate at which distance is changing w.r.t time. i.e. speed. A +ve slope means speed is increasing and a −ve slope means it is decreasing.



50.2 Rate of Change Problems

- One of the primary uses of differential calculus
- Rates of change generally relate to a change w.r.t time
- Rate of increase is +ve
- Rate of decrease is −ve
- Often problems involve two variables changing with time - hence the chain rule is required:

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$$

- This means that y is a function of u and u is a function of t
- When answering these problems, state:
  - What has been given
  - What is required
  - Find the connection between variables
  - Make sure units are compatible
- Recall these formulae:
  - Volume of sphere $\frac{4}{3}\pi r^3$
  - Surface area of sphere $4\pi r^2$
  - Volume of a cone $\frac{1}{3}\pi r^2 h$

50.2.1 Example:

1 An objects speed varies according to the equation $y = 4 sin\, 2\theta$ and θ increases at a constant rate of 3 radians / sec. Find the rate at which y is changing when $\theta = \frac{15\pi}{18}$

Given: $y = 4\, sin\, 2\theta; \quad \frac{d\theta}{dt} = 3$

Required: $\frac{dy}{dt}$ when $\theta = \frac{15\pi}{18}$

Connection: $\frac{dy}{dt} = \frac{d\theta}{dt} \times \frac{dy}{d\theta}$

$$\frac{dy}{d\theta} = 8\, cos\, 2\theta$$

$$\therefore \quad \frac{dy}{dt} = 3 \times 8\, cos\, 2\theta = 24\, cos\, 2\theta$$

When $\theta = \frac{15\pi}{18}$ $\quad \frac{dy}{dt} = 24\, cos\left(2 \times \frac{15\pi}{18}\right) = 24 \times \frac{1}{2} = 12\, units\, /\, sec$

2

3 A spherical balloon (specially designed for exams) is being inflated. When the diameter is 10 cm, its volume is increasing at 200 cm^3 / sec. What rate is the surface area increasing.

Given: Volume of sphere: $V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 200$

Surface area of sphere: $A = 4\pi r^2$

Required: $\frac{dA}{dt}$ when $r = 5$

Connection: $\frac{dA}{dt} = \frac{dV}{dt} \times \frac{dA}{dV}$

To find $\frac{dA}{dV}$ will require a connection between Volume and Area which is the radius.
Using the chain rule to connect all the variables: $\frac{dA}{dV} = \frac{dr}{dV} \times \frac{dA}{dr}$
Extending the first chain connection we get:

$$\frac{dA}{dt} = \frac{dV}{dt} \times \frac{dr}{dV} \times \frac{dA}{dr}$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\therefore \quad \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\frac{dA}{dr} = 8\pi r$$

$$\frac{dA}{dt} = 200 \times \frac{1}{4\pi r^2} \times 8\pi r$$

$$\frac{dA}{dt} = \frac{400}{r}$$

When $r = 5$ $$\frac{dA}{dt} = \frac{400}{5} = 80\ cm^2/\ sec$$

4 The same balloon has its volume increased by 4 m^3/ sec. Find the rate at which the radius changes when $r = 4$ cm.

Given: Volume of sphere: $V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4$

Required: $\frac{dr}{dt}$ when $r = 4$

Connection: $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$$\frac{dV}{dr} = 4\pi r^2 \quad \therefore \quad \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$$

$$\frac{dr}{dt} = 4 \times \frac{1}{4\pi r^2} = \frac{1}{\pi r^2}$$

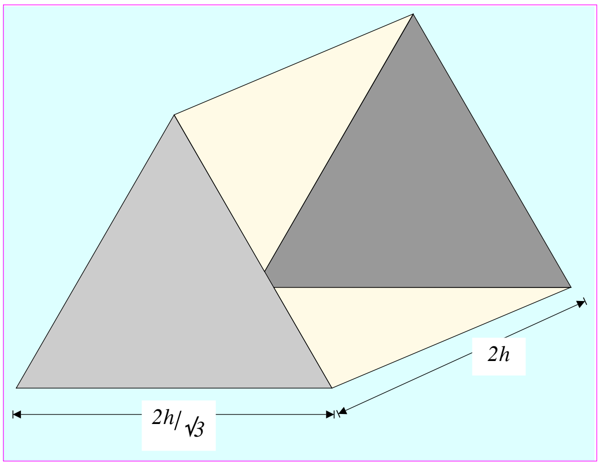
When $r = 4$ $$\frac{dr}{dt} = \frac{1}{16\pi}\ cm/\ sec$$

5 A prism, with a regular triangular base has length $2h$ and each side of the triangle measures $\frac{2h}{\sqrt{3}}$ cm. If h is increasing at 2 m/sec what is the rate of increase in the volume when $h = 9$?

Given: Volume of prism: $V = \frac{1}{2}bhl$; $\frac{dh}{dt} = 2$

Required: $\frac{dV}{dt}$ when $h = 9$

Connection: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$



Volume of prism: $V = \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h \times 2h = \frac{2h^3}{\sqrt{3}}$ $\quad \therefore \frac{dV}{dh} = \frac{6h^2}{\sqrt{3}}$

$$\frac{dV}{dt} = \frac{6h^2}{\sqrt{3}} \times 2 = \frac{12h^2}{\sqrt{3}}$$

When $h = 9$ $\quad \frac{dV}{dt} = \frac{12 \times 9^2}{\sqrt{3}} = 561{\cdot}2 \ (4\ sf)$

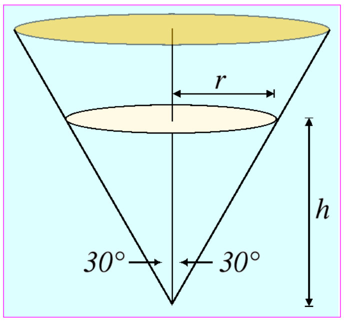
6 A conical vessel with a semi vertical angle of 30° is collecting fluid at the rate of 2 cm^3/ sec. At what rate is the fluid rising when the depth of the fluid is 6 cm, and what rate is the surface area of the fluid increasing?

Given: Volume of cone: $V = \frac{1}{3}\pi r^2 h$; $\frac{dV}{dt} = 2$

Radius of fluid: $r = h \tan 30 = \frac{h}{\sqrt{3}}$

Required, part 1: $\frac{dh}{dt}$ when $h = 6$

Connection, part 1: $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$



Volume of cone in terms of h: $V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{1}{9}\pi h^3$

$$\frac{dV}{dh} = \frac{3}{9}\pi h^2 = \frac{\pi h^2}{3}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\pi h^2} \times 2$$

When : $h = 6$ $\quad \frac{dh}{dt} = \frac{6}{\pi 36} = \frac{1}{6\pi}\ cm / sec$

Required, part 2: $\frac{dA}{dt}$ when $h = 6$

Connection, part 2: $\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

Area of fluid surface: $A = \pi r^2 = \pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{\pi h^2}{3}$

$$\frac{dA}{dh} = \frac{2\pi h}{3}$$

$\therefore$ $\quad \frac{dA}{dt} = \frac{2\pi h}{3} \times \frac{6}{\pi h^2} = \frac{4}{h}$

When : $h = 6$ $\quad \frac{dA}{dt} = \frac{4}{6} = \frac{2}{3} = 0{\cdot}333\ cm^2 /\ sec$

7

51 • C3 • Integration: Exponential Functions

51.1 Integrating e^x

Recall that:

$$\frac{d}{dx}e^x = e^x$$

and since integration is the reverse of differentiation, (i.e. integrate the RHS) we derive:

$$\int e^x dx = e^x + C$$

Similarly:

$$\frac{d}{dx}ae^x = ae^x \qquad \text{and} \qquad \frac{d}{dx}e^{(ax+b)} = ae^{(ax+b)}$$

$$\int ae^x dx = ae^x + C \qquad \text{and} \qquad \int e^{(ax+b)}dx = \frac{1}{a}e^{(ax+b)} + C$$

Note: to integrate an exponential with a different base that is not e, then the base must be converted to base e. A good reason to use base e at all times for calculus!

51.2 Integrating $1/x$

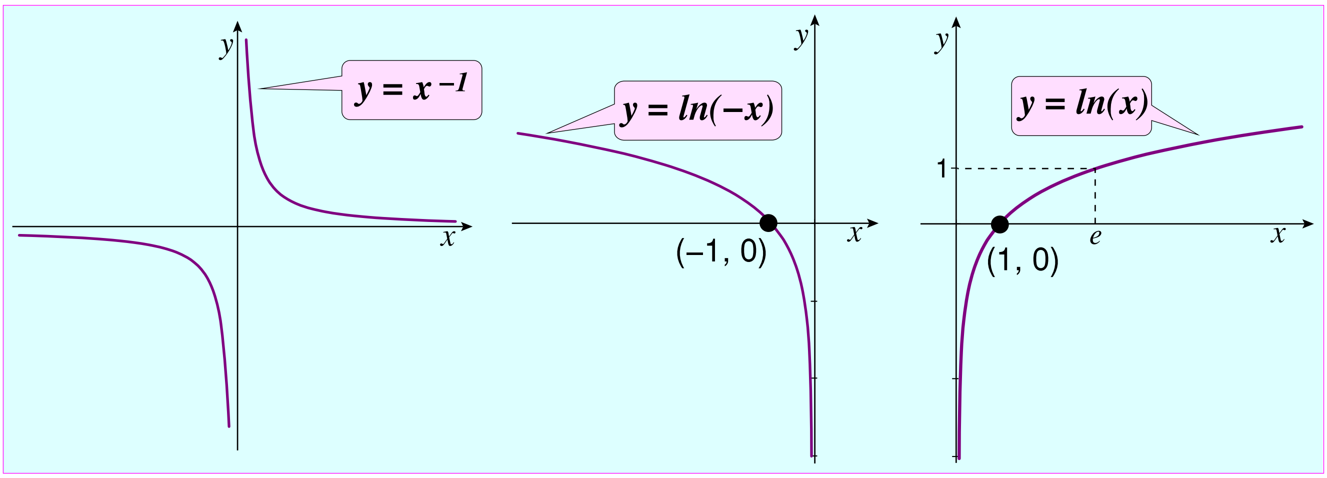
If you try to use the standard method of integration on a reciprocal function you end up in a mess, such as:

$$\int \frac{1}{x}dx = \int x^{-1}dx = \frac{1}{-1+1}x^{-1+1} + C = \frac{1}{0}x^0 + C \ ?\ ?\ ?\ ?\ ?\ ?$$

Now recall the work on differentiating $ln\,x$:

$$\frac{d}{dx}ln\,x = \frac{1}{x} \qquad \text{valid for} \qquad x > 0$$

and review the graphs for $ln\,x$ and $\frac{1}{x}$:



Graphs of 1/x, ln(−x) & ln x

Since $ln\,x$ is only valid for positive values of x (see graph above) and taking the reverse of the differential of $ln\,x$, (i.e. integrate the RHS) and provided $x > 0$ we derive:

$$\frac{d}{dx} ln\,x = \frac{1}{x} \quad \Leftrightarrow \quad \int \frac{1}{x}\,dx = ln\,x + C \quad \text{valid for } x > 0$$

However, from the graph of $y = x^{-1}$ we can see that solutions exist for negative values of x, so it must be possible to integrate $y = \frac{1}{x}$ for all values of x except for $x = 0$. The problem is dealing with $x < 0$.

From the graph, we can see that $ln\,(-x)$ is defined for negative values of x and so using the chain rule it can be shown that:

$$\frac{d}{dx} ln\,kx = \frac{1}{x} \qquad \text{where } k \text{ is a constant}$$

$$\text{If } k = -1 \qquad \text{then} \qquad \frac{d}{dx} ln\,(-x) = \frac{-1}{-x} = \frac{1}{x}$$

Hence:

$$\frac{d}{dx} ln\,(-x) = \frac{1}{x} \quad \Leftrightarrow \quad \int \frac{1}{x}\,dx = ln\,(-x) + C \quad \text{valid for } x < 0$$

Combining these two results using modulus notation we have:

$$\int \frac{1}{x}\,dx = ln\,|x| + C \qquad \text{provided} \qquad x \neq 0$$

With the restriction of $x \neq 0$, you cannot find the area under a curve with limits that span $x \neq 0$.

51.3 Integrating other Reciprocal Functions

Similar arguments can be made for reciprocals of the form $\frac{1}{ax+b}$.

Recall that:

$$\frac{d}{dx} ln\,(ax + b) = \frac{a}{ax + b} \quad \therefore \quad \int \frac{1}{ax + b}\,dx = \frac{1}{a} ln\,|ax + b| + C$$

52 • C3 • Integration: By Inspection

52.1 Integration by Inspection

Recall that integration is the reverse of differentiation such that:

$$\frac{d}{dx}f(x) = f'(x) \qquad \Leftrightarrow \qquad \int f'(x)\,dx = f(x) + C$$

This reversal of the process leads to a number of standard integrals (many of which can be found in the appendix).

Recognising and using standard integrals is often called

52.2 Integration of $(ax+b)^n$ by Inspection

Recall that using the chain rule:

$$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$$

and since integration is the reverse of differentiation, (i.e. integrate the RHS) we can derive the following standard integral:

$$\int (ax+b)^n = \frac{1}{a(n+1)}(ax+b)^{n+1} + C \qquad n \neq -1$$

52.2.1 Example:

1

If $y = (2x-1)^6$ then $\frac{dy}{dx} = 12(2x-1)^5$

$$\therefore \qquad \int (2x-1)^5\,dx = \frac{1}{12}(2x-1)^6 + C$$

$$\therefore \text{ Formula:} \qquad \int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C$$

2

Find the integral of $\sqrt{2-7x}$

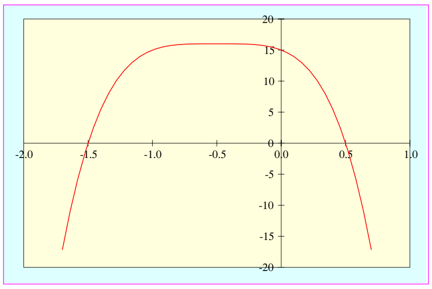
Solution:

$$\int \sqrt{2-7x}dx = \int (2-7x)^{\frac{1}{2}}\,dx = \frac{1}{-7 \times \frac{3}{2}}(2-7x)^{\frac{3}{2}} + C$$

$$= -\frac{1}{\frac{21}{2}}(2-7x)^{\frac{3}{2}} + C$$

$$= -\frac{2}{21}(2-7x)^{\frac{3}{2}} + C$$

3 Find the area between the curve $y = 16 - (2x + 1)^4$ and the x axis:



The curve crosses the x-axis at 2 points when:

$$16 - (2x + 1)^4 = 0$$

$$\therefore (2x + 1)^4 = 16 \qquad \Rightarrow (2x + 1) = \pm 2$$

$$\therefore x = \tfrac{1}{2} \text{ or } x = -\tfrac{3}{2}$$

$$\text{Area} = \int_{-\frac{3}{2}}^{\frac{1}{2}} 16 - (2x + 1)^4 \, dx = \left[16x - \frac{(2x + 1)^5}{10}\right]_{-\frac{3}{2}}^{\frac{1}{2}}$$

$$= \left[\frac{16}{2} - \frac{(1 + 1)^5}{10}\right] - \left[-\frac{16 \times 3}{2} - \frac{(-3 + 1)^5}{10}\right]$$

$$= \left[8 - \frac{(2)^5}{10}\right] - \left[-24 - \frac{(-2)^5}{10}\right]$$

$$= \left[8 - \frac{32}{10}\right] - \left[-24 - \frac{(-32)}{10}\right]$$

$$= 4.8 - (-20.8) = 25.6$$

52.3 Integration of $(ax+b)^{-1}$ by Inspection

The standard integral also applies to $(ax + b)^n$ for all values of n, except $n = 1$, which is a special case.

$$\int (ax + b)^{-n} dx = \frac{1}{a(-n + 1)} (ax + b)^{-n+1} + C \qquad \text{Not valid for } n = 1$$

The standard integral for $(ax + b)^{-1}$ is:

$$\int (ax + b)^{-1} dx = \frac{1}{a} \ln(ax + b) + C$$

52.3.1 Example:

1 Find the integral of $\dfrac{1}{(3x - 5)}$

$$\int (3x - 5)^{-1} \, dx = \frac{1}{3} \ln(3x - 5) + C$$

2 Find the integral of $\dfrac{1}{(3x - 5)^3}$

$$\int (3x - 5)^{-3} \, dx = \frac{1}{3 \times (-2)} (3x - 5)^{-2} + C$$

$$= -\frac{1}{6} (3x - 5)^{-2} + C$$

$$= -\frac{1}{6(3x - 5)^2} + C$$

53 • C3 • Integration: Linear Substitutions

See also the C4 topic on Substitution. Integration by Substitution

53.1 Integration by Substitution Intro

Although integrating something like $\int 3x(2x - 1)^3 \, dx$ could be solved by laboriously multiplying out the brackets and terms, an easier way is to use substitution, which is the integrals version of the chain rule.

Sometimes this is known as changing the variable.

We let u equal an expression in the integral and change all instances of x to u, (since we cannot integrate mixed variables).

$$\int (ax + b)^n \, dx = \int u^n dx = \int u^n \frac{dx}{du} \, du$$

Substitution method as follows:

- Choose the expression to be substituted and make equal to u
- Differentiate to find $\frac{du}{dx}$ and hence $\frac{dx}{du}$
- Substitute the new variable into the original integral
- Integrate w.r.t u
- Write the answer in terms of x.

53.2 Integration of $(ax+b)^n$ by Substitution

Although these types can be done by inspection, substitution can also be used if required.

53.2.1 Example:

1

$$\int (2x - 1)^5 \, dx$$

Let $u = 2x - 1$ $\qquad \frac{du}{dx} = 2 \qquad \frac{dx}{du} = \frac{1}{2}$

$$\therefore \int (2x - 1)^5 \, dx = \int (u)^5 \frac{dx}{du} \, du = \int u^5 \frac{1}{2} \, du$$

$$= \frac{1}{2 \times 6} u^6 + C$$

$$= \frac{1}{12} (2x - 1)^6 + C$$

2 Find the integral of $\sqrt{2 - 7x}$

$$\int \sqrt{2 - 7x} \, dx = \int (2 - 7x)^{\frac{1}{2}} \, dx$$

Let $u = 2 - 7x$ $\qquad \frac{du}{dx} = -7 \qquad \frac{dx}{du} = -\frac{1}{7}$

$$\therefore \int (2 - 7x)^{\frac{1}{2}} \, dx = \int (u)^{\frac{1}{2}} \frac{dx}{du} \, du = \int (u)^{\frac{1}{2}} \left(-\frac{1}{7}\right) du$$

$$= \left(-\frac{1}{7} \times \frac{2}{3}\right) (u)^{\frac{3}{2}} + C$$

$$= -\frac{2}{21} (2 - 7x)^{\frac{3}{2}} + C$$

3 Find the integral of $\dfrac{1}{(3x - 5)^3}$

Solution:

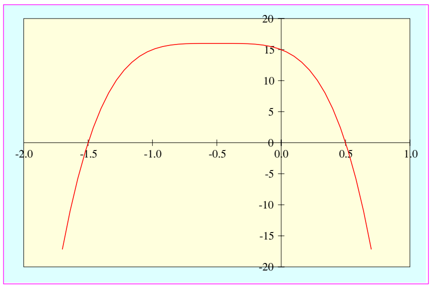
Let $u = 3x - 5$ $\qquad \dfrac{du}{dx} = 3 \qquad \dfrac{dx}{du} = \dfrac{1}{3}$

$$\therefore \int (3x - 5)^{-3}\, dx = \int (u)^{-3} \frac{dx}{du}\, du = \int (u)^{-3}\left(\frac{1}{3}\right) du$$

$$= \frac{1}{3}\left(-\frac{1}{2}\right)(u)^{-2}\, du$$

$$= -\frac{1}{6\,(3x - 5)^2} + C$$

4 Find the area between the curve $y = 16 - (2x + 1)^4$ and the x axis:



The curve crosses the x-axis at 2 points when:

$$16 - (2x + 1)^4 = 0$$

$$\therefore (2x + 1)^4 = 16 \qquad \Rightarrow (2x + 1) = \pm 2$$

$$\therefore x = \tfrac{1}{2} \text{ or } x = -\tfrac{3}{2}$$

$$\text{Area } A = \int_{-\frac{3}{2}}^{\frac{1}{2}} 16 - (2x + 1)^4\, dx$$

Let $u = 2x + 1$ $\qquad \dfrac{du}{dx} = 2 \qquad \dfrac{dx}{du} = \dfrac{1}{2}$

Change the limits to be in terms of u:

$x = \dfrac{1}{2},\ u = 2 \qquad\qquad x = -\dfrac{3}{2},\ u = -2$

$$A = \int_{-2}^{2} 16 - (u)^4 \frac{dx}{du}\, du$$

$$A = \frac{1}{2}\int_{-2}^{2} 16 - (u)^4\, du$$

$$= \frac{1}{2}\left[16u - \frac{(u)^5}{5}\right]_{-2}^{2}$$

$$= \left[8u - \frac{(u)^5}{10}\right]_{-2}^{2}$$

$$= \left[16 - \frac{(2)^5}{10}\right] - \left[-16 - \frac{(-2)^5}{10}\right]$$

$$= 12.8 - (-12.8) = 25.6$$

53.3 Integration Worked Examples

53.3.1 Example:

1 $\int 4x(6x+5)^4\,dx$

Solution:

Let $u = 6x + 5$ $\qquad \frac{du}{dx} = 6 \qquad \frac{dx}{du} = \frac{1}{6}$

$\therefore \quad x = \frac{u-5}{6}$

$$\therefore \int 4x(6x+5)^4\,dx = \int 4x(u)^4 \frac{dx}{du}\,du = \int 4x(u)^4 \frac{1}{6}\,du$$
$$= \frac{4}{6}\int x\,u^4\,du$$
$$= \frac{2}{3}\int\left(\frac{u-5}{6}\right)u^4\,du$$
$$= \frac{2}{18}\int (u-5)\,u^4\,du$$
$$= \frac{1}{9}\int\left(u^5 - 5u^4\right)du$$
$$= \frac{1}{9}\left[\frac{u^6}{6} - \frac{5u^5}{5}\right] + C$$
$$= \frac{1}{9}\left[\frac{u^6}{6} - u^5\right] + C$$
$$= \frac{1}{9}\left[\frac{u^6}{6} - \frac{6}{6}u^5\right] + C$$
$$= \frac{1}{54}\left[u^6 - 6u^5\right] + C$$
$$= \frac{1}{54}u^5[u-6] + C$$
$$= \frac{1}{54}(6x+5)^5[6x+5-6] + C$$
$$= \frac{1}{54}(6x+5)^5(6x-1) + C$$

2 $\int x\sqrt{2-x^2}\,dx = \int x(2-x^2)^{\frac{1}{2}}\,dx$

Let $u = 2 - x^2$ $\qquad \frac{du}{dx} = -2x \qquad \frac{dx}{du} = -\frac{1}{2x}$

$$\int x(2-x^2)^{\frac{1}{2}}\,dx = \int x(u)^{\frac{1}{2}}\frac{dx}{du}\,du = \int x(u)^{\frac{1}{2}}\left(-\frac{1}{2x}\right)du$$
$$= -\frac{1}{2}\int (u)^{\frac{1}{2}}\,du$$
$$= \left(-\frac{1}{2}\times\frac{2}{3}\right)(u)^{\frac{3}{2}} + C$$
$$= -\frac{1}{3}(2-x^2)^{\frac{3}{2}} + C$$

3 $\displaystyle\int (x+1)(3x-4)^4\,dx$

Solution:

Let $u = 3x - 4 \qquad \dfrac{du}{dx} = 3 \qquad \dfrac{dx}{du} = \dfrac{1}{3}$

$\therefore \quad x = \dfrac{u+4}{3} \quad \Rightarrow \quad x + 1 = \dfrac{u+4}{3} + 1 \Rightarrow \quad x+1 = \dfrac{u+7}{3}$

$$\int (x+1)(3x-4)^4\,dx = \int \left(\frac{u+7}{3}\right)(u)^4 \frac{dx}{du}\,du = \int \left(\frac{u+7}{3}\right)(u)^4 \frac{1}{3}\,du$$

$$= \frac{1}{9}\int (u+7)\,u^4\,du$$

$$= \frac{1}{9}\int \left(u^5 + 7u^4\right)du$$

$$= \frac{1}{9}\left[\frac{u^6}{6} - \frac{7u^5}{5}\right] + C$$

$$= \frac{1}{9}\left[\frac{5u^6}{30} - \frac{42u^5}{30}\right] + C$$

$$= \frac{1}{270}u^5(u-42) + C$$

$$= \frac{1}{270}(3x-4)^5(3x-4-42) + C$$

$$= \frac{1}{270}(3x-4)^5(3x-46) + C$$

4 Find the integral of $\dfrac{1}{\sqrt{x}\,(3+\sqrt{x})}$ using $u = \sqrt{x}$ as the substitution.

Solution:

Let $u = \sqrt{x} = x^{\frac{1}{2}} \qquad \dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} \qquad \dfrac{dx}{du} = 2\sqrt{x} = 2u$

$$\int \frac{1}{\sqrt{x}\,(3+\sqrt{x})}\,dx = \int \left(\frac{1}{u(3+u)}\right)\frac{dx}{du}\,du = \int \left(\frac{1}{u(3+u)}\right)2u\,du$$

$$= \int \frac{2u}{u(3+u)}\,du = 2\int \frac{1}{3+u}\,du$$

$$= 2\ln(3+u) + C$$

$$= 2\ln\left(3+\sqrt{x}\right) + C$$

5 $\int (6x + 3)(6x - 3)^5 \, dx$

Solution:

Let $u = 6x - 3$ $\quad \frac{du}{dx} = 6$ $\quad \frac{dx}{du} = \frac{1}{6}$

$\therefore \quad x = \frac{u + 3}{6} \quad \Rightarrow \quad 6x + 3 = 6\left(\frac{u + 3}{6}\right) + 3 \quad \Rightarrow \quad 6x + 3 = u + 6$

$$\int (6x + 3)(6x - 3)^5 \, dx = \int (u + 6)(u)^5 \frac{dx}{du} du = \int (u + 6)(u)^5 \frac{1}{6} du$$

$$= \frac{1}{6} \int (u + 6)(u)^5 \, du$$

$$= \frac{1}{6} \int (u^6 + 6u^5) du$$

$$= \frac{1}{6}\left[\frac{u^7}{7} + \frac{6u^6}{6}\right] + C$$

$$= \frac{u^6}{6}\left[\frac{u}{7} + 1\right] + C$$

$$= \frac{u^6}{42}[u + 7] + C$$

$$= \frac{1}{42}(6x - 3)^6 (6x - 3 + 7) + C$$

$$= \frac{1}{42}(6x - 3)^6 (6x + 4) + C$$

$$= \frac{2}{42}(3x + 2)(6x - 3)^6 + C$$

$$= \frac{1}{21}(3x + 2)(6x - 3)^6 + C$$

6 Find the integral of $\dfrac{e^x}{(2e^x + 3)}$

Solution:

Let $u = 2e^x + 3 \qquad \dfrac{du}{dx} = 2e^x \qquad \dfrac{dx}{du} = \dfrac{1}{2e^x}$

$e^x = \dfrac{u-3}{2} \quad \Rightarrow \quad 2e^x = u - 3 \quad \therefore \quad \dfrac{dx}{du} = \dfrac{1}{u-3}$

$$\int \frac{e^x}{(2e^x + 3)}\,dx = \int \left(\frac{u-3}{2}\right) \times \frac{1}{u}\frac{dx}{du}\,du = \int \left(\frac{u-3}{2}\right)\frac{1}{u}\left(\frac{1}{u-3}\right)\,du$$

$$= \int \frac{1}{2u}\,du$$

$$= \frac{1}{2}\,ln(u) + C$$

$$= 2\,ln\left(2e^x + 3\right) + C$$

7 Find the integral of $\dfrac{1}{6x+3}$ between $x = 0$ & $x = 1$

Solution:

Let $u = 6x + 3 \qquad \dfrac{du}{dx} = 6 \qquad \dfrac{dx}{du} = \dfrac{1}{6}$

$$\int_0^1 \frac{1}{6x+3}dx = \int_{x=0}^{x=1} \frac{1}{u}\frac{dx}{du}\,du = \int_{x=0}^{x=1} \frac{1}{6u}\,du$$

$$= \frac{1}{6}\int_{x=0}^{x=1} \frac{1}{u}\,du$$

$$= \frac{1}{6}\,\left[ln(u)\right]_{x=0}^{x=1}$$

$$= \frac{1}{6}\left[\,ln(6x+3)\right]_{x=0}^{x=1}$$

$$= \frac{1}{6}\left(ln(6+3) - ln(0+3)\right)$$

$$= \frac{1}{6}\left(ln(9) - ln(3)\right)$$

$$= \frac{1}{6}\left(ln\,\frac{9}{3}\right) = \frac{1}{6}\,ln\,3$$

Alternatively - change the limits to be in terms of u:

$x = 1 \quad \Rightarrow \quad u = 9$

$x = 0 \quad \Rightarrow \quad u = 3$

$$= \frac{1}{6}\int_3^9 \frac{1}{u}\,du$$

$$= \frac{1}{6}\,\left[ln(u)\right]_3^9$$

$$= \frac{1}{6}\left(ln(9) - ln(3)\right)$$

$$= \frac{1}{6}\,ln\,3$$

8 $\int 4x\left(x^2 - 5\right)^4\, dx$

Solution:

Let $u = x^2 - 5$ $\qquad \frac{du}{dx} = 2x \qquad \frac{dx}{du} = \frac{1}{2x}$

$$\therefore \int 4x\left(x^2 - 5\right)^4 dx = \int 4x\,(u)^4 \frac{dx}{du}\, du = \int 4x\,(u)^4 \frac{1}{2x}\, du$$

$$= \int 2\,(u)^4\, du$$

$$= \frac{2}{5}\,(u)^5 + C$$

$$= \frac{2}{5}\,(x^2 - 5)^5 + C$$

53.4 Derivation of Substitution Method

The argument goes something like this:

Given that:

$$x = g(u) \qquad \& \qquad f(x) = f[g(u)]$$

and $$y = \int f(x)\,dx \qquad (1)$$

Differentiating both sides of (1): $$\frac{dy}{dx} = f(x) \qquad (2)$$

From the chain rule: $$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

From (2) $$\frac{dy}{du} = f(x) \times \frac{dx}{du}$$

Integrating both sides w.r.t u: $$y = \int f(x)\frac{dx}{du}\,du$$

But $f(x) = f[g(u)]$ $$\therefore \quad y = \int f[g(u)]\frac{dx}{du}\,du$$

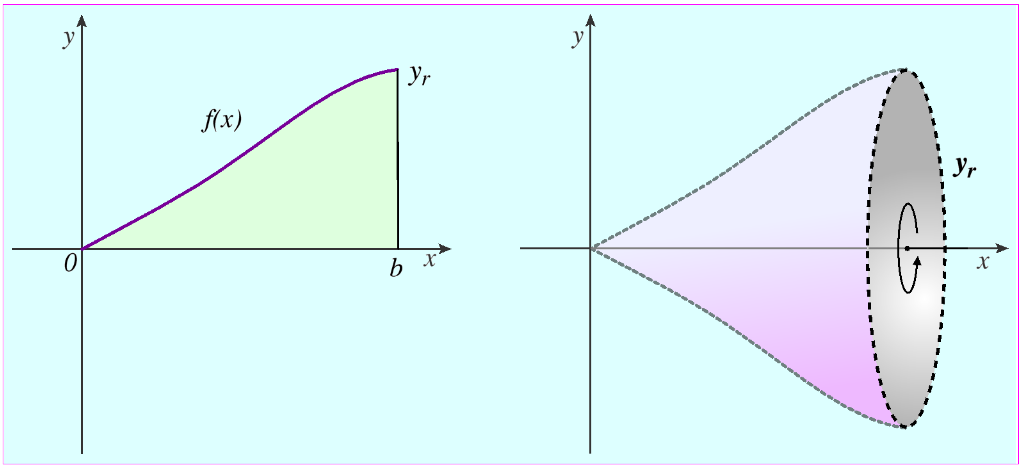
From (1) $$\int f(x)\,dx = \int f[g(u)]\frac{dx}{du}\,du$$

Note that in integrating $f(x)$ w.r.t. x, dx is replaced by $\frac{dx}{du}\,du$ and the rest of the integral is expressed in terms of u.

54 • C3 • Integration: Volume of Revolution

54.1 Intro to the Solid of Revolution

Integration gives us a convenient method for finding the area under a curve. Now consider the effect of rotating that area through 2π radians about the x-axis. This will produce a 'solid of revolution', as the example below illustrates. It then becomes possible to calculate the volume of this solid shape.

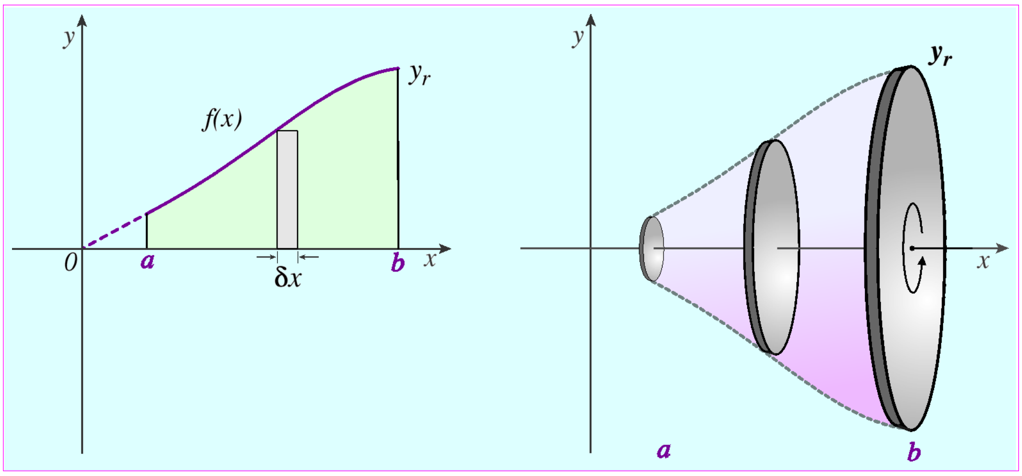


Solid of Revolution

54.2 Volume of Revolution about the x-axis

In a similar method to that of finding the area under a curve, we will put the solid shape though a bacon slicer, and produce a very large number of very thin slices. By assuming that each slice is a perfect cylinder of thickness δx the volume of each slice can be found. Summing all these slices together will give us the volume of the solid, or the 'Volume of Revolution'.

For the rotation of a curve $y = f(x)$ about the y-axis we have:



Volume of Revolution

Recall that the volume of a cylinder is $\pi r^2 d$, where r is the radius and d is the depth of the cylinder.

The volume of a thin slice, δV, is given by:

$$\delta V \approx \pi y^2 \delta x$$

Hence, the total volume of revolution about the x-axis is approximated by adding these slices together:

$$V \approx \sum \delta V \approx \sum_{i=1}^{n} \pi y^2 \delta x$$

Accuracy improves as δx becomes ever smaller and tends towards zero, hence the volume is the limit of the sum of all the slices as $\delta x \to 0$.

$$V = \lim_{\delta x \to 0} \sum_{i=1}^{n} \pi y^2 \delta x$$

Since $y = f(x)$ we can write:

$$V = \lim_{\delta x \to 0} \sum_{i=1}^{n} \pi [f(x)]^2 \delta x$$

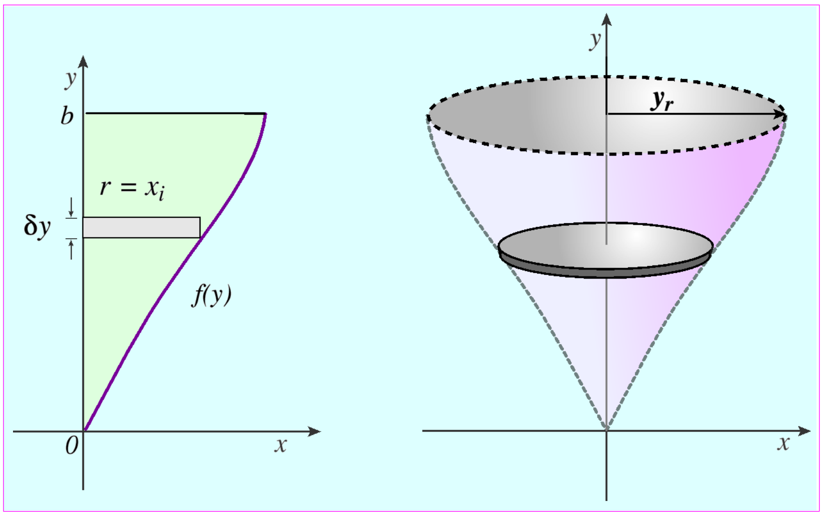
So the volume between the points $x = a$ and $x = b$ is given by:

$$\therefore \quad V = \int_a^b \pi [f(x)]^2 \, dx \equiv \int_a^b \pi y^2 dx$$

Note that since integration is done w.r.t x, then the limits are for $x = a$, & $x = b$.

54.3 Volume of Revolution about the y-axis

A similar argument can be made for the rotation of a curve $x = g(y)$ about the y-axis.



The volume of a slice is given by:

$$\delta V \approx \pi x^2 \delta y$$

Hence total volume of revolution about the y-axis is approximated by:

$$V \approx \sum \delta V \approx \sum_{i=1}^{n} \pi x^2 \delta y$$

Hence:

$$V = \lim_{\delta y \to 0} \sum_{i=1}^{n} \pi x^2 \delta y$$

$$V = \lim_{\delta y \to 0} \sum_{i=1}^{n} \pi [f(y)]^2 \delta y$$

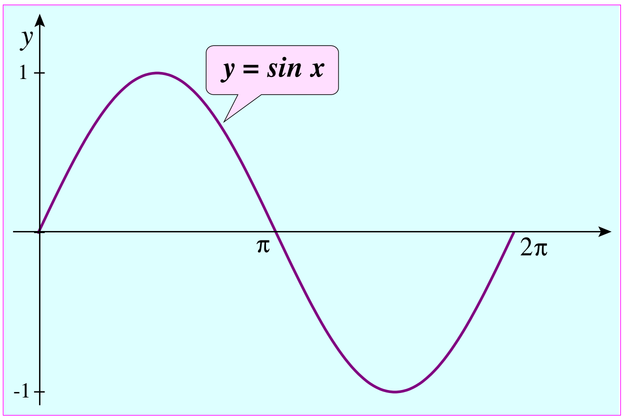
$$\therefore \quad V = \int_a^b \pi [f(y)]^2 \, dy$$

Note that since integration is done w.r.t y, then the limits are for $y = a$, & $y = b$.

54.4 Volume of Revolution Worked Examples

54.4.1 Example:

1



Find the volume of the solid generated by rotating the area under the curve of $y = \sin x$ when rotated through 2π radians about the x-axis, and between the y-axis and the line $x = \pi$.

Solution:

$$V = \int_a^b \pi y^2 dx$$

$$\therefore \quad V = \int_0^\pi \pi \sin^2 x \, dx$$

Now: $\quad 2\sin^2 x = 1 - \cos 2x$

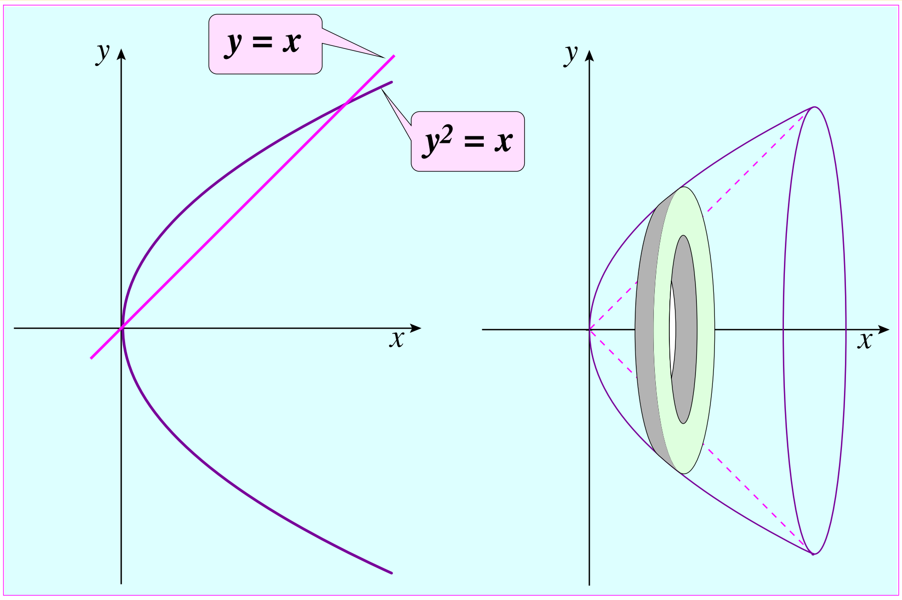
$$\therefore \quad V = \frac{\pi}{2}\int_0^\pi 1 - \cos 2x \; dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$$

$$= \frac{\pi}{2}\left[(\pi - 0) - \frac{1}{2}(0 - 0)\right]$$

$$= \frac{\pi^2}{2}\text{ units}^3$$

2



Find the volume of the solid generated by rotating the area between the curve $y^2 = x$ and the line $y = x$ through 2π radians, about the x-axis.

Solution:

In general, the solid of rotation of similar shapes is the difference between the solids of rotation of the two separate curves or lines.

Note that the limits are found from the intersection of the straight line and the curve. The intersection points are easy found to be (0, 0) and (1, 1).

$$V = \int_a^b \pi \left(y_1{}^2 - y_2{}^2\right) dx$$

In this case: $y_1{}^2 = x$ and $y_2 = x$

$$\therefore \qquad V = \int_0^1 \pi \left(x - x^2\right) dx$$

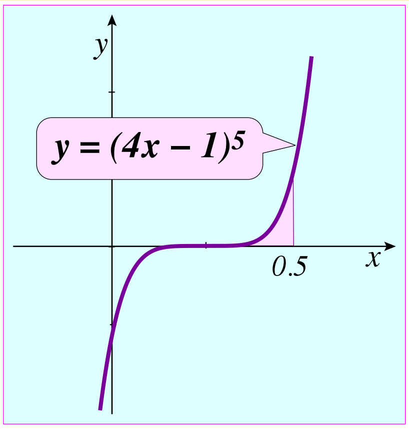
$$= \pi \int_0^1 \left(x - x^2\right) dx$$

$$= \pi \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \pi \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \pi \left[\frac{3}{6} - \frac{2}{6}\right] = \frac{1}{6}\pi \ units^3$$

3



The shaded region is rotated about the x-axis, find the volume of the solid.

Solution:

The limits of the shaded region are found when $y = 0$ and $x = 0{\cdot}5$ (given)

When $y = 0$ then $(4x - 1)^5 = 0$

$4x - 1 = 0$

$4x = 1$

$x = 0{\cdot}25$

To find the volume:

$$V = \int_a^b \pi y^2 dx$$

$$\therefore \quad V = \int_{0\cdot25}^{0\cdot5} \pi\left[(4x - 1)^5\right]^2 dx$$

let $u = 4x - 1$ and $\frac{du}{dx} = 4$, $\therefore \quad dx = \frac{du}{4}$

Changing the limits: $x = 0{\cdot}5 \quad u = 4 \times 0{\cdot}5 - 1 = 1$

$x = 0{\cdot}25 \quad u = 4 \times 0{\cdot}25 - 1 = 0$

$$\therefore \quad V = \int_0^1 \pi\left(u^5\right)^2 \frac{du}{4}$$

$$= \frac{\pi}{4}\int_0^1 u^{10}\, du$$

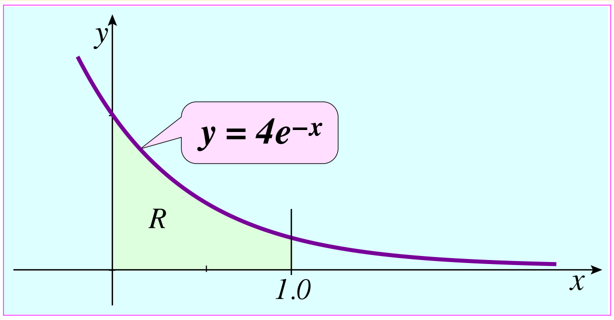
$$= \frac{\pi}{4}\left[\frac{1}{11}u^{11}\right]_0^1$$

$$= \frac{\pi}{44}\left[u^{11}\right]_0^1$$

$$= \frac{\pi}{44}\left[1 - 0\right]$$

$$= \frac{\pi}{44}\, units^3$$

4



The shaded region R enclosed by the curve $y = 4e^{-x}$ is rotated about the x-axis. Find the volume of the solid when the curve is bounded by the lines $x = 0$, $x = 1$ and $y = 0$.

Solution:

To find the volume:

$$V = \int_a^b \pi y^2 dx$$

$$\therefore \quad V = \int_0^1 \pi \left[4e^{-x}\right]^2 dx$$

$$= \int_0^1 16\pi\, e^{-2x} dx$$

$$= 16\pi \int_0^1 e^{-2x} dx$$

$$= 16\pi \left[-\frac{1}{2}e^{-2x}\right]_0^1$$

$$= \frac{16}{2}\pi \left[\left(-e^{-2}\right) - (-1)\right]$$

$$= 8\pi \left[\left(-e^{-2}\right) + 1\right]$$

$$= 8\pi \left(1 - e^{-2}\right)$$

54.5 Volume of Revolution Digest

Volume of Revolution about the *x*-axis:

$$V = \int_a^b \pi y^2 dx = \int_a^b \pi \left[f(x)\right]^2 dx$$

$$V = \pi \int_a^b y^2 dx$$

Note that since integration is done w.r.t x, then the limits are for $x = a$, & $x = b$.

Volume of Revolution about the *y*-axis:

$$V = \int_a^b \pi x^2 dy = \int_a^b \pi \left[f(y)\right]^2 dy$$

$$V = \pi \int_a^b x^2 dy$$

Note that since integration is done w.r.t y, then the limits are for $y = a$, & $y = b$.

Module C4

Core 4 Basic Info

Algebra and graphs; Differentiation and integration; Differential equations; Vectors.

The C4 exam is 1 hour 30 minutes long and is in two sections, and worth 72 marks (75 AQA).
(That's about a minute per mark allowing some time for over run and checking at the end)
Section A (36 marks) 5 – 7 short questions worth at most 8 marks each.
Section B (36 marks) 2 questions worth about 18 marks each.

OCR Grade Boundaries.

These vary from exam to exam, but in general, for C4, the approximate raw mark boundaries are:

Grade	100%	*A* *	*A*	*B*	*C*
Raw marks	72	62 ± 3	55 ± 3	48 ± 3	41 ± 3
UMS %	100%	90%	80%	70%	60%

The raw marks are converted to a unified marking scheme and the UMS boundary figures are the same for all exams.

C4 Contents

Plus other minor editoral alterations and corrections. * means latest items to be updated

C4 Brief Syllabus

1 Algebra and Graphs

- divide a polynomial, (degree $\leqslant 4$), by a linear or quadratic polynomial, & give quotient & any remainder
- express rational functions as partial fractions, and carry out decomposition, where the denominator is no more complicated than $(ax+b)(cx+d)(ex+f)$ or $(ax+b)(cx+d)^2$, and not top heavy.
- use the expansion of $(1+x)^n$ where n is a rational number and x <1 (finding a general term is not included, but adapting the standard series to expand, e.g. $(2-\frac{1}{2}x)^{-1}$ is included)
- understand the use of a pair of parametric equations to define a curve, and use a given parametric representation of a curve in simple cases
- convert the equation of a curve between parametric and Cartesian forms.

2 Differentiation and Integration

- use the derivatives of $\sin x$, $\cos x$, $\tan x$ together with sums, differences and constant multiples
- find and use the first derivative of a function which is defined parametrically or implicitly
- extend the idea of 'reverse differentiation' to include the integration of trig functions (e.g. $\sin x$, $\sec^2 2x$)
- use trig identities (e.g. double angle formulae) in the integration of functions such as $\cos^2 x$
- integrate rational functions by decomposition into partial fractions
- recognise an integrand of the form $\frac{kf'(x)}{f(x)}$, and integrate, for example $\frac{x}{x^2+1}$
- recognise when an integrand can be regarded as a product, and use integration by parts to integrate, for example, $x\sin 2x$, x^2e^2, $\ln x$ (understand the relationship between integration by parts and differentiation of a product)
- use a given substitution to simplify and evaluate either a definite or an indefinite integral

  (understand the relationship between integration by substitution and the chain rule).

3 First Order Differential Equations

- derive a differential equation from a simple statement involving rates of change (with a constant of proportionality if required)
- find by integration a general form of solution for a differential equation in which the variables are separable
- use an initial condition to find a particular solution of a differential equation
- interpret the solution of a differential equation in the context of a problem being modelled by the equation.

4 Vectors

- use of standard notations for vectors
- carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms
- use unit vectors, position vectors and displacement vectors
- calculate the magnitude of a vector, and identify the magnitude of a displacement vector $\overrightarrow{AB}$ as being the distance between the points A and B
- calculate the scalar product of two vectors (in either two or three dimensions), and use the scalar product to determine the angle between two directions and to solve problems concerning perpendicularity of vectors
- understand the significance of all the symbols used when the equation of a straight line is expressed in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$
- determine whether two lines are parallel, intersect or are skew
- find the angle between two lines, and the point of intersection of two lines when it exists.

C4 Assumed Basic Knowledge

Knowledge of C1, C2 and C3 is assumed, and you may be asked to demonstrate this knowledge in C4.
You should know the following formulae, (which are NOT included in the Formulae Book).

1 Differentiation and Integration

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
$\tan kx$	$k \sec^2 kx$

Function $f(x)$	Integral $\int f(x)\,dx$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert + c$

x in radians!

$$\int f'(g(x))\,g'(x)\;dx = f(g(x)) + c$$

2 Vectors

$$\lvert x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\rvert = \sqrt{x^2 + y^2 + z^2}$$

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \bullet (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = ax + by + cz$$

$$\mathbf{p} \bullet \mathbf{q} = \lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\cos\theta$$

$$\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz = \left(\sqrt{a^2 + b^2 + c^2}\right)\left(\sqrt{x^2 + y^2 + z^2}\right)\cos\theta$$

$$\mathbf{r} = \mathbf{a} + t\mathbf{p}$$

3 Trig

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 1 - 2\sin^2 A$$

$$= 2\cos^2 A - 1$$

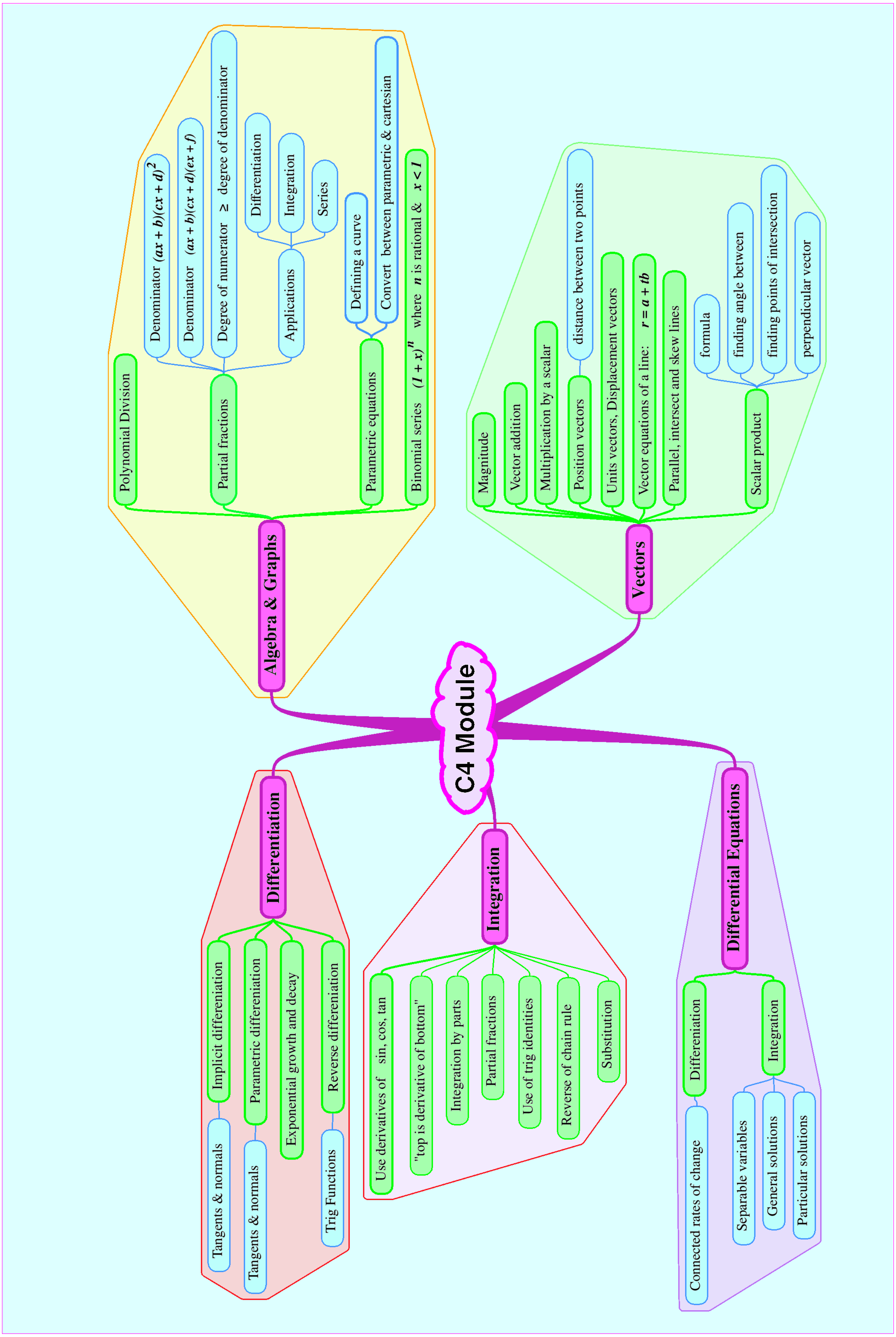
$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

$$a\sin x \pm b\cos x \equiv R\sin(x \pm \alpha)$$

$$a\cos x \pm b\sin x \equiv R\cos(x \mp \alpha)$$ (watch the signs)

$R = \sqrt{a^2 + b^2}$ $R\cos\alpha = a$ $R\sin\alpha = b$

$\tan\alpha = \frac{b}{a}$ $0 < a < \frac{\pi}{2}$


C4 Module
Algebra & Graphs
Polynomial Division
Partial fractions
Denominator $(ax+b)(cx+d)^2$
Denominator $(ax+b)(cx+d)(ex+f)$
Degree of numerator ≥ degree of denominator
Applications
Differentiation
Integration
Series
Parametric equations
Defining a curve
Convert between parametric & cartesian
Binomial series $(1+x)^n$ where n is rational & $|x|<1$
Vectors
Magnitude
Vector addition
Multiplication by a scalar
Position vectors
distance between two points
Units vectors, Displacement vectors
Vector equations of a line: $r = a + tb$
Parallel, intersect and skew lines
Scalar product
formula
finding angle between
finding points of intersection
perpendicular vector
Differentiation
Implicit differentiation
Tangents & normals
Parametric differentiation
Tangents & normals
Exponential growth and decay
Reverse differentiation
Trig Functions
Integration
Use derivatives of sin, cos, tan
"top is derivative of bottom"
Integration by parts
Partial fractions
Use of trig identities
Reverse of chain rule
Substitution
Differential Equations
Differeniation
Connected rates of change
Integration
Separable variables
General solutions
Particular solutions

56 • C4 • Differentiating Trig Functions

56.1 Defining other Trig Functions

This depends on 3 ideas:

- Definitions of $\tan x$, $\cot x$, $\sec x$ & $\operatorname{cosec} x$ in terms of $\sin x$ & $\cos x$
- The differential of $\sin x$ & $\cos x$
- Product and Quotient rules of differentiation.

From previous module: (*Note the coloured letters in bold - an easy way to remember them*).

$$\sec x \equiv \frac{1}{\cos x} \qquad \operatorname{cosec} x \equiv \frac{1}{\sin x} \qquad \cot x \equiv \frac{1}{\tan x}$$

Function $f(x)$	*Differential* $\frac{dy}{dx} = f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

true for x in radians

Product and Quotient rules:

Product rule: if $y = uv$ then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

Quotient rule: if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

We can use these results to find the differentials of the other trig functions:

1 ***tan x***

$$y = \tan x$$

$$y = \frac{\sin x}{\cos x} \Rightarrow \frac{u}{v}$$

Quotient rule: if $\quad y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x \times -\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

2 **sec *x***

$$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}$$

$$u = \cos x \qquad \frac{du}{dx} = -\sin x$$

$$y = u^{-1} \qquad \frac{dy}{du} = -u^{-2}$$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = -\sin x \times (-u^{-2}) = \frac{\sin x}{\cos^2 x} = \tan x \sec x$$

3 **cosec x**

$$y = cosec\,x = \frac{1}{\sin x} = (\sin x)^{-1}$$

$$u = \sin x \qquad \frac{du}{dx} = \cos x$$

$$y = u^{-1} \qquad \frac{dy}{du} = -u^{-2}$$

use the Chain rule: $$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = \cos x \times (-u^{-2})$$

$$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\tan x \sin x} = -\cot x\,cosec\,x$$

or use the Quotient rule:

$$u = 1 \qquad \frac{du}{dx} = 0$$

$$v = \sin x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\cot x\,cosec\,x$$

4 **cot x**

$$y = \cot x = \frac{1}{\tan x} = (\tan x)^{-1}$$

$$u = \tan x \qquad \frac{du}{dx} = \sec x$$

$$v = u^{-1} \qquad \frac{dv}{dx} = -u^{-2} = -\frac{\sec x}{\tan^2 x}$$

$$\frac{dy}{dx} = \frac{-1 - \tan x}{\tan^2 x} = \frac{-1}{\tan^2 x} \times \frac{-\tan x}{\tan^2 x}$$

$$\frac{dy}{dx} = -\cot^2 x - 1 = -cosec^2 x$$

Summary so far:

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-cosec\,x$
$cosec\,x$	$-cosec\,x\cot x$
$\sec x$	$\sec x \tan x$

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$

56.2 Worked Trig Examples

56.2.1 Example: Differentiate the following:

1

$y = x^3 \sin x$ [product rule]

$u = x^3 \qquad v = \sin x$

$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = x^3 \times \cos x + \sin x \times 3x^2$

$\frac{dy}{dx} = x^2 (x \cos x + 3 \sin x)$

2

$y = \frac{1}{x} \cos x \Rightarrow \frac{\cos x}{x}$ [quotient rule]

Let: $u = \cos x \qquad v = x$

$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{x \times (-\sin x) - \cos x}{x^2}$

$\frac{dy}{dx} = \frac{-x \sin x - \cos x}{x^2} = -\frac{x \sin x + \cos x}{x^2}$

3

$y = \cos^4 x$ [chain rule]

$u = \cos x \qquad y = u^4$

$\frac{du}{dx} = -\sin x \qquad \frac{dy}{du} = 4u^3$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 4u^3 \times (-\sin x)$

$\frac{dy}{dx} = 4 \cos^3 x (-\sin x) = -4 \cos^3 x \sin x$

4

$y = \cos^4 x$ [quick method - diff out - diff in]

$y = (\cos x)^4$

$\frac{dy}{dx} = 4 (\cos x)^3 \times (-\sin x)$ [differentiate outside bracket - differentiate inside bracket]

$\frac{dy}{dx} = -4 \cos^3 x \sin x$

5

$y = \ln \sec x$ [chain rule]

$u = \sec x \qquad y = \ln u$

$\frac{du}{dx} = \sec x \tan x \qquad \frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = \sec x \tan x \times \frac{1}{\sec x} = \tan x$

6

$$y = \sin\left(3x - \frac{\pi}{4}\right)$$ [chain rule]

$u = 3x - \frac{\pi}{4}$ $\qquad$ $y = \sin u$

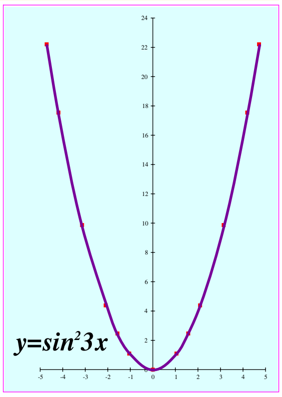
$\frac{du}{dx} = 3$ $\qquad$ $\frac{dy}{du} = \cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos(u) \times 3$$

$$\frac{dy}{dx} = 3\cos\left(3x - \frac{\pi}{4}\right)$$

7



$y = \sin^2 3x$

$u = 3x$ $\qquad$ $y = \sin^2 u$

$v = \sin u$

$\therefore y = v^2$

$\frac{du}{dx} = 3$ $\qquad$ $\frac{dy}{dv} = 2v$ $\qquad$ $\frac{dv}{du} = \cos u$ $\qquad$ [extended chain rule]

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{du}{dx} \times \frac{dv}{du}$$

$$\frac{dy}{dx} = 2v \times 3 \times \cos u$$

$$\frac{dy}{dx} = 2\sin u \times 3 \times \cos u$$

$$\frac{dy}{dx} = 6\cos 3x \times \sin 3x$$

$\frac{dy}{dx} = 3\sin 6x$ $\qquad$ double angle formula

8 Alternative approach to above problem

$y = \sin^2 3x = (\sin 3x)^2$ [chain rule]

$u = \sin 3x \qquad y = u^2$

$\frac{du}{dx} = 3\cos 3x \qquad \frac{dy}{du} = 2u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 2u \times 3\cos 3x$

$\frac{dy}{dx} = 6\sin 3x \,.\, \cos 3x$

$\frac{dy}{dx} = 3\sin 6x$ [double angle formula]

9 $y = \sin^5 x \cos^3 x$

$u = \sin^5 x \qquad v = \cos^3 x$

Use chain rule on u

If $z = \sin x$

$u = z^5$

$\frac{du}{dz} = 5z^4 \qquad \frac{dz}{dx} = \cos x$

$\frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx}$

$\frac{du}{dx} = 5z^4 \times \cos x$

$\frac{du}{dx} = 5\sin^4 x \cos x$

Use chain rule on v

If $w = \cos x$

$v = w^3$

$\frac{dv}{dw} = 3w^2 \qquad \frac{dw}{dx} = -\sin x$

$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$

$\frac{dv}{dx} = 3w^2 \times (-\sin x)$

$\frac{dv}{dx} = -3\cos^2 x \sin x$

Use product rule:

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{dy}{dx} = \cos^3 x \times 5\sin^4 x \times \cos x - \sin^5 x \times 3\cos^2 x \times \sin x$

$\frac{dy}{dx} = 5\cos^4 x \times \sin^4 x - 3\sin^6 x \times \cos^2 x$

$\frac{dy}{dx} = \sin^4 x \cos^2 x\,(5\cos^2 x - 3\sin^2 x)$

10

$y = \ln\sqrt{\sin x}$ [chain rule]

$= \ln(\sin x)^{\frac{1}{2}}$

$= \frac{1}{2}\ln(\sin x)$ [log laws]

Let $u = \sin x$

$\therefore \quad y = \frac{1}{2}\ln u$

$\frac{dy}{du} = \frac{1}{2} \times \frac{1}{u} \qquad \frac{du}{dx} = \cos x$

$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{u} \times \cos x = \frac{\cos x}{2\sin x}$

11

$y = 4x^6 \sin x$ [product rule]

Let $u = 4x^6 \qquad v = \sin x$

$\therefore \frac{du}{dx} = 24x^5 \qquad \frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{dy}{dx} = \sin x \times 24x^5 + 4x^6 \times \cos x$

$\frac{dy}{dx} = 4x^5(6\sin x + x\cos x)$

12

$y = \tan^3 x - 3\tan x$ [chain rule]

$= (\tan x)^3 - 3\tan x$

Let $u = \tan x \qquad \therefore \frac{du}{dx} = \sec^2 x$

$y = u^3 - 3u$

$\frac{dy}{du} = 3u^2 - 3$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (3u^2 - 3) \times \sec^2 x$

$\frac{dy}{dx} = (3\tan^2 x - 3) \times \sec^2 x$

$\frac{dy}{dx} = 3\sec^2 x(\tan^2 x - 1)$

13

$y = \dfrac{\cos 3x}{e^{3x}}$ [quotient rule]

Let $u = \cos 3x \qquad v = e^{3x}$

$\therefore \dfrac{du}{dx} = -3\sin 3x \qquad \dfrac{dv}{dx} = 3e^{3x}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{e^{3x}(-3\sin 3x) - \cos 3x \times 3e^{3x}}{(e^{3x})^2}$$

$$= \frac{-3\sin 3x - \cos 3x}{(e^{3x})}$$

$$= -\frac{3(\sin 3x + \cos 3x)}{e^{3x}}$$

14

$y = \operatorname{cosec} 3x$ [quick method - diff out - diff in]

$y = \operatorname{cosec}(3x)$

$\dfrac{dy}{dx} = -\operatorname{cosec}(3x)\cot(3x) \times 3$ [differentiate outside bracket - differentiate inside bracket]

$\dfrac{dy}{dx} = -3\operatorname{cosec} 3x \cot 3x$

15

$y = \cot^2 3x$ [quick method - diff out - diff in]

$y = (\cot\ (3x))^2$

$\dfrac{dy}{dx} = 2(\cot(3x))^1 \times (-\operatorname{cosec}^2 3x) \times 3$

[differentiate outside bracket - differentiate inside bracket – used twice]

$\dfrac{dy}{dx} = -6\cot 3x \operatorname{cosec}^2 3x$

16

Find the smallest value of θ for which the curve $y = 2\theta - 3\sin\theta$ has a gradient of 0.5

$y = 2\theta - 3\sin\theta$

$\dfrac{dy}{d\theta} = 2 - 3\cos\theta$

When $\dfrac{dy}{d\theta} = 0{\cdot}5 \qquad \Rightarrow \qquad 2 - 3\cos\theta = 0{\cdot}5$

$\therefore \cos\theta = 0{\cdot}5$

$\therefore$ smallest +ve value of $\theta = \dfrac{\pi}{3}$

Note: the answer is given in radians, differentiation and integration are valid only if angles are measured in radians.

17

$$y = \frac{sin^4 3x}{6x} \qquad \text{[quotient rule \& chain rule]}$$

$$y = \frac{(sin\ (3x))^4}{6x}$$

Let $z = sin\ (3x) \qquad \frac{dz}{dx} = 3\,cos\ (3x)$

$u = (z)^4 \qquad v = 6x$

$\therefore \quad \frac{du}{dz} = 4\,z^3 \qquad \frac{dv}{dx} = 6$

$$\frac{dz}{dx} = 3\,cos\ (3x)$$

$$\frac{du}{dx} = \frac{dz}{dx} \times \frac{du}{dz} = 3\,cos\ (3x) \times 4\,z^3 = 12\,cos\ (3x)\ sin^3(3x)$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{6x\,[12\,cos\,(3x)\ sin^3(3x)] - sin^4(3x) \times 6}{(6x)^2}$$

$$\frac{dy}{dx} = \frac{6x\,[4\,(sin\ (3x))^3 \times 3\,cos\,(3x)] - sin^4(3x) \times 6}{(6x)^2}$$

[or differentiate outside bracket - differentiate inside bracket]

$$= \frac{72x\,sin^3(3x)\ cos\,(3x) - 6\,sin^4(3x)}{36x^2}$$

$$= \frac{sin^3(3x)\,[12\,cos\,(3x) - sin\,(3x)]}{6x^2}$$

18

$$y = sin^2 x\ cos\ 3x$$

Need product rule and chain rule:

$$y = (sin\,x)^2 \times cos\,(3x)$$

$$\frac{dy}{dx} = sin^2 x \times -3sin\,(3x) + cos\ 3x \times 2\,(sin\,x)\ cos\,x$$

$$\frac{dy}{dx} = sin\,x\,[2\,cos\ 3x\ cos\,x - 3sin\,x\ sin\ 3x]$$

56.3 Differentiation of Log Functions

These can be used to find the integrals by reversing the process.

1 $y = ln|\sin x|$ [chain rule]

Let $u = \sin x \quad \therefore \frac{du}{dx} = \cos x$

$y = ln|u| \quad \therefore \frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times \cos x$$

OR

If $y = ln|f(x)| \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \times \cos x$$

$$\frac{dy}{dx} = \cot x$$

2 $y = ln|\sec x|$ [chain rule]

$$\frac{dy}{dx} = \frac{1}{\sec x} \times \frac{d}{dx}(\sec x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \times \sec x \tan x$$

$$\frac{dy}{dx} = \tan x$$

3 $y = ln|\sec x + \tan x|$ [chain rule]

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \frac{d}{dx}(\sec x + \tan x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \sec x \tan x + \sec^2 x$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \sec x\,(\tan x + \sec x)$$

$$\frac{dy}{dx} = \sec x$$

4 $y = -\ln|\operatorname{cosec} x + \cot x|$ [chain rule]

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} x + \cot x} \times \frac{d}{dx}(\operatorname{cosec} x + \cot x)$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} x + \cot x} \times (-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} x + \cot x} \times (-\operatorname{cosec} x\ (\cot x + \operatorname{cosec} x)$$

$$\frac{dy}{dx} = -\frac{-\operatorname{cosec} x\ (\cot x + \operatorname{cosec} x)}{\operatorname{cosec} x + \cot x}$$

$$\frac{dy}{dx} = \operatorname{cosec} x$$

57 • C4 • Integrating Trig Functions

57.1 Intro

Integrating trig functions is mainly a matter of recognising the standard derivative and reversing it to find the standard integral. You need a very good working knowledge of the trig identities and be able to use the chain rule. Although this chapter has been divided up into a number of smaller sections to aid recognition of the different types of integral, most of the methods used are similar to each other.

57.2 Integrals of $\sin x$, $\cos x$ and $\sec^2 x$

From the standard derivative of the basic trig functions, the integral can be found by reversing the process. Thus:

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\, dx = \sin x + c$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad \Rightarrow \quad \int \sin x\, dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\, dx = \tan x + c$$

Only valid for x in radians

57.3 Using Reverse Differentiation:

In a similar manner the following can be found:

Function $y = f(x)$	Integral $\int f(x)\,dx$	
$\sin x$	$-\cos x + c$	* *
$\cos x$	$\sin x + c$	* *
$\sin kx$	$-\frac{1}{k}\cos kx +$	* *
$\cos kx$	$\frac{1}{k}\sin kx +$	* *
$\sec^2 kx$	$\frac{1}{k}\tan kx + c$	*
$\sec x \tan x$	$\sec x + c$	
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x + c$	
$\operatorname{cosec}^2 x$	$-\cot x + c$	
$\cot x$	$\ln\lvert \sin x \rvert$	

Valid for x in radians

E.g.

$$\int \operatorname{cosec} 2x \cot 2x\, dx = -\frac{1}{2}\operatorname{cosec} 2x + c$$

57.3.2 Example:

1

$$\int \tan(2x - \pi)\, dx$$

$$= \frac{1}{2} \ln \left| \sec\,(2x - \pi) \right| + c$$

2

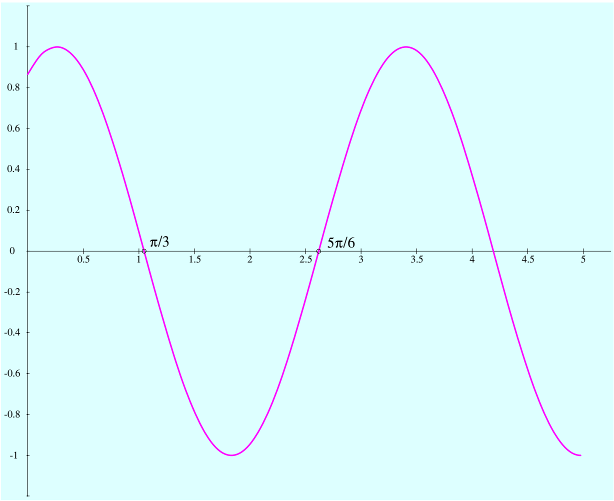
$$\int \tan^2 x\, dx$$

use $\quad 1 + \tan^2 x = \sec^2 x \quad$ Avoids use of *ln* in the answer …

$$\therefore \int (\sec^2 x - 1)\, dx = \tan x - x + c$$

3

Find the area under the curve $y = \sin(2x + \frac{1}{3}\pi)$ from $x = 0$ to the first point at which the graph cuts the positive x-axis.



To find limits of function:

Axis is cut when: $\sin\left(2x + \frac{1}{3}\pi\right) = 0$

$$\therefore \quad 2x + \frac{1}{3}\pi = 0,\ \pi,\ etc$$

$$2x = -\frac{\pi}{3},\ \frac{2\pi}{3},\ etc$$

$\therefore \quad x = -\frac{\pi}{6}$ (which can be ignored as it is outside the range required)

or $\quad x = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} \sin\left(2x + \frac{\pi}{3}\right) dx = \left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \left[-\frac{1}{2}\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)\right] - \left[-\frac{1}{2}\cos\left(\frac{\pi}{3}\right)\right] \qquad [\cos\pi = -1]$$

$$= \left(\frac{1}{2}\right) - \left(-\frac{1}{4}\right) = \frac{3}{4} \qquad [\cos\pi/3 = \tfrac{1}{2}]$$

57.4 Integrals of tan x and cot x

To find the integrals recognise the standard integral type:

$$\int \frac{kf'(x)}{f(x)}\,dx = k\ln|f(x)| + c$$

Derive $\int \tan x$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\int \tan x = \int \frac{\sin x}{\cos x}\,dx$$

$$= -\int \frac{-\sin x}{\cos x}\,dx$$

$$= -\ln|\cos x| + c$$

$$= \ln|(\cos x)^{-1}| + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c$$

$$\int \tan x = -\ln|\cos x| + c = \ln|\sec x| + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$$

For the general case:

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + c$$

This is often asked for in the exam.

Similarly it can be shown that:

$$\int \cot x = \int \frac{\cos x}{\sin x}\,dx$$

$$= \ln|\sin x| + c$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin x| + c$$

NB the modulus sign means you can't take the natural log of a negative number.

57.5 Recognising the Opposite of the Chain Rule

Reversing the derivatives (found using the chain rule), the following can be derived:

$$\frac{d}{dx} sin\ (ax + b) = a\, cos\ (ax + b) \quad \Rightarrow \quad \int cos\ (ax + b)\ dx = \frac{1}{a} sin\ (ax + b) + c$$

$$\frac{d}{dx} cos\ (ax + b) = -a\, sin\ (ax + b) \quad \Rightarrow \quad \int sin\ (ax + b)\ dx = -\frac{1}{a} cos\ (ax + b) + c$$

$$\frac{d}{dx} tan\ (ax + b) = a\, sec^2 (ax + b) \quad \Rightarrow \quad \int sec^2 (ax + b)\ dx = \frac{1}{a} tan\ (ax + b) + c$$

57.5.1 Example:

1 Show that $\int_0^{\frac{\pi}{4}} sec^2\left(2x - \frac{\pi}{4}\right) = 1$

Solution:

$$\int_0^{\frac{\pi}{4}} sec^2\left(2x - \frac{\pi}{4}\right) = \left[\frac{1}{2} tan\left(2x - \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} tan\left(2\frac{\pi}{4} - \frac{\pi}{4}\right) - \frac{1}{2} tan\left(0 - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left[tan\left(\frac{\pi}{4}\right) - tan\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}[1 + 1] = 1$$

2

57.6 Integrating with Trig Identities

This covers many of the sub topics in this chapter. You really, really need to know these, the most useful of which are:

Pythag:

$$\cos^2 A + \sin^2 A \equiv 1$$

$$1 + \tan^2 A \equiv \sec^2 A$$

Double angle

$$\cos 2A \equiv 2\cos^2 A - 1 \qquad \therefore \qquad \cos^2 A \equiv \tfrac{1}{2}(1 + \cos 2A)$$

$$\cos 2A \equiv 1 - 2\sin^2 A \qquad \therefore \qquad \sin^2 A \equiv \tfrac{1}{2}(1 - \cos 2A)$$

$$\sin 2A \equiv 2\sin A\cos A$$

Addition or compound angle formulae

$$\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A - B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$$

* * * *

From the Addition or compound angle formulae

$$2\sin A\cos B \equiv \sin(A - B) + \sin(A + B)$$

$$2\cos A\cos B \equiv \cos(A - B) + \cos(A + B)$$

$$2\sin A\sin B \equiv \cos(A - B) - \cos(A + B)$$

$$2\sin A\cos A \equiv \sin 2A$$

$$\therefore \qquad \sin A\cos B \equiv \tfrac{1}{2}\left(\sin(A - B) + \sin(A + B)\right)$$

$$\therefore \qquad \cos A\cos B \equiv \tfrac{1}{2}\left(\cos(A - B) + \cos(A + B)\right)$$

$$\therefore \qquad \sin A\sin B \equiv \tfrac{1}{2}\left(\cos(A - B) - \cos(A + B)\right)$$

$$\therefore \qquad \sin A\cos A \equiv \tfrac{1}{2}\sin 2A$$

Factor formulae

$$\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

* * * *

57.6.1 Example:

1

$$\int cos^2 3x\, dx$$

$$cos^2 A \equiv \tfrac{1}{2}(1 + cos\, 2A) \qquad \text{[Double angle]}$$

$$\int cos^2 3x\, dx = \tfrac{1}{2}\int (1 + cos\, 6x)\, dx$$

$$= \tfrac{1}{2}(x + \tfrac{1}{6} sin\, 6x) + c$$

2

$$\int sin\, 3x\, cos\, 3x\, dx$$

$$2\, sin\, A\, cos\, B \equiv sin\,(A - B) + sin\,(A + B) \qquad \text{[Compound angle]}$$

$$\int sin\, 3x\, cos\, 3x\, dx = \tfrac{1}{2}\int sin\,(3x - 3x) + sin\,(3x + 3x)\, dx$$

$$= \tfrac{1}{2}\int sin\,(6x)\, dx$$

$$= \frac{1}{2}\left(-\frac{1}{6} cos\, 6x\right) + c$$

$$= -\frac{1}{12} cos\, 6x + c$$

3

$$\int_0^{4\pi} sin^2\left(\tfrac{1}{2}x\right)\, dx$$

$$\int_0^{4\pi} sin^2\left(\tfrac{1}{2}x\right)\, dx = \tfrac{1}{2}\int_0^{4\pi} (1 - cos\, \tfrac{2x}{2})\, dx \qquad \text{[Double angle]}$$

$$= \tfrac{1}{2}\left[x - sin\, x\right]_0^{4\pi}$$

$$= \tfrac{1}{2}\left[(4\pi - sin\, 4\pi) - (0 - sin\, 0)\right]$$

$$= \tfrac{1}{2}(4\pi - 0)$$

$$= 2\pi$$

4

57.7 Integrals of Type: $\cos A \cos B$, $\sin A \cos B$ & $\sin A \sin B$

This type of problem covers the most common questions. Use the addition (compound angle) trig identities.

57.7.1 Example:

1 Integrate $\sin 3x \cos 4x$

Use formula: $2 \sin A \cos B \equiv \sin(A - B) + \sin(A + B)$

Let: $A = 3x,\ B = 4x$

$$\therefore \quad 2 \sin 3x \cos 4x = \sin(3x - 4x) + \sin(3x + 4x)$$

$$= \sin(-x) + \sin 7x$$

$$\therefore \quad \sin 3x \cos 4x = \tfrac{1}{2}\left(\sin(-x) + \sin 7x\right)$$

$$\int \sin 3x \cos 4x \, dx = \int \tfrac{1}{2}\left(\sin(-x) + \sin 7x\right) dx$$

$$= \tfrac{1}{2}\left(\cos x - \tfrac{1}{7}\cos 7x\right) + c$$

$$= \frac{1}{2}\cos x - \frac{1}{14}\cos 7x + c$$

2 Integrate $\sin 4x \cos 4x$

Use formula: $\sin A \cos A \equiv \tfrac{1}{2}\left(\sin(2A)\right)$

Let: $A = 4x$

$$\int \sin 4x \cos 4x = \tfrac{1}{2}\int \sin 8x \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x\right] + c = -\frac{1}{16}\cos 8x$$

57.8 Integrating EVEN powers of $\sin x$ & $\cos x$

For this we need to adapt the double angle cosine identities:

$$\cos 2A \equiv 2\cos^2 A - 1$$

$$\cos 2A \equiv 1 - 2\sin^2 A$$

$$\therefore \quad \cos^2 A \equiv \tfrac{1}{2}(1 + \cos 2A)$$

$$\therefore \quad \sin^2 A \equiv \tfrac{1}{2}(1 - \cos 2A)$$

This technique can be used for any even power of $\sin x$ or $\cos x$, and also $\sin^2(ax + b)$ etc.

57.8.1 Example:

1

Find: $\int \sin^2 x \, dx$

Recognise: $\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$

$$\therefore \quad \int \sin^2 x \, dx = \tfrac{1}{2}\int (1 - \cos 2x)\, dx$$

$$= \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) + c$$

2

Find: $\int \cos^2 x \, dx$

Recognise: $\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$

$$\int \cos^2 x \, dx = \tfrac{1}{2}\int (1 + \cos 2x)\, dx$$

$$= \tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right) + c$$

3

Find: $\int_0^{\frac{\pi}{4}} \sin^2 2x \, dx$

now $\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$ Let $A = 2x$, $\therefore \sin^2 2x = \tfrac{1}{2}(1 - \cos 4x)$

$$\therefore \quad \int_0^{\frac{\pi}{4}} \sin^2 2x \, dx = \tfrac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \cos 4x)\, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \sin\frac{4\pi}{4}\right) - (0 - 0)\right]$$

$$= \frac{\pi}{8}$$

4

Find: $\int \cos^4 x\,dx$

$$\int \cos^4 x\,dx = \int \cos^2 x \cos^2 x\,dx$$
$$= \int \frac{1}{2}(1 + \cos 2x) \times \frac{1}{2}(1 + \cos 2x)\,dx$$
$$= \frac{1}{4}\int (1 + \cos 2x)(1 + \cos 2x)\,dx$$
$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$
$$= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\,dx$$
$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right] + c$$
$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

5

Find: $\int \sin^2(2x + 3)\,dx$

Recognise: $\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$

$$\int \sin^2(2x + 3)\,dx = \tfrac{1}{2}\int (1 - \cos 2(2x + 3))\,dx$$
$$= \tfrac{1}{2}\int (1 - \cos(4x + 6))\,dx$$

Recall: $\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + c$

$$\int \sin^2(2x + 3)\,dx = \tfrac{1}{2}\left[x - \frac{1}{4}\cos(4x + 6)\right] + c$$

57.9 Integrals of Type: $\cos^n A \sin A$, $\sin^n A \cos A$

Another example of applying the reverse of the differentiation and the chain rule:

From the chain rule, the derivative required is

$$\frac{d}{dx}(\sin^n x) = n \sin^{n-1} x \cos x$$

In reverse

$$\int \sin^n x \cos x \, dx = \frac{1}{n+1} \sin^{n+1} + c$$

Similarly:

$$\int \cos^n x \sin x \, dx = -\frac{1}{n+1} \cos^{n+1} + c$$

57.9.1 Example:

1

$$\int \sin^4 x \cos x \, dx = \frac{1}{5} \sin^5 x + c$$

2

$$\int \cos^7 x \sin x \, dx = -\frac{1}{8} \cos^8 x + c$$

3 Three ways of integrating $\sin x \cos x$:

$$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + c \qquad \text{See } \sin^n x \cos x \text{ above}$$

$$= -\frac{1}{2} \cos^2 x + c \qquad \text{See } \cos^n x \sin x \text{ above}$$

$$= \int \frac{1}{2} \sin 2x \, dx \qquad \text{See addition formula}$$

$$= -\frac{1}{4} \cos 2x + c$$

57.10 Integrating ODD powers of $\sin x$ & $\cos x$

This technique is entirely different - change all but one of the *sin*/*cos* functions to the opposite by using the pythag identity:

$$\cos^2 x + \sin^2 x = 1$$

Hence:

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

57.10.1 Example:

1

Find: $\int \sin^3 x\, dx$

$$\int \sin^3 x\, dx = \int \sin x \sin^2 x\ dx$$

$$\int \sin x \sin^2 x\ dx = \int \sin x\ (1 - \cos^2 x)\, dx$$

$$= \int (\sin x - \cos^2 x \sin x)\ dx$$

Recognise standard type $\int \cos^n x \sin x\, dx$ [from previous section]

$$= -\cos x + \frac{1}{3}\cos^3 x + c$$

2

Find: $\int \sin^5 x\, dx$

$$\int (\sin x \sin^2 x \sin^2 x)\ dx = \int \sin x (1 - \cos^2 x)(1 - \cos^2 x)\ dx$$

$$= \int \sin x\ (1 - 2\cos^2 x + \cos^4 x)\, dx$$

$$= \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x)\ dx$$

Recognise standard type $\int \cos^n x \sin x\ dx$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

57.11 Integrals of Type: sec x, cosec x & cot x

From the standard derivative of these functions, the integral can be found by reversing the process. Thus:

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x \, dx = \sec x + c$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \quad \Rightarrow \quad \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \quad \Rightarrow \quad \int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

57.11.1 Example:

1

$$\int \frac{\cos 3x}{\sin^2 3x}$$

Rewrite integral as: $\displaystyle\int \frac{\cos 3x}{\sin^2 3x} = \int \frac{1}{\sin 3x} \times \frac{\cos 3x}{\sin 3x}\, dx$

$$= \int \operatorname{cosec} 3x \cot 3x \, dx$$

$$= -\frac{1}{3}\operatorname{cosec} 3x + c$$

2

$$\int \cot^2 x \, dx$$

Recognise identity: $1 + \cot^2 x = \operatorname{cosec}^2 x$

$$\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\cot x - x + c$$

57.12 Integrals of Type: $sec^n x \tan x$, $tan^n x \sec^2 x$

From the standard derivative of these functions, the integral can be found by reversing the process. Thus:

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x \, dx = \sec x + c$$

and

$$\frac{d}{dx}(\sec^n x) = n \sec^{n-1} x (\sec^n \tan x)$$

$$= n \sec^n x \tan x$$

Reversing the derivative gives $\Rightarrow \quad \int \sec^n x \tan x \, dx = \frac{1}{n} \sec^n x + c$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x \, dx = \tan x + c$$

and

$$\frac{d}{dx}(\tan^{n+1} x) = (n + 1) \tan^n x \sec^2 x$$

Reversing the derivative gives $\Rightarrow \quad \int \tan^n x \sec^2 x \, dx = \frac{1}{n+1} \tan^{n+1} x + c$

57.12.1 Example:

1 Find: $\int \tan^2 x \sec^2 x \, dx$

$$\int \tan^2 x \sec^2 x \, dx = \frac{1}{3} \tan^3 x + c$$

2 Find: $\int \tan^2 x \, dx$

$$\int \tan^2 x \, dx = \int (\sec^2 - 1) \, dx$$

$$= \tan x - x + c$$

3 Find: $\int \tan^3 x \, dx$

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$

$$= \int \tan x \, (\sec^2 x - 1) \, dx$$

$$= \int (\tan x \sec^2 x - \tan x) \, dx$$

$$= \tfrac{1}{2} \tan^2 x + \ln(\cos x) + c$$

4 Alternatively

$$\int \tan^3 x \, dx = \int (\tan x \sec^2 x - \tan x) \, dx$$

$$= \tfrac{1}{2} \sec^2 x + \ln(\cos x) + c$$

57.13 Standard Trig Integrals (radians only)

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x \, dx = \sin x + c$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad \Rightarrow \quad \int \sin x \, dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x \, dx = \tan x + c$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x \, dx = \sec x + c$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \quad \Rightarrow \quad \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \quad \Rightarrow \quad \int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

$$\frac{d}{dx}(\sin(ax + b)) = \cos(ax + b) \quad \Rightarrow \quad \int \cos(ax + b) \, dx = \frac{1}{a}\sin(ax + b) + c$$

$$\frac{d}{dx}(\cos(ax + b)) = -\sin(ax + b) \quad \Rightarrow \quad \int \sin(ax + b) \, dx = -\frac{1}{a}\cos(ax + b) + c$$

$$\frac{d}{dx}(\tan(ax + b)) = \sec^2(ax + b) \quad \Rightarrow \quad \int \sec^2(ax + b) \, dx = \frac{1}{a}\tan(ax + b) + c$$

$$\frac{d}{dx}(\sin f(x)) = f'(x)\cos f(x) \quad \Rightarrow \quad \int f'(x)\cos f(x) \, dx = \sin f(x) + c$$

$$\frac{d}{dx}(\cos f(x)) = -f'(x)\sin f(x) \quad \Rightarrow \quad \int f'(x)\sin f(x) \, dx = -\cos f(x) + c$$

$$\frac{d}{dx}(\tan f(x)) = f'(x)\sec^2 f(x) \quad \Rightarrow \quad \int f'(x)\sec^2 f(x) \, dx = \tan f(x) + c$$

$$\int \tan x \, dx = -\ln|\cos x| + c = \ln|\sec x| + c$$

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln|\sin x| + c$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$

$$\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + c$$

$$\int \sin^n x \cos x \, dx = \frac{1}{n + 1}\sin^{n+1} x + c$$

$$\int \cos^n x \sin x \, dx = -\frac{1}{n + 1}\cos^{n+1} x + c$$

$$\int \sec^n x \tan x \, dx = \frac{1}{n}\sec^n x + c$$

$$\int \tan^n x \sec^2 x \, dx = \frac{1}{n + 1}\tan^{n+1} x + c$$

58 • C4 • Integration by Inspection

OCR C4 / AQA C3

58.1 Intro to Integration by Inspection

This covers two forms of integration which involve a function combined with its differential, either as a product or a quotient. These include:

- Integrals of the form $\int \frac{k f'(x)}{f(x)} dx$
- Integrals of the form $\int k f'(x) [f(x)]^n dx$
- Integrals of the form $\int k f'(x) e^{f(x)} dx$

Integration of these types is often called 'integration by inspection' or 'integration by recognition', because once proficient in using this method, you should be able to just write down the answer by 'inspecting' the function.

It is derived from reversing the 'function of a function' rule for differentiation, i.e. the chain rule.

The key to using this method is recognising that one part of the integrand is the differential (or scalar multiple) of the other part.

There are several methods of integrating fractions and products, depending of the form of the original function, and recognition of this form will save a good deal of calculations. A common alternative to this method is 'integration by substitution'.

58.2 Method of Integration by Inspection

The basic method for any of these types is the same:

- Guess — at a suitable integral by inspecting the function
- Test — your guess by differentiating
- Reverse — if $\frac{d}{dx}(guess) = z$, then $\int z\, dx = (guess) + c$, since differentiation & integration are inverse processes
- Adapt — compare your $\int z\, dx$ with original question and adapt the answer accordingly.
  Note that any adjustment must be a number only, not a function of x.
  (This step not required if $f'(x)$ is the exact differential of $f(x)$)

58.3 Integration by Inspection — Quotients

Integrals of the form $\int \frac{k f'(x)}{f(x)} dx$ are basically fractions with a function in the denominator and a multiple of its differential in the numerator, assuming the function is rational.

E.g.

$$\int \frac{4x}{x^2 + 1} dx \Rightarrow \frac{2 \times \text{differential of the denominator}}{\text{a function with a differential of } 2x}$$

$$\int \frac{4 \sin x}{\cos x + 1} dx \Rightarrow \frac{-4 \times \text{differential of the denominator}}{\text{a function with a differential of } -\sin x}$$

From C3 work, using the chain rule, recall that:

If $\quad y = \ln x \quad$ then $\quad \frac{dy}{dx} = \frac{1}{x}$

and if $\quad y = \ln f(x) \quad$ then $\quad \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x)$

Reversing the differential by integrating we get:

$$\int \frac{k f'(x)}{f(x)} dx \Rightarrow k \ln|f(x)| + c$$

Note that the modulus sign indicates that you cannot take the natural log of a negative number.

Following our method, our first guess should, therefore, be: $(guess) = \ln|denominator|$.

Note that the numerator has to be an exact derivative of the denominator and not just a derivative of a function inside the denominator.

E.g.

$$\int \frac{x}{\sqrt{x+2}} dx \neq \ln\left|\sqrt{x+2}\right| + c$$

In this case use substitution to evaluate the integral.

Recall the following standard integrals and differential:

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + c$$

$$\int \frac{k}{ax+b} dx = \frac{k}{a} \ln|ax+b| + c$$

$$\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n f'(x)\left[f(x)\right]^{n-1} \qquad \text{[chain rule]}$$

58.3.3 Example:

1

$$\int \frac{x^2}{1+x^3} dx$$

Guess: $\ln|1+x^3|$

Test: $\frac{d}{dx}\left[\ln|1+x^3|\right] = \frac{1}{1+x^3} \times 3x^2 = \frac{3x^2}{1+x^3}$

Reverse: $\int \frac{3x^2}{1+x^3} dx = \ln|1+x^3| + c$

Adapt: $\int \frac{x^2}{1+x^3} dx = \frac{1}{3} \ln|1+x^3| + c$

Note: Adjustment has to be a number only.

2

$$\int \frac{2e^x}{e^x+4} dx$$

Guess: $\ln|e^x+4|$

Test: $\frac{d}{dx}\left[\ln|e^x+4|\right] = \frac{1}{e^x+4} \times e^x = \frac{e^x}{e^x+4}$

Reverse: $\int \frac{e^x}{e^x+4} dx = \ln|e^x+4| + c$

Adapt: $\int \frac{2e^x}{e^x+4} dx = 2\ln|e^x+4| + c$

$= \ln(e^x+4)^2 + c$ — Squared term is +ve

3

$$\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$

Guess: $\ln|\sin x + \cos x|$

Test: $\frac{d}{dx}\left[\ln|\sin x + \cos x|\right] = \frac{1}{\sin x + \cos x} \times (\cos x - \sin x) = \frac{\cos x - \sin x}{\sin x + \cos x}$

Reverse: $\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \ln|\sin x + \cos x| + c$

Adapt: Not required because the numerator is the exact differential of the denominator.

4

$$\int \frac{2x}{x^2 + 9}\,dx$$

Of the form $\int \frac{f'(x)}{f(x)}\,dx$

$$\therefore \quad \int \frac{2x}{x^2 + 9}\,dx = \ln|x^2 + 9| + c$$

$$= \ln(x^2 + 9) + c$$

Note: for all real values of x, $(x^2 + 9) > 0$, hence modulus sign not required.

5

$\int \tan x\,dx$ Often comes up in the exam!

Think $\tan x = \frac{\sin x}{\cos x}$ and $\frac{d}{dx}(\cos x) = -\sin x$

Guess: $\ln|\cos x|$

Test: $\frac{d}{dx}\left[\ln|\cos x|\right] = \frac{1}{\cos x} \times (-\sin x) = \frac{-\sin x}{\cos x}$

Reverse: $\int \frac{-\sin x}{\cos x}\,dx = \ln|\cos x| + c$

Adapt: $\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$

$$\therefore \quad \int \tan x\,dx = -\ln|\cos x| + c$$

$$= \ln|\cos x|^{-1} + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c$$

6

$\int \cot 2x\,dx$

Think $\cot 2x = \frac{1}{\tan 2x} = \frac{\cos 2x}{\sin 2x}$ and $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

Guess: $\ln|\sin 2x|$

Test: $\frac{d}{dx}\left[\ln|\sin 2x|\right] = \frac{1}{\sin 2x} \times (2\cos 2x) = \frac{2\cos 2x}{\sin 2x}$

Reverse: $\int \frac{2\cos 2x}{\sin 2x}\,dx = \ln|\sin 2x| + c$

Adapt: $\int \frac{\cos 2x}{\sin 2x}\,dx = \frac{1}{2}\ln|\sin 2x| + c$

$$\int \cot 2x\,dx = \frac{1}{2}\ln|\sin 2x| + c$$

7 $\displaystyle\int \frac{x^3}{x^4+9}dx$

Guess: $ln\left|x^4+9\right|$

Test: $\displaystyle\frac{d}{dx}ln\left|x^4+9\right| = \frac{1}{x^4+9}\times 4x^3 = \frac{4x^3}{x^4+9}$

Reverse: $\displaystyle\int \frac{4x^3}{x^4+9}\,dx = ln\left|x^4+9\right| + c$

Adapt: $\displaystyle\int \frac{x^3}{x^4+9}\,dx = \frac{1}{4}ln\left|x^4+9\right| + c$

$\displaystyle = \frac{1}{4}\,ln\left(x^4+9\right) + c$ $\qquad$ x term is +ve

58.4 Integration by Inspection — Products

Integrals of the form $\int k\,f'(x)\,(f(x))^n\,dx$ and $\int k\,f'(x)\,e^{f(x)}\,dx$ involves a function raised to a power or e raised to the power of the function, multiplied by a multiple of the differential of $f(x)$. Note that many of these examples can also be solved by other methods like substitution.

E.g.

$\int x\left(x^2+1\right)^2 dx \qquad f(x) = x^2+1 \quad\Rightarrow\quad f'(x) = 2x$

$\int x^2\left(3x^3+1\right)^4 dx \qquad f(x) = 3x^3+1 \quad\Rightarrow\quad f'(x) = 9x^2$

$\int x\,e^{x^2} \qquad f(x) = x^2 \quad\Rightarrow\quad f'(x) = 2x$

$\int 3x^4\,e^{x^5+6}dx \qquad f(x) = x^5+6 \quad\Rightarrow\quad f'(x) = 5x^4$

Some quotients have to be treated as a product:

$\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \int x\left(x^2+1\right)^{-\frac{1}{2}}\,dx \quad : f(x) = x^2+1 \quad\Rightarrow\quad f'(x) = 2x$

From earlier work with the chain rule, recall that:

If $y = \left[f(x)\right]^n$ then $\displaystyle\frac{dy}{dx} = n\,f'(x)\left[f(x)\right]^{n-1}$

If $y = e^{f(x)}$ then $\displaystyle\frac{dy}{dx} = f'(x)\,e^{f(x)}$

Reversing the differentials by integrating we get:

$$\int f'(x)\left[f(x)\right]^n dx \Rightarrow \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

$$\int f'(x)\,e^{f(x)}\,dx \Rightarrow e^{f(x)} + c$$

58.4.2 Example:

1 $\int x\,(x^2+1)^2\,dx$

Guess: $(x^2+1)^{2+1} \Rightarrow (x^2+1)^3$

Test: $\frac{d}{dx}\left[(x^2+1)^3\right] = 3(x^2+1)^2 \times 2x = 6x(x^2+1)^2$

Reverse: $\int 6x(x^2+1)^2\,dx = (x^2+1)^3 + c$

Adapt: $\int x\,(x^2+1)^2\,dx = \frac{1}{6}(x^2+1)^3 + c$

2 $\int \cos x \sin^3 x\,dx \Rightarrow \int \cos x\,(\sin x)^3\,dx$

Guess: $(\sin x)^4$

Test: $\frac{d}{dx}\left[(\sin x)^4\right] = 4(\sin x)^3 \times \cos x = 4\cos x\,(\sin x)^3$

Reverse: $\int 4\cos x\,(\sin x)^3\,dx = (\sin x)^4 + c$

Adapt: $\int \cos x \sin^3 x\,dx = \frac{1}{4}\sin^4 x + c$

3 $\int x^2\sqrt{(x^3+5)}\,dx \Rightarrow \int x^2(x^3+5)^{\frac{1}{2}}\,dx$

Guess: $(x^3+5)^{\frac{3}{2}}$

Test: $\frac{d}{dx}\left[(x^3+5)^{\frac{3}{2}}\right] = \frac{3}{2}(x^3+5)^{\frac{1}{2}} \times 3x^2 = \frac{9}{2}x^2(x^3+5)^{\frac{1}{2}}$

Reverse: $\int \frac{9}{2}x^2(x^3+5)^{\frac{1}{2}}\,dx = (x^3+5)^{\frac{3}{2}} + c$

Adapt: $\int x^2(x^3+5)^{\frac{1}{2}}\,dx = \frac{2}{9}(x^3+5)^{\frac{3}{2}} + c$

4 $\int x\,e^{x^2}dx$

Guess: e^{x^2}

Test: $\frac{d}{dx}\left[e^{x^2}\right] = e^{x^2} \times 2x = 2x\,e^{x^2}$

Reverse: $\int 2x\,e^{x^2}dx = e^{x^2} + c$

Adapt: $\int x\,e^{x^2}dx = \frac{1}{2}e^{x^2} + c$

5 $\int \cos x\,e^{\sin x}\,dx$

Guess: $e^{\sin x}$

Test: $\frac{d}{dx}\left[e^{\sin x}\right] = e^{\sin x} \times \cos x = \cos x\,e^{\sin x}$

Reverse: $\int \cos x\,e^{\sin x} = e^{\sin x} + c$

Adapt: not required

6

$$\int \frac{x}{(3x^2 - 4)^5}\,dx \;\Rightarrow\; \int x\left(3x^2 - 4\right)^{-5} dx$$

Guess: $\left(3x^2 - 4\right)^{-4}$

Test: $\frac{d}{dx}\left[\left(3x^2 - 4\right)^{-4}\right] = -4 \times 6x\left(3x^2 - 4\right)^{-5} = -24x\left(3x^2 - 4\right)^{-5}$

Reverse: $\int -24x\left(3x^2 - 4\right)^{-5} dx = \left(3x^2 - 4\right)^{-4} + c$

Adapt: $\int x\left(3x^2 - 4\right)^{-5} dx = -\frac{1}{24}\left(3x^2 - 4\right)^{-4} + c$

7

After a while it becomes easier to write the answer down, but always check the possible answer by differentiating.

$$\int e^x\left(6e^x - 5\right)^2 dx$$

Note: $f(x) = 6e^x - 5 \quad\Rightarrow\quad f'(x) = 6e^x$

Adapt: $\int e^x\left(6e^x - 5\right)^2 dx = \frac{1}{6}\int 6e^x\left(6e^x - 5\right)^2 dx$

Inspect: $\frac{1}{6}\int 6e^x\left(6e^x - 5\right)^2 dx = \frac{1}{6} \times \frac{1}{3}\left(6e^x - 5\right)^3$

$= \frac{1}{18}\left(6e^x - 5\right)^3$

Test: $\frac{d}{dx}\left[\frac{1}{18}\left(6e^x - 5\right)^3\right] = \frac{1}{18} \times 6e^x \times 3\left(6e^x - 5\right)^2 = e^x\left(6e^x - 5\right)^2$

58.5 Integration by Inspection Digest

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{1}{f(x)} \times \left(f'(x)\right)$$

$$\int \frac{k f'(x)}{f(x)} dx = k \ln\left|f(x)\right| + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int \sin^n x \cos x\,dx = \frac{1}{n+1}\sin^{n+1} + c$$

$$\frac{d}{dx}\left[\left(f(x)\right)^n\right] = n f'(x)\left[f(x)\right]^{n-1}$$

$$\int f'(x)\left[f(x)\right]^n dx \Rightarrow \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

$$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)\,e^{f(x)}$$

$$\int e^x\,dx = e^x + c$$

$$\int f'(x)\,e^{f(x)}\,dx \Rightarrow e^{f(x)} + c$$

59 • C4 • Integration by Parts

OCR C4 / AQA C3

This is the equivalent of the product rule for integration. It is usually used when the product we want to integrate is not of the form $f'(x)(f(x))^n$ and so cannot be integrated with this standard method, or by recognition or by substitution.

Integrating by Parts is particularly useful for integrating the product of two types of function, such as a polynomial with a trig, exponential or log function, (e.g. $x \sin x$, $x^2 e^x$, $\ln x$).

59.1 Rearranging the Product rule:

The rule for Integrating by Parts comes from integrating the product rule.

Product rule: $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating w.r.t x to get: $$\int \frac{d}{dx}(uv)\, dx = \int u\frac{dv}{dx}\, dx + \int v\frac{du}{dx}\, dx$$

$$uv = \int u\frac{dv}{dx}\, dx + \int v\frac{du}{dx}\, d$$

Rearranging: $$\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$$

$$\int u\,\frac{dv}{dx}\, dx = uv - \int v\,\frac{du}{dx}\, dx$$

59.2 Choice of u & dv/dx

Care must be taken over the choice of u *and* dv/dx.

The aim is to ensure that it is simpler to integrate $v\frac{du}{dx}$ than the original $u\frac{dv}{dx}$. So we choose u to be easy to differentiate and when differentiated to become simpler. Choose dv to be easy to integrate.

Normally, u is assigned to any polynomial in x, and if any exponential function is involved, assign this to $\frac{dv}{dx}$. However, if $\ln x$ is involved make this u, as it is easier to differentiate the $\ln$ function than to integrate it.

59.3 Method

- Let u = the bit of the product which will differentiate to a constant, even if it takes 2 or 3 turns, such as polynomials in x, (e.g. x^3 differentiates to $3x^2 \rightarrow 6x \rightarrow 6$)
- If this is not possible or there is a difficult part to integrate let this be u. e.g. $\ln x$.
- Differentiate to find $\frac{du}{dx}$.
- Let the other part of the product be $\frac{dv}{dx}$, like e^{ax} which is easy to integrate.
- Integrate to find v.
- Substitute into the rule and finish off.
- Add the constant of integration at the end.
- Sometimes integrating by parts needs to be applied more than once (see special examples). Do not confuse the use of u in the second round of integration.
- This is the method used to integrate $\ln x$.

59.4 Evaluating the Definite Integral by Parts

Use this for substituting the limits:

$$\int_a^b u\frac{dv}{dx}\,dx = \Big[uv\Big]_a^b - \int_a^b v\frac{du}{dx}dx$$

59.5 Handling the Constant of Integration

The method listed above suggests adding the constant of integration at the end of the calculation. Why is this?

The best way to explain this is to show an example of adding a constant after each integration, and you can see that the first one cancels out during the calculation.

Example : Find: $\int x \sin x \, dx$

Solution:

Let: $u = x$ & $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1$ $\qquad v = \int \frac{dv}{dx} = -\cos x + k$

where k is the constant from the first integration and c is the constant from the second integrations.

Recall: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$$\int x \cos x \, dx = x(-\cos x + k) - \int(-\cos x + k) \times 1 \, dx$$

$$= -x\cos x + kx + \int \cos x \, dx - \int k \, dx$$

$$= -x\cos x + kx + \sin x - kx + c$$

$$= -x\cos x + \sin x + c$$

59.6 Integration by Parts: Worked examples

59.6.1 Example:

1 Find: $\int x \cos x \, dx$

Solution:

Let: $u = x$ & $\frac{dv}{dx} = \cos x$ Note: $u = x$ becomes simpler when differentiated.

$\frac{du}{dx} = 1$ $v = \int \frac{dv}{dx} = \int \cos x = \sin x$

$$\int x \cos x \, dx = x \sin x - \int \sin x \times 1 \, dx$$

$$= x \sin x + \cos x + c$$

Alternative (longer) Solution:

Let: $u = \cos x$ & $\frac{dv}{dx} = x$

$\frac{du}{dx} = -\sin x$ $v = \frac{x^2}{2}$

$$\int x \cos x \, dx = \sin x \, . \frac{x^2}{2} - \int \frac{x^2}{2} (-\sin x) \, dx$$

$$= \frac{x^2}{2} \sin x + \int \frac{x^2}{2} \sin x \, dx \qquad etc\ etc$$

As you can see, this gives a more involved solution, that has to have another round of integration by parts. This emphasises the importance of choosing u wisely. In this case it would be prudent to start again with $u = x$.

2 Find: $\int x \sec^2 x \, dx$

Solution:

Let: $u = x$ & $\frac{dv}{dx} = \sec^2 x$

$\frac{du}{dx} = 1$ $v = \int \frac{dv}{dx} = \tan x$ Standard tables

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \times 1 \, dx$$

$$= x \sin x + \ln(\cos x) + c$$ Standard tables

3 Find: $\int (4x + 2) \sin 4x \, dx$

Solution:

Let: $u = 4x + 2$ & $\dfrac{dv}{dx} = \sin 4x$

$\dfrac{du}{dx} = 4$ $v = -\dfrac{1}{4}\cos 4x$

$$\int (4x + 2) \sin 4x \, dx = (4x + 2)(-\frac{1}{4}\cos 4x) - \int -\frac{1}{4}\cos 4x \, . \, 4 \, dx$$

$$= -\frac{1}{4}(4x + 2) \cos 4x + \int \cos 4x \; dx$$

$$= -\frac{1}{4}(4x + 2) \cos 4x + \frac{1}{4}\sin 4x + c$$

4 Find: $\int x^2 \sin x \, dx$

Solution:

Let: $u = x^2$ & $\dfrac{dv}{dx} = \sin x$

$\dfrac{du}{dx} = 2x$ $v = \displaystyle\int \frac{dv}{dx} = -\cos x$

$$\int x^2 \sin x \, dx = x^2(-\cos x) - \int -\cos x \, . \, 2x \, dx$$

$$= -x^2 \cos x + \int 2x \cos x \; dx$$

Now integrate by parts again and then one final integration to give…

Now let $u = 2x$ & $\dfrac{dv}{dx} = \cos x$

$\dfrac{du}{dx} = 2$ $v = \sin x$

$$\int x^2 \sin x \, dx = -x^2 \cos x + \left[2x \sin x - \int \sin x \times 2 \; dx\right]$$

$$= -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x - 2\,(-\cos x) + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

$$= 2 \cos x - x^2 \cos x + 2x \sin x + c$$

$$= (2 - x^2) \cos x + 2x \sin x + c$$

Note: Integrating any function of the form $x^n \sin x$ or $x^n \cos x$, will require n rounds of integration by parts.

5

Find: $\int_0^{\pi} x^2 \cos x \, dx$

Solution:

Let: $u = x^2$ & $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2x$ $v = \int \frac{dv}{dx} = \sin x$

$$\int_0^{\pi} x^2 \cos x \, dx = \left[x^2 \sin x\right]_0^{\pi} - \int_0^{\pi} \sin x \, . \, 2x \, dx$$

$$= [0 - 0] - \int_0^{\pi} 2x \sin x \, dx$$

Now integrate by parts again, and then one final integration to give…

Now let: $u = 2x$ & $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2$ $v = -\cos x$

$$\int_0^{\pi} x^2 \cos x \, dx = 0 - \left\{\left[2x\,(-\cos x)\right]_0^{\pi} - \int_0^{\pi} -\cos x \, . \, 2 \, dx\right\}$$

$$= 0 - \left\{\left[-2x \cos x\right]_0^{\pi} + \int_0^{\pi} 2 \cos x \, dx\right\}$$

$$= -\left\{[2\pi - 0] + \int_0^{\pi} 2 \cos x \, dx\right\}$$

$$= -2\pi - \int_0^{\pi} 2 \cos x \, dx$$

$$= -2\pi - \left[2 \sin x\right]_0^{\pi}$$

$$= -2\pi - [0 - 0] = -2\pi$$

6

Find: $\int 2x \sin(3x - 1) \, dx$

Solution:

Let: $u = 2x$ & $\frac{dv}{dx} = \sin(3x - 1)$

$\frac{du}{dx} = 2$ $v = -\frac{1}{3}\cos(3x - 1)$

$$\int 2x \sin(3x - 1) \, dx = 2x\left(-\frac{1}{3}\cos(3x - 1)\right) - \int -\frac{1}{3}\cos(3x - 1) \, . \, 2 \, dx$$

$$= -\frac{2}{3} x \cos(3x - 1) + \frac{2}{3}\int \cos(3x - 1) \, dx$$

$$= -\frac{2}{3} x \cos(3x - 1) + \frac{2}{3} \times \frac{1}{3} \sin(3x - 1) + c$$

$$= -\frac{2}{3} x \cos(3x - 1) + \frac{2}{9} \sin(3x - 1) + c$$

$$= \frac{2}{9} \sin(3x - 1) - \frac{2}{3} x \cos(3x - 1) + c$$

7

Solve by parts $\int x(2x+3)^5\,dx$

This can be solved by inspection, but is included here for completeness.

Solution:

Let: $u = x \qquad \frac{dv}{dx} = (2x+3)^5$

$\frac{du}{dx} = 1$

Recall: $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$

$$\therefore v = \int (2x+3)^5\,dx = \frac{1}{2(6)}(2x+3)^6 + c$$

$$= \frac{1}{12}(2x+3)^6 + c$$

Recall: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$$\int x(2x+3)^5\,dx = x.\frac{1}{12}(2x+3)^6 - \int \frac{1}{12}(2x+3)^6 \times 1\,dx$$

$$= \frac{x}{12}(2x+3)^6 - \frac{1}{12}\int (2x+3)^6\,dx$$

$$= \frac{x}{12}(2x+3)^6 - \frac{1}{12} \times \frac{1}{2 \times 7}(2x+3)^7 + c$$

$$= \frac{x}{12}(2x+3)^6 - \frac{1}{12} \times \frac{1}{14}(2x+3)^7 + c$$

$$= \frac{1}{12}(2x+3)^6\left[x - \frac{1}{14}(2x+3)\right] + c$$

$$= \frac{1}{12}(2x+3)^6\left[\frac{14x}{14} - \frac{(2x+3)}{14}\right] + c$$

$$= \frac{1}{12}(2x+3)^6\left[\frac{14x - 2x - 3)}{14}\right] + c$$

$$= \frac{1}{12}(2x+3)^6\left[\frac{12x - 3}{14}\right] + c$$

$$= \frac{3}{12}(2x+3)^6\left[\frac{4x - 1}{14}\right] + c$$

$$= \frac{1}{56}(2x+3)^6[4x - 1] + c$$

$$= \frac{1}{56}(2x+3)^6(4x - 1) + c$$

8

Find: $\int x\, e^{3x}\, dx$

Solution:

Let: $u = x$ & $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1$ $v = \frac{1}{3}e^{3x}$

$$\int x\, e^{3x}\, dx = x \,.\, \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} \times 1\, dx$$

$$\int x\, e^{3x}\, dx = \frac{1}{3}x\, e^{3x} - \frac{1}{9}e^{3x} + c$$

$$= \frac{1}{3}\, e^{3x}\left(x - \frac{1}{3}\right) + c$$

$$= \frac{1}{9}\, e^{3x}\,(3x - 1) + c$$

9

Find: $\int x^2\, e^{4x}\, dx$

Solution:

Let: $u = x^2$ & $\frac{dv}{dx} = e^{4x}$

$\frac{du}{dx} = 2x$ $v = \frac{1}{4}e^{4x}$

$$\int x^2\, e^{4x}\, dx = x^2 \,.\, \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x} \,.\, 2x\, dx$$

$$= \frac{1}{4}x^2\, e^{4x} - \frac{1}{2}\int x\, e^{4x}\, dx$$

$$= \frac{1}{4}x^2\, e^{4x} - \frac{1}{2}\int u\, \frac{dv}{dx}\, dx$$

Now integrate by parts again and then one final integration to give…

Now let: $u = x$ & $\frac{dv}{dx} = e^{4x}$

$\frac{du}{dx} = 1$ $v = \frac{1}{4}e^{4x}$

$$\therefore \quad \int x\, e^{4x}\, dx = x \,.\, \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\, dx$$

$$= \frac{1}{4}\, x\, e^{4x} - \frac{1}{16}e^{4x}$$

Substituting back into the original...

$$\therefore \quad \int x^2\, e^{4x}\, dx = \frac{1}{4}x^2\, e^{4x} - \frac{1}{2}\left(\frac{1}{4}\, x\, e^{4x} - \frac{1}{16}e^{4x}\right) + c$$

$$= \frac{1}{4}x^2\, e^{4x} - \frac{1}{8}\, x\, e^{4x} + \frac{1}{32}e^{4x} + c$$

$$= e^{4x}\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right) + c$$

$$\int x^2\, e^{4x}\, dx = \frac{1}{32}\, e^{4x}\left(8x^2 - 4x + 1\right) + c$$

10 Infinite integral example. Find: $\int_0^{\infty} x\, e^{-ax}\, dx$

Solution:

Let: $u = x \quad \& \quad \frac{dv}{dx} = e^{-ax}$

$\frac{du}{dx} = 1 \qquad v = -\frac{1}{a}e^{-ax}$

$$\int_0^{\infty} x\, e^{-ax}\, dx = \left[-\frac{x}{a}e^{-ax}\right]_0^{\infty} - \int_0^{\infty} 1 \times \left(-\frac{1}{a}e^{-ax}\right) dx$$

$$= \left[-\frac{x}{a}e^{-ax}\right]_0^{\infty} + \frac{1}{a}\int_0^{\infty} e^{-ax}\, dx$$

$$= \left[-\frac{x}{a}e^{-ax} + \frac{1}{a} \times \left(-\frac{1}{a}e^{-ax}\right)\right]_0^{\infty}$$

$$= \left[-\frac{x}{a}e^{-ax} - \frac{1}{a^2}e^{-ax}\right]_0^{\infty}$$

$$= \left[-\frac{x}{ae^{ax}} - \frac{1}{a^2e^{ax}}\right]_0^{\infty}$$

As $x \to \infty, \quad \frac{x}{ae^{ax}} \to 0 \quad$ and $\quad \frac{1}{a^2e^{ax}} \to 0$

$$\therefore \int_0^{\infty} x\, e^{-ax}\, dx = [0 - 0] - \left[0 - \frac{1}{a^2}\right]$$

$$= \frac{1}{a^2}$$

Alternatively, you can evaluate the bracketed part early, thus:

$$\int_0^{\infty} x\, e^{-ax}\, dx = \left[-\frac{x}{a}\, e^{-ax}\right]_0^{\infty} + \frac{1}{a}\int_0^{\infty} e^{-ax}\, dx$$

$$= \left[-\frac{x}{ae^{ax}}\right]_0^{\infty} + \frac{1}{a}\int_0^{\infty} e^{-ax}\, dx$$

$$= [0 - 0] + \frac{1}{a}\int_0^{\infty} e^{-ax}\, dx$$

$$\int_0^{\infty} x\, e^{-ax}\, dx = \frac{1}{a}\int_0^{\infty} e^{-ax}\, dx$$

$$= \left[-\frac{1}{a^2e^{ax}}\right]_0^{\infty}$$

$$= [0] - \left[-\frac{1}{a^2}\right]$$

$$\int_0^{\infty} x\, e^{-ax}\, dx = \frac{1}{a^2}$$

59.7 Integration by Parts: $ln\ x$

So far we have found no means of integrating $ln\, x$, but now, by regarding $ln\, x$ as the product $ln\, x \times 1$, we can now apply integration by parts. In this case make $u = ln\, x$ as $ln\, x$ is hard to integrate and we know how to differentiate it.

The 'trick' of multiplying by 1 can be used elsewhere, especially for integrating inverse trig functions.

59.7.1 Example:

1 **Integrating** $ln\, x$

$\int ln\, x \times 1\, dx$ Multiply by 1 to give a product to work with.

Let: $u = ln\, x$ & $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$ $v = x$

$$\int ln\, x \times 1\, dx = x\, ln\, x - \int x \frac{1}{x}\, dx$$

$$= x\, ln\, x - \int dx$$

$$= x\, ln\, x - x + c$$

$$\int ln\, x = x\,(ln\, x - 1) + c$$

2 Find: $\int x^4\, ln\, x\, dx$

Solution:

Following the guidelines on choice of u & dv, then we would let $u = x$ and $\frac{dv}{dx} = ln\, x$

However, $ln\, x$ is difficult to integrate, so choose $u = ln\, x$

Let: $u = ln\, x$ & $\frac{dv}{dx} = x^4$

$\frac{du}{dx} = \frac{1}{x}$ $v = \frac{1}{5}x^5$

$$\int x^4\, ln\, x\, dx = ln\, x \,.\, \frac{1}{5}x^5 - \int \frac{1}{5}x^5 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{5}x^5\, ln\, x - \frac{1}{5}\int x^4\, dx$$

$$= \frac{1}{5}x^5\, ln\, x - \frac{1}{5} \times \frac{1}{5}x^5 + c$$

$$\int x^4\, ln\, x\, dx = \frac{1}{5}x^5\, ln\, x - \frac{1}{25}x^5 + c$$

$$= \frac{1}{5}x^5\left(ln\, x - \frac{1}{5}\right) + c$$

$$= \frac{1}{25}x^5\,(5\, ln\, x - 1) + c$$

3 Evaluate: $\int_2^8 x \ln x \, dx$

Solution:

As above, choose $u = \ln x$

Let: $u = \ln x$ & $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = \frac{x^2}{2}$

$$\int_2^8 x \ln x \, dx = \left[\frac{x^2}{2} \ln x\right]_2^8 - \int_2^8 \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \left[\frac{x^2}{2} \ln x\right]_2^8 - \int_2^8 \frac{x}{2} \, dx$$

$$= \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_2^8$$

$$= (32 \ln 8 - 16) - (2 \ln 2 - 1)$$

$$= 32 \ln 8 - 2 \ln 2 - 15$$

$$= 32 \ln 2^3 - 2 \ln 2 - 15$$

$$= 96 \ln 2 - 2 \ln 2 - 15$$

$$\int_2^8 x \ln x \, dx = 94 \ln 2 - 15$$

4 Find: $\int \sqrt{x} \ln x \, dx$

Solution:

Let: $u = \ln x$ & $\frac{dv}{dx} = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = \int \sqrt{x} = \frac{2}{3} x^{\frac{3}{2}}$

$$\int \sqrt{x} \ln x \, dx = \ln x \,.\, \frac{2}{3} x^{\frac{3}{2}} - \int \frac{2}{3} x^{\frac{3}{2}} \cdot \frac{1}{x} \, dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int x^{\frac{1}{2}} \, dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \times \frac{2}{3} x^{\frac{3}{2}} + c$$

$$= \frac{2}{9} \sqrt{x^3} \left(3 \ln x - 2\right) + c$$

59.8 Integration by Parts: Special Cases

These next examples use the integration by parts twice, which generates a term that is the same as the original integral. This term can then be moved to the LHS, to give the final result by division.

Generally used for integrals of the form $e^{ax} \sin bx$ or $e^{ax} \cos bx$. In this form, the choice of u & dv does not matter.

59.8.1 Example:

1 Find: $\int \frac{\ln x}{x}\, dx$

Solution:

Let: $u = \ln x$ & $\frac{dv}{dx} = \frac{1}{x}$

$\frac{du}{dx} = \frac{1}{x}$ $v = \ln x$

$$\int \frac{\ln x}{x}\, dx = \ln x \,.\, \ln x - \int \ln x \,.\, \frac{1}{x}\, dx$$

$$= (\ln x)^2 - \int \frac{\ln x}{x}\, dx$$

$$2\int \frac{\ln x}{x}\, dx = (\ln x)^2$$

$$\therefore \int \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln x)^2 + c$$

Note: $(\ln x)^2$ is not the same as $\ln x^2$

2 Find: $\int e^x \sin x\, dx$

Solution 1:

Let: $u = \sin x$ & $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = \cos x$ $v = e^x$

$$\int e^x \sin x\, dx = \sin x \,.\, e^x - \int e^x \cos x\, dx$$

Now integrate by parts again, which changes $\cos x$ to $\sin x$ to give…

$u = \cos x$ & $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = -\sin x$ $v = e^x$

$$\int e^x \sin x\, dx = e^x \sin x - \left[\cos x \,.\, e^x - \int e^x (-\sin x)\, dx\right]$$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x\, dx\right]$$

$$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

$$\therefore 2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + c$$

$$\therefore \int e^x \sin x\, dx = \frac{1}{2} e^x(\sin x - \cos x) + c$$

Solution 2:

$$\text{Let: } u = e^x \qquad \& \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = e^x \qquad\qquad v = -\cos x$$

$$\int e^x \sin x \, dx = e^x (-\cos x) - \int -\cos x \, . \, e^x \, dx$$

$$\int e^x \sin x \, dx = -e^x \cos x + \int \cos x \, . \, e^x \, dx$$

Now integrate by parts again to give…

$$u = e^x \qquad \& \qquad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = e^x \qquad\qquad v = \sin x$$

$$\int e^x \sin x \, dx = -e^x \cos x + \left[e^x . \sin x - \int \sin x \, . \, e^x \, dx \right]$$

$$= -e^x \sin x + e^x \cos x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + c$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

3 Find: $\int e^x \cos x \, dx$

Solution:

Let: $u = \cos x$ & $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = -\sin x$ $v = e^x$

$$\int e^x \cos x \, dx = \cos x \,.\, e^x - \int e^x \, (-\sin x) \, dx$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Now integrate by parts again and then one final integration to give…

$u = \sin x$ & $\frac{dv}{dx} = e^x$

$\therefore$ $\frac{du}{dx} = \cos x$ $v = e^x$

$$\int e^x \cos x \, dx = e^x \cos x + \left[\sin x \,.\, e^x - \int e^x \cos x \, dx \right]$$

$$\int e^x \cos x \, dx = e^x (\cos x + \sin x) - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x (\cos x + \sin x) + c$$

$$\therefore \int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x) + c$$

4 Find: $\int e^{2x} \sin 4x \, dx$

Solution:

Let: $u = \sin 4x$ & $\dfrac{dv}{dx} = e^{2x}$

$\dfrac{du}{dx} = 4 \cos 4x$ $\qquad v = \dfrac{1}{2} e^{2x}$

$$\int e^{2x} \sin 4x \, dx = \sin 4x \,.\, \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \,.\, 4 \cos 4x \, dx$$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - 2 \int e^{2x} \cos 4x \, dx$$

Now integrate by parts again and then one final integration to give…

$u = \cos 4x$ & $\dfrac{dv}{dx} = e^{2x}$

$\therefore \quad \dfrac{du}{dx} = -4 \sin 4x$ $\qquad v = \dfrac{1}{2} e^{2x}$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - 2\left[\cos 4x \,.\, \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \,.\, (-4 \sin 4x) \, dx\right]$$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - 2\left[\frac{1}{2} e^{2x} \cos 4x + 2 \int e^{2x} \sin 4x \, dx\right]$$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - e^{2x} \cos 4x - 4 \int e^{2x} \sin 4x \, dx$$

$$5 \int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - e^{2x} \cos 4x + c$$

$$= \frac{1}{2} e^{2x} \sin 4x - \frac{2}{2} e^{2x} \cos 4x + c$$

$$5 \int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} (\sin 4x - 2 \cos 4x) + c$$

$$\therefore \int e^{2x} \sin 4x \, dx = \frac{1}{10} e^{2x} (\sin 4x - 2 \cos 4x) + c$$

59.9 Integration by Parts Digest

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

$$\int_a^b u \frac{dv}{dx} \, dx = \Big[uv\Big]_a^b - \int_a^b v \frac{du}{dx} dx$$

60 • C4 • Integration by Substitution

60.1 Intro to Integration by Substitution

Also known as integration by change of variable. This is the nearest to the chain rule that integration can get. It is used to perform integrations that cannot be done by other methods, and is also an alternative method to some other methods. It is worth checking if the integration can be done by inspection, which may be simpler.

Substitution is often used to define some standard integrals.

The object is to substitute some inner part of the function by a second variable u, and change all the instances of x to be in terms of u, including dx.

The basic argument for Integration by Substitution is:

If $$y = \int f(x)\, dx$$

$$\frac{dy}{dx} = f(x)$$

From the chain rule, if u is a function of x

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = f(x)\frac{dx}{du}$$

$$\int \frac{dy}{du} du = \int f(x)\frac{dx}{du}\, du$$

$$y = \int f(x)\frac{dx}{du}\, du$$

$$\therefore \quad \int f(x)\, dx = \int f(x)\frac{dx}{du}\, du$$

60.2 Substitution Method

- Used for integrating products and quotients,
- Let u = part of the expression, usually the messy part in brackets or the denominator of a fraction,
- If necessary, express any other parts of the function in terms of u,
- Differentiate u to find $\frac{du}{dx}$,
- Re-arrange $\frac{du}{dx}$ to find dx in terms of du as we need to replace dx if we are to integrate an expression w.r.t u, i.e. we need to find $dx = (z)\, du$,
- Substitute the expressions, found above, for x and dx, back into the original integral and integrate in terms of u. It should be reasonable to integrate, or allow the use of standard integrals,
- If the integration is a definite integral, change the x limits to limits based on u,
- Put your x's back in again at the end, and finish up,
- If the substitution is not obvious, then it should be given to you in the exam,
- There is often more that one substitution that could be chosen, practise makes perfect,
- All integrals that can be done by inspection, can also be done by substitution.

60.3 Required Knowledge

From C3 module recall:

$$\int (ax + b)^n = \frac{1}{a(n + 1)}(ax + b)^{n+1} + c$$

$$\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln|ax + b| + c$$

$$\int e^{(ax+b)}\,dx = \frac{1}{a}e^{(ax+b)} + c$$

60.4 Substitution: Worked Examples

60.4.1 Example:

Examples 1 & 2 are based on the form of: $\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du$

1 Use substitution to find: $\int (5x - 3)^3\,dx$

Solution:

Let: $u = 5x - 3$

$$\frac{du}{dx} = 5 \quad \Rightarrow \quad \frac{dx}{du} = \frac{1}{5}$$

$$\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du$$

Substituting:

$$\int (5x - 3)^3\,dx = \int (u)^3\frac{1}{5}\,du \Rightarrow \frac{1}{5}\int (u)^3\,du$$

$$= \frac{1}{5}\times\frac{1}{4}u^4 + c$$

$$= \frac{1}{20}(5x - 3)^4 + c$$

2 Use substitution to find: $\int \frac{1}{4x + 2}\,dx$

Solution:

Let: $u = 4x + 2$

$$\frac{du}{dx} = 4 \quad \Rightarrow \quad \frac{dx}{du} = \frac{1}{4}$$

Substituting

$$\int \frac{1}{4x + 2}\,dx = \int \frac{1}{u}\frac{1}{4}\,du = \frac{1}{4}\int\frac{1}{u}\,du$$

$$= \frac{1}{4}\ln u + c$$

$$= \frac{1}{4}\ln(4x + 2) + c$$

This is a standard result:

$$\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln(ax + b) + c$$

The integration process can be streamlined somewhat if we find dx in terms of u and du, rather that find $\frac{dx}{du}$ specifically each time, as in the following examples.

3 Use substitution to find:

$$\int \frac{1}{x + \sqrt{x}}\, dx$$

Solution:

Let: $u = \sqrt{x}, \quad \Rightarrow \quad u = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$du = \frac{1}{2\sqrt{x}}\, dx \quad \Rightarrow \quad dx = 2\sqrt{x}\, du$

but we still have x involved, so substitute for x $\quad \therefore\ dx = 2u\, du$

Substituting into the original:

$$\int \frac{1}{x + \sqrt{x}}\, dx = \int \frac{1}{u^2 + u} 2u\, du \Rightarrow \int \frac{2u}{u(u + 1)}\, du \Rightarrow \int \frac{2}{(u + 1)}\, du$$

$$= 2\, ln|u + 1| + c$$

$$= 2\, ln|\sqrt{x} + 1| + c$$

4 Use substitution to find:

$$\int 3x\sqrt{1 + x^2}\, dx$$

Solution:

Let: $u = x^2$

$\frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x}$

Substituting:

$$\int 3x\sqrt{1 + x^2}\, dx = 3\int x(1 + u)^{\frac{1}{2}} \frac{du}{2x} \Rightarrow \frac{3}{2}\int (1 + u)^{\frac{1}{2}}\, du$$

$$= \frac{3}{2} \times \frac{2}{3}(1 + u)^{\frac{3}{2}} + c = (1 + u)^{\frac{3}{2}} + c$$

$$= (1 + x^2)^{\frac{3}{2}} + c$$

Alternative solution:

Let: $u = 1 + x^2 \quad \Rightarrow \quad x^2 = u - 1 \quad \Rightarrow \quad x = (u - 1)^{\frac{1}{2}}$

$\frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x} = \frac{du}{2(u - 1)^{\frac{1}{2}}}$

Substituting:

$$\int 3x\sqrt{1 + x^2}\, dx = 3\int (u - 1)^{\frac{1}{2}} (u)^{\frac{1}{2}} \frac{du}{2(u - 1)^{\frac{1}{2}}} \Rightarrow \frac{3}{2}\int (u)^{\frac{1}{2}}\, du$$

$$= \frac{3}{2} \times \frac{2}{3}(u)^{\frac{3}{2}} + c = (u)^{\frac{3}{2}} + c$$

$$= (1 + x^2)^{\frac{3}{2}} + c$$

5 Use substitution to find: $\int 3x\left(1 + x^2\right)^5 dx$

Solution:

Let: $u = (1 + x^2)$ $\quad \frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x}$

Substituting:

$$\int 3x\left(1 + x^2\right)^5 dx = 3\int x\,(u)^5 \frac{du}{2x} \Rightarrow \frac{3}{2}\int (u)^5\, du$$

$$= \frac{3}{2} \times \frac{1}{6}u^6 + c = \frac{1}{4}u^6 + c$$

$$= ¼\left(1 + x^2\right)^6 + c$$

6 Use substitution to find: $\int \frac{6x}{\sqrt{2x + 1}}\, dx$

Solution:

Let: $u = (2x + 1)$ $\quad \frac{du}{dx} = 2 \quad \Rightarrow \quad dx = \frac{du}{2}$

Substituting

$$\int \frac{6x}{\sqrt{2x + 1}}\, dx = \int \frac{6x}{\sqrt{u}} \frac{du}{2} \Rightarrow \int \frac{3x}{u^{\frac{1}{2}}}\, du$$

We have an x left, so go back to the substitution and find x

$$u = 2x + 1 \quad \Rightarrow \quad x = \frac{u - 1}{2}$$

$$\therefore \int \frac{3x}{u^{\frac{1}{2}}}\, du = 3\int x\,\frac{1}{u^{\frac{1}{2}}}\, du = 3\int \frac{u - 1}{2} \times \frac{1}{u^{\frac{1}{2}}} \cdot du$$

$$= \frac{3}{2}\int \frac{u - 1}{u^{\frac{1}{2}}}\, du = \frac{3}{2}\int \left[\frac{u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}}\right] du$$

$$= \frac{3}{2}\int \left[u^{+\frac{1}{2}} - u^{-\frac{1}{2}}\right] du$$

$$= \frac{3}{2}\left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right] + c$$

$$= u^{\frac{3}{2}} - 3u^{\frac{1}{2}} + c \qquad = u^{\frac{1}{2}}\left[u - 3\right] + c$$

$$= (2x + 1)^{\frac{1}{2}}(2x + 1 - 3) + c$$

$$= 2(x - 1)(2x + 1)^{\frac{1}{2}} + c$$

7 Find: $\int 2x\, e^{x^2} dx$

Solution:

Let: $u = x^2$ $\quad \frac{du}{dx} = 2x \quad \Rightarrow dx = \frac{du}{2x}$

$$\int 2x\, e^{x^2} dx = \int 2x\, e^u \frac{du}{2x}$$

$$= \int e^u\, du = e^u + c$$

$$= e^{x^2} + c$$

8 Find:

$$\int \frac{e^x}{(1 - e^x)^2}\,dx$$

Note e^x is a derivative of $1 - e^x$ not $(1 - e^x)^2$, so use substitution.

Solution:

Let: $u = 1 - e^x$

$$\frac{du}{dx} = -e^x \qquad \Rightarrow \; dx = -\frac{du}{e^x}$$

Substituting

$$\int \frac{e^x}{(1 - e^x)^2}\,dx = -\int \frac{e^x}{(u)^2}\,\frac{du}{e^x} \Rightarrow -\int \frac{1}{u^2}\,du$$

$$= -\int u^{-2}du$$

$$= u^{-1} + c$$

$$= \frac{1}{1 - e^x} + c$$

9 Use substitution to find:

$$\int (x + 5)(3x - 1)^5\,dx$$

Solution:

Let: $u = 3x - 1$

$$\frac{du}{dx} = 3 \qquad \Rightarrow \qquad dx = \frac{du}{3}$$

Substituting

$$\int (x + 5)(3x - 1)^5\,dx = \int (x + 5)(u)^5 \cdot \frac{1}{3}\,du$$

We have an x left, so go back to the substitution and find x

$$u = 3x - 1 \qquad \Rightarrow \qquad x = \frac{u + 1}{3}$$

$$\int (x + 5)(3x - 1)^5\,dx = \frac{1}{3}\int \left(\frac{u + 1}{3} + 5\right)u^5\,du$$

$$= \frac{1}{3}\int \left(\frac{u + 1 + 15}{3}\right)u^5\,du$$

$$= \frac{1}{9}\int (u + 16)\,u^5\,du = \frac{1}{9}\int \left(u^6 + 16u^5\right)du$$

$$= \frac{1}{9}\left[\frac{u^7}{7} + \frac{16u^6}{6}\right] + c = \frac{1}{9}\left[\frac{u^7}{7} + \frac{8u^6}{3}\right] + c$$

$$= \frac{1}{9}\left[\frac{3u^7 + 56u^6}{21}\right] + c = \frac{1}{189}\left(3u^7 + 56u^6\right) + c$$

$$= \frac{u^6}{189}(3u + 56) + c$$

$$= \frac{(3x - 1)^6}{189}\left[3(3x - 1) + 56\right] + c$$

$$= \frac{(3x - 1)^6}{189}(9x + 53) + c$$

10 Use substitution to find:

$$\int \sqrt{4 - x^2}\, dx \qquad \left[\text{Has the form } \sqrt{a^2 - b^2x^2} \text{ use } x = a/b \sin u\right]$$

Solution:

Let: $x = 2 \sin u$

$$\frac{dx}{du} = 2 \cos u \qquad \therefore\ dx = 2 \cos u\, du$$

Substituting

$$\int \sqrt{4 - x^2}\, dx = \int \sqrt{4 - (2 \sin u)^2} \times 2 \cos u\, du$$

$$= \int \sqrt{4 - 4\sin^2 u} \times 2 \cos u\, du = \int \sqrt{4(1 - \sin^2 u)} \times 2 \cos u\, du$$

but $\cos^2 u = 1 - \sin^2 u$

$$= \int \sqrt{4 \cos^2 u} \times 2 \cos u\, du = \int 2 \cos u \times 2 \cos u\, du$$

$$= 4 \int \cos^2 u\, du$$

but: $2 \cos^2 u = 1 + \cos 2u$

$$= 4 \int \frac{1}{2} (1 + \cos 2u)\, du$$

$$= 2 \int (1 + \cos 2u)\, du$$

$$= 2\left[u + \frac{1}{2}\sin 2u\right] + c$$

$$= 2u + \sin 2u + c$$

Substituting back:

Given: $x = 2\sin u$

Identity: $\sin^2 u = 1 - \cos^2 u$

Identity: $\sin 2u = 2 \sin u \cos u$

$\therefore$ Need to find $\sin u$ & $\cos u$

$$\therefore \quad \sin u = \frac{x}{2} \qquad \& \qquad \sin^2 u = \frac{x^2}{4}$$

$$\sin^{-1}\left(\frac{x}{2}\right) = u$$

$$\cos^2 u = 1 - \sin^2 u$$

$$\cos^2 u = 1 - \frac{x^2}{4} \qquad = \frac{4 - x^2}{4}$$

$$\cos u = \frac{1}{2}\sqrt{4 - x^2}$$

Substituting: $= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right) \times \frac{1}{2}\sqrt{4 - x^2} + c$

$$\therefore \int \sqrt{4 - x^2}\, dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)\sqrt{4 - x^2} + c$$

60.5 Definite Integration using Substitutions

Because of the substitution, you must also change the limits into the new variable, so we can then evaluate the integral as soon as we have done the integration. This saves you having to put the x's back in at the end and using the original limits.

60.5.1 Example:

1 Use substitution to find:

$$\int_0^1 \sqrt{4 - x^2}\, dx \qquad \left[\text{Has form } \sqrt{a^2 - b^2x^2} \text{ : use } x = a/b \sin u\right]$$

Solution:

Let: $x = 2\sin u$

$$\frac{dx}{du} = 2\cos u \qquad \therefore dx = 2\cos u\, du$$

Limits:

x	$2\sin u$	$\sin u$	u
1	1	$\frac{1}{2}$	$\frac{\pi}{6}$
0	0	0	0

From previous example

$$\int \sqrt{4 - x^2}\, dx = 2u + \sin 2u + c$$

$$\int_0^1 \sqrt{4 - x^2}\, dx = \Big[2u + \sin 2u\Big]_0^{u = \frac{\pi}{6}} = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) - 0$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} = \frac{\pi + \sqrt{3}}{6}$$

2 Use substitution to find:

$$\int_0^1 \frac{1}{1 + x^2}\, dx$$

Solution:

Let: $x = \tan u$

$$\frac{dx}{du} = \sec^2 u \qquad \therefore dx = \sec^2 u\, du$$

Limits:

x	$\tan u$	u
1	1	0
0	0	$\frac{\pi}{4}$

Substituting:

$$\int_0^1 \frac{1}{1 + x^2}\, dx = \int_0^{x = 1} \frac{1}{1 + \tan^2 u} \sec^2 u\, du$$

$$= \int_0^{x = 1} 1\, du \qquad \text{Since: } 1 + \tan^2 u = \sec^2 u$$

$$= \Big[u\Big]_{u = 0}^{u = \frac{\pi}{4}} = \frac{\pi}{4}$$

3

$$\int_0^2 x(2x-1)^6\,dx$$

Solution:

Let: $u = 2x - 1 \quad \Rightarrow \quad x = \tfrac{1}{2}(u+1)$

$\dfrac{du}{dx} = 2 \quad \Rightarrow \quad dx = \tfrac{1}{2}\,du$

Limits:

x	$u = 2x - 1$
2	3
1	−1

Substituting:

$$\int_0^2 x(2x-1)^6\,dx = \int_{u=-1}^{u=3} \tfrac{1}{2}(u+1)(u^6)\tfrac{1}{2}\,du$$

$$= \frac{1}{4}\int_{-1}^{3} u^7 + u^6\,du$$

$$= \frac{1}{4}\left[\frac{1}{8}u^8 + \frac{1}{7}u^7\right]_{-1}^{3}$$

$$= \frac{1}{4}\left[\frac{1}{8}u^8 + \frac{1}{7}u^7\right]_{-1}^{3}$$

$$= \frac{1}{4}\left[\frac{1}{8}3^8 + \frac{1}{7}3^7\right] - \frac{1}{4}\left[\frac{1}{8}(-1)^8 + \frac{1}{7}(-1)^7\right]$$

$$= \tfrac{1}{4}(1132{\cdot}57) = 283{\cdot}14$$

4

$$\int_{-1}^2 x^2\sqrt{(x^3+1)}\,dx$$

Solution:

Let: $u = x^3 + 1 \quad \Rightarrow \quad x = \tfrac{1}{2}(u+1)$

$\dfrac{du}{dx} = 3x^2 \quad \Rightarrow \quad dx = \dfrac{1}{3x^2}\,du$

Limits:

x	$u = x^3 + 1$
2	9
−1	0

Substituting:

$$\int_{-1}^2 x^2\sqrt{(x^3+1)}\,dx = \int_{u=0}^{u=9} x^2 u^{\frac{1}{2}}\frac{1}{3x^2}\,du$$

$$= \frac{1}{3}\int_0^9 u^{\frac{1}{2}}\,du$$

$$= \frac{1}{3}\left[\frac{1}{\frac{3}{2}}u^{\frac{3}{2}}\right]_0^9 = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^9$$

$$= \frac{1}{3}\left[\frac{2}{3}9^{\frac{3}{2}}\right] - 0 = \frac{2}{9}\left[9^{\frac{3}{2}}\right]$$

$$= 6$$

60.6 Reverse Substitution

This is where we have to recognise the substitution by ourselves, by recognising the reverse chain rule.

60.6.1 Example:

1 Use substitution to find: $\int \frac{1}{x} \ln x \, dx$

Solution:

Let: $x = e^u$

$\frac{dx}{du} = e^u \qquad \therefore dx = e^u \, du$

Substituting

$$\int \frac{1}{x} \ln x \, dx = \int \frac{1}{e^u} \ln e^u \times e^u \, du \Rightarrow \int \ln e^u \, du$$

but: $\quad u \ln e = u \qquad \therefore = \int u \, du = \frac{u^2}{2} + c$

$x = e^u \qquad \therefore u = \ln x$

$$\int \frac{1}{x} \ln x \, dx = \frac{(\ln x)^2}{2} + c$$

2 Find: $\int \frac{6x}{\sqrt{1 + x^2}} \, dx$

Solution:

Let: $u = 1 + x^2$

$\frac{du}{dx} = 2x \qquad \Rightarrow \qquad dx = \frac{du}{2x}$

$$\int \frac{6x}{\sqrt{1 + x^2}} \, dx = \int \frac{6x}{\sqrt{u}} \times \frac{du}{2x} = \int \frac{3}{\sqrt{u}} \, du$$

$$= \int 3u^{-\frac{1}{2}} \, du = \frac{3u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 6u^{\frac{1}{2}} + c$$

$$= 6\left(1 + x^2\right)^{\frac{1}{2}} + c \qquad \Rightarrow 6\sqrt{1 + x^2} + c$$

3 Use reverse substitution to find: $\int x^2 \sqrt{1 + x^3} \, dx$

Solution:

Let: $u = 1 + x^3$

$\frac{du}{dx} = 3x^2 \qquad \Rightarrow \qquad dx = \frac{du}{3x^2}$

$$\int x^2 \sqrt{1 + x^3} \, dx = \int x^2 \sqrt{u} \times \frac{du}{3x^2} = \frac{1}{3} \int u^{\frac{1}{2}} \, du$$

$$= \frac{1}{3}\left[\frac{2}{3} u^{\frac{3}{2}}\right] + c$$

$$= \frac{2}{9}\left[\left(1 + x^3\right)^{\frac{3}{2}}\right] + c$$

4 Consider:

$$\int \frac{7x}{(1 + 2x^2)^3}\, dx$$

Solution:

Let: $u = 1 + 2x^2$

$$\frac{du}{dx} = 4x \qquad \Rightarrow \qquad dx = \frac{du}{4x}$$

$$\int \frac{7x}{(1 + 2x^2)^3}\, dx = \int \frac{7x}{(u)^3} \cdot \frac{du}{4x} = \int \frac{7}{4u^3}\, du$$

$$= \int \frac{7}{4} u^{-3}\, du = \frac{7}{4} \int u^{-3}\, du$$

$$= \frac{7}{4}\left[\frac{1}{-2} u^{-2}\right] + c = -\frac{7}{8} u^{-2} + c \Rightarrow -\frac{7}{8u^2} + c$$

$$= -\frac{7}{8}\left(1 + 2x^2\right)^{-2} + c \Rightarrow -\frac{7}{8\,(1 + 2x^2)^2} + c$$

In the following two questions, note that we have a fraction, of which the top is the differential of the denominator, or a multiple thereof.

$$\int \frac{f'(x)}{f(x)}\, dx = \ln\left| f(x) \right| + c$$

5 Try:

$$\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$$

Solution:

Let: $u = \sin x + \cos x$

$$\frac{du}{dx} = \cos x - \sin x \qquad \Rightarrow \qquad dx = \frac{du}{\cos x - \sin x}$$

$$\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx = \int \frac{\cos x - \sin x}{u} \times \frac{du}{\cos x - \sin x}$$

$$= \int \frac{1}{u}\, du$$

$$= \ln u + c$$

$$= \ln\left|\sin x + \cos x\right| + c$$

6 Try:

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$$

Solution:

Let: $u = e^x + e^{-x}$

$$\frac{du}{dx} = e^x - e^{-x} \quad \Rightarrow \quad dx = \frac{du}{e^x - e^{-x}}$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \int \frac{e^x - e^{-x}}{u} \times \frac{du}{e^x - e^{-x}} = \int \frac{1}{u} du$$

$$= \ln u + c$$

$$= \ln\left(e^x + e^{-x}\right) + c$$

$\left(e^x + e^{-x}\right)$ is always +ve

7 Try:

$$\int \frac{\sec^2 x}{\tan^3 x}\, dx$$

Solution:

Note that $\sec^2 x$ is the derivative of $\tan x$ not $\tan^3 x$

Let: $u = \tan x$

$$\frac{du}{dx} = \sec^2 x \quad \Rightarrow \quad dx = \frac{du}{\sec^2 x}$$

$$\int \frac{\sec^2 x}{\tan^3 x}\, dx = \int \frac{\sec^2 x}{u^3} \times \frac{du}{\sec^2 x}$$

$$= \int \frac{1}{u^3} du$$

$$= \int u^{-3} du$$

$$= -\frac{1}{2} u^{-2} + c$$

$$= -\frac{1}{2u^2} + c$$

$$= -\frac{1}{2\tan^2 x} + c$$

The common trig functions that are of the form $\int \frac{f'(x)}{f(x)}\, dx$ are:

$Function\ y = f(x)$	$Integral\ \int f(x)\, dx$	
$\tan x$	$\ln\lvert\sec x\rvert + c$	*
$\cot x$	$\ln\lvert\sin x\rvert + c$	*
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert + c$	*
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + c$	*

60.7 Harder Integration by Substitution

If the integrand contains $a^2 + b^2x^2$ use $x = \frac{a}{b} \tan u$

If the integrand contains $a^2 - b^2x^2$ use $x = \frac{a}{b} \sin u$

N.B. integrand = the bit to be integrated.

60.7.1 Example:

1

$$\int \frac{1}{25 + 16x^2}\, dx$$

i.e. $a = 5, \; b = 4$

Let: $x = \frac{5}{4} \tan u \qquad \Rightarrow \qquad \frac{dx}{du} = \frac{5}{4} \sec^2 u$

$$\therefore \quad dx = \frac{5}{4} \sec^2 u \, du$$

$$\int \frac{1}{25 + 16x^2}\, dx = \int \frac{1}{25 + 16\left(\frac{5}{4} \tan u\right)^2} \times \frac{5}{4} \sec^2 u \, du$$

$$= \int \frac{1}{25 + 25 \tan^2 u} \times \frac{5}{4} \sec^2 u \, du$$

$$= \int \frac{1}{25\left(1 + \tan^2 u\right)} \times \frac{5}{4} \sec^2 u \, du$$

$$= \int \frac{1}{25 \sec^2 u} \times \frac{5}{4} \sec^2 u \, du$$

$$= \int \frac{1}{5} \times \frac{1}{4}\, du = \int \frac{1}{20}\, du$$

$$= \frac{1}{20} u + c \qquad \text{Note: } \tan u = \frac{4x}{5}$$

$$= \frac{1}{20} \tan^{-1}\left(\frac{4x}{5}\right) + c \qquad \text{Note: } u = \tan^{-1}\left(\frac{4x}{5}\right)$$

2

$$\int_0^1 \sqrt{(1 - x^2)}\, dx$$

Let: $x = \sin u \quad \Rightarrow \quad \frac{dx}{du} = \cos u \qquad \therefore dx = \cos u\, du$

Limits: $\Rightarrow \qquad u = \sin^{-1} x$

x	u
1	$\sin^{-1} x = \frac{\pi}{2}$
0	$\sin^{-1} 0 = 0$

$$\int_0^1 \sqrt{(1 - x^2)}\, dx = \int_0^{u = \frac{\pi}{2}} \sqrt{(1 - \sin^2 u)} \times \cos u\, du$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(\cos^2 u)} \times \cos u\, du$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 u\, du$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2u)\, du$$

$$= \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin \pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{4}$$

60.8 Options for Substitution

Substitution allows a wide range of functions to be integrated, but it is not always obvious which one should be used. The following table attempts to give some clues as to which to choose as the appropriate substitution.

For :	*Try* :
$(ax + b)^n$	$u = ax + b$
$\sqrt[n]{(ax + b)}$	$u^n = ax + b$
$a - bx^2$	$x = \sqrt{\frac{a}{b}}\sin u$
$a + bx^2$	$x = \sqrt{\frac{a}{b}}\tan u$
$bx^2 - a$	$x = \sqrt{\frac{a}{b}}\sec u$
e^x	$u = e^x \;:\; x = \ln u$
$\ln(ax + b)$	$ax + b = e^u \;:\; x = \frac{1}{a}e^u - \frac{b}{a}$

60.9 Some Generic Solutions

1 Use substitution to find: $\int x(ax + b)^n \, dx$

Solution:

Let: $u = ax + b \qquad \frac{du}{dx} = a \qquad \Rightarrow dx = \frac{du}{a}$

$ax = u - b \qquad x = \frac{u - b}{a}$

Substituting:

$$\int x(ax + b)^n \, dx = \int \frac{u - b}{a} (u)^n \frac{du}{a}$$

$$= \frac{1}{a^2} \int (u - b) u^n \, du$$

$$= \frac{1}{a^2} \int (u^{n+1} - bu^n) \, du$$

$$= \frac{1}{a^2} \left[\frac{u^{n+2}}{n + 2} - \frac{bu^{n+1}}{n + 1} \right] + c$$

$$= \frac{1}{a^2} \left[\frac{(ax + b)^{n+2}}{n + 2} - \frac{b(ax + b)^{n+1}}{n + 1} \right] + c$$

61 • C4 • Partial Fractions

61.1 Intro to Partial Fractions

$$\text{If:} \quad \frac{3}{2x+1} + \frac{2}{x-2} = \frac{7x-8}{(2x+1)(x-2)}$$

then we ought to be able to convert

$$\frac{7x-8}{(2x+1)(x-2)} \text{ back into its partial fractions of: } \frac{3}{2x+1} + \frac{2}{x-2}$$

The process is often called decomposition of a fraction. To do this, we create an identity that is valid for all values of x and then find the missing constants of the partial fractions.

To decompose a fraction we need to start with a proper fraction. Improper fractions (see later) have to be converted into a whole number part with a proper fraction remainder. Later on, partial fractions will be useful in integration, differentiation and the binomial theorem.

There are four different types of decomposition based on the sort of factors in the denominator. These are:

- Linear factors in the denominator:

$$\frac{x}{(ax+b)(cx+d)} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

- Squared terms in the denominator (includes quadratics that will not factorise easily)

$$\frac{x}{(ax+b)(cx^2+d)} \equiv \frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}$$

- Repeated Linear factors in the form:

$$\frac{x}{(ax+b)(cx+d)^3} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{D}{(cx+d)^3}$$

- Improper (top heavy) fractions in the form:

$$\frac{x^{n+m}}{ax^n + bx + d}$$

To solve for the unknown constants, A, B & C *etc*., we can use one or more of the following four methods:

- Equating coefficients
- Substitution in the numerator
- Separating the unknown by multiplication and substituting
- Cover up method (only useful for linear factors)

61.2 Type 1: Linear Factors in the Denominator

This the simplest of them all. The denominator factorises into two or more different linear factors of the form $(ax+b)$, $(cx+d)$ etc. Recognise that each of these linear factors is a root of the expression in the denominator. We set up the partial fractions on the RHS and each root must have its own 'unknown constant' assigned to it.

e.g. $$\frac{7x-8}{2x^2-5x+2} = \frac{7x-8}{(2x-1)(x-2)} = \frac{A}{(2x-1)} + \frac{B}{(x-2)}$$

e.g. $$\frac{8x}{(2x-1)(x-2)(x+4)} = \frac{A}{(2x-1)} + \frac{B}{(x-2)} + \frac{C}{(x+4)}$$

61.3 Solving by Equating Coefficients

Taking the first example from above:

61.3.1 Example:

The first task is to factorise the denominator:

$$\frac{7x - 8}{2x^2 - 5x + 2} = \frac{7x - 8}{(2x - 1)(x - 2)}$$

Then set up the identity with the correct number of partial fractions:

$$\frac{7x - 8}{(2x - 1)(x - 2)} \equiv \frac{A}{2x - 1} + \frac{B}{x - 2}$$

Add the fractions on the RHS to give:

$$\frac{7x - 8}{(2x - 1)(x - 2)} \equiv \frac{A}{2x - 1} + \frac{B}{x - 2} \equiv \frac{A(x - 2) + B(2x - 1)}{(2x - 1)(x - 2)}$$

$$\therefore \quad 7x - 8 \equiv A(x - 2) + B(2x - 1)$$

$$7x - 8 \equiv Ax - 2A + 2Bx - B$$

Equate the terms in: x : $7x = Ax + 2Bx$ $\quad \therefore 7 = A + 2B \quad \Rightarrow \quad A = 7 - 2B$

Equate the Constants : $-8 = -2A - B$ $\quad \therefore 2A + B = 8$

Substituting or using simultaneous equations:

$$\therefore \quad 2(7 - 2B) + B = 8$$

$$14 - 4B + B = 8 \quad \Rightarrow \quad -3B = -6$$

$$3B = 6 \quad \therefore \quad B = 2 \quad \& \quad A = 3$$

$$\therefore \quad \frac{7x - 8}{(2x - 1)(x - 2)} = \frac{3}{2x - 1} + \frac{2}{x - 2}$$

61.4 Solving by Substitution in the Numerator

Using the same example as above:

61.4.1 Example:

$$\frac{7x - 8}{(2x - 1)(x - 2)} \equiv \frac{A}{2x - 1} + \frac{B}{x - 2} \Rightarrow \frac{A(x - 2) + B(2x - 1)}{(2x - 1)(x - 2)}$$

$$\therefore \quad 7x - 8 \equiv A(x - 2) + B(2x - 1)$$

Find B by choosing $x = 2$ (to make the A term zero)

$$14 - 8 = B(4 - 1)$$

$$3B = 6 \quad \therefore \quad B = 2$$

Find A by choosing $x = \frac{1}{2}$ (to make 2nd (B) term zero)

$$\frac{7}{2} - 8 = A\left(\frac{1}{2} - 2\right)$$

$$\therefore \quad -\frac{9}{2} = -\frac{3}{2}A \quad \therefore A = 3$$

$$\therefore \quad \frac{7x - 8}{(2x - 1)(x - 2)} = \frac{3}{2x - 1} + \frac{2}{x - 2}$$

61.5 Solving by Separating an Unknown

A variation on the substitution method which involves multiplying by one of the factors, and then using substitution. In some cases this can be used if the other two methods don't work.

61.5.1 Example:

$$\frac{4x}{x^2 - 4} \equiv \frac{A}{x + 2} + \frac{B}{x - 2}$$

Multiply both sides by one of the factors, say $(x + 2)$

$$\frac{4x(x + 2)}{x^2 - 4} \equiv \frac{A(x + 2)}{x + 2} + \frac{B(x + 2)}{x - 2}$$

Cancel common terms:

$$\frac{4x}{x - 2} \equiv A + \frac{B(x + 2)}{x - 2}$$

$$\therefore \quad A = \frac{4x}{x - 2} - \frac{B(x + 2)}{x - 2}$$

Now substitute a value for x such that the B term is zero:

If $\quad x = -2 \;:\; A = \frac{-8}{-4} - 0 = 2$

Now multiply both sides by one of the other factors, $(x - 2)$ in this case:

$$\frac{4x(x - 2)}{x^2 - 4} \equiv \frac{A(x - 2)}{x + 2} + \frac{B(x - 2)}{x - 2}$$

Cancel common terms:

$$\frac{4x}{x + 2} \equiv \frac{A(x - 2)}{x + 2} + B$$

$$\therefore B = \frac{4x}{x + 2} + \frac{A(x - 2)}{x + 2}$$

Now substitute a value for x such that the A term is zero:

If $\quad x = 2 \;:\; B = \frac{8}{4} = 2$

Hence:

$$\frac{4x}{x^2 - 4} \equiv \frac{2}{x + 2} + \frac{2}{x - 2}$$

Test this by substituting a value for x on both sides. Don't use the values chosen above, as we need to check it is valid for all values of x.

If $\quad x = 1 \;:$

$$\frac{4}{1 - 4} \equiv \frac{2}{1 + 2} + \frac{2}{1 - 2}$$

$$-\frac{4}{3} \equiv \frac{2}{3} - \frac{2}{1}$$

$$\equiv \frac{2}{3} - \frac{6}{3} = -\frac{4}{3}$$

61.6 Type 2: Squared Terms in the Denominator

This is where we have a squared term in the denominator that cannot be factorised into linear factors. It can be of the form of $(ax^2 + b)$ or it may be a traditional quadratic, such as $x^2 + bx + c$, that cannot be factorised. In either case we have to take this into account. The general form of the partial fractions are:

$$\frac{x}{(ax + b)(cx^2 + d)} \equiv \frac{A}{(ax + b)} + \frac{Bx + C}{(cx^2 + d)}$$

Note that the numerator is always one degree less than the denominator.

61.6.1 Example:

1

$$\frac{4x}{(x + 1)(x^2 - 3)} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 3} \equiv \frac{A(x^2 - 3) + (Bx + C)(x + 1)}{(x + 1)(x^2 - 3)}$$

$\therefore \quad 4x \equiv A(x^2 - 3) + (Bx + C)(x + 1)$

To eliminate the $(Bx + C)$ term, let $x = -1$

$\therefore \quad -4 = A(1 - 3) + 0 \qquad \therefore A = 2$

Equate the terms in x

$$4x \equiv A(x^2 - 3) + (Bx + C)(x + 1)$$

$$4x \equiv Ax^2 - 3A + Bx^2 + Bx + Cx + C$$

$$4 = B + C$$

Equate the constants terms :

$$0 = -3A + C$$

But $\quad A = 2$

$\therefore \quad C = 6$

$\therefore \quad B = 4 - C = 4 - 6 = -2$

Hence:

$$\frac{4x}{(x + 1)(x^2 - 3)} = \frac{2}{x + 1} + \frac{6 - 2x}{x^2 - 3}$$

Check result by substituting any value for x, except -1 used above. So let $x = 1$

$$\frac{4}{(1 + 1)(1 - 3)} = \frac{2}{1 + 1} + \frac{6 - 2}{1 - 3}$$

$$\frac{4}{(2)(-2)} = \frac{2}{2} + \frac{4}{-2}$$

$$-1 = 1 - 2 = -1$$

2 A trick question - know your factors (difference of squares)!

$$\frac{4x}{x^2 - 4} \equiv \frac{A}{x + 2} + \frac{B}{x - 2} \equiv \frac{A(x - 2) + B(x + 2)}{(x + 2)(x - 2)}$$

$\therefore 4x \equiv A(x - 2) + B(x + 2)$

If $x = 2$: $\quad 8 = A(0) + B(4) \qquad B = 2$

If $x = -2$: $\quad -8 = A(-4) + B(0) \qquad A = 2$

$$\therefore \quad \frac{4x}{x^2 - 4} \equiv \frac{2}{x + 2} + \frac{2}{x - 2}$$

61.7 Type 3: Repeated Linear Factors in the Denominator

A factor raised to a power such as $(x + 2)^3$ gives rise to repeated factors of $(x + 2)$. Handling these repeated factors requires a partial fraction for each power of the factor, up to the highest power of the factor.

Thus, a cubed factor requires three fractions using descending powers of the factor.

e.g. $$\frac{x}{(x+2)^3} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$$

Similarly, factors of $(x + 2)^4$ would be split into fractions with $(x + 2)^4$, $(x + 2)^3$, $(x + 2)^2$, $(x + 2)$

e.g. $$\frac{x}{(x+2)^4} = \frac{A}{(x+2)^4} + \frac{B}{(x+2)^3} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)}$$

The general rule is that the number of unknowns on the RHS must equal the degree of the denominators polynomial on the left. In the example below, the degree of the expression in the denominator is four. Hence:

e.g. $$\frac{x}{(x+1)(x+4)(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+4)} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$$

The reasoning behind the use of different powers of a factor requires an explanation that is really beyond the scope of these notes. Suffice it to say that anything else does not provide a result that is true for **all values of *x***, which is what we require. In addition, we need the same number of equations as there are unknowns in order to find a unique answer.

An alternative way to view this problem, is to treat the problem in the same way as having a squared term in the denominator. For example:

$$\frac{x}{(x+2)^2} = \frac{Ax + B}{(x+2)^2}$$

However, the whole point of partial fractions is to simplify the original expression as far as possible, ready for further work such as differentiation or integration. In the exam, repeated linear factors need to be solved as discussed above.

61.7.1 Example:

1

$$\frac{x}{(x+1)(x+2)^2}$$

$$\frac{x}{(x+1)(x+2)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+2)^2} + \frac{C}{x+2}$$

$$\therefore \quad x \equiv A(x+2)^2 + B(x+1) + C(x+1)(x+2)$$

$x = -2$ $\quad -B = -2 \quad \Rightarrow \quad B = 2$

$x = -1$ $\quad A = -1 \quad \Rightarrow \quad A = -1$

Look at x^2 term: $\quad A + C = 0 \quad \therefore C = 1$

$$\frac{x}{(x+1)(x+2)^2} = -\frac{1}{x+1} + \frac{2}{(x+2)^2} + \frac{1}{x+2}$$

2 Same problem, but treated as a squared term (interest only).

$$\frac{x}{(x+1)(x+2)^2}$$

$$\frac{x}{(x+1)(x+2)^2} \equiv \frac{A}{x+1} + \frac{Bx+C}{(x+2)^2}$$

$$\therefore \quad x \equiv A(x+2)^2 + (Bx+C)(x+1)$$

$$x = -1$$

$$-1 = A(1)^2 + 0 \qquad \Rightarrow \quad A = -1$$

$$x = -2$$

$$-2 = 0 + (-2B + C)(-1)$$

$$-2 = 2B - C$$

Equate constant terms

$$0 = 4A + C \qquad \Rightarrow \qquad C = 4$$

$$\therefore \quad 2B = C - 2 \qquad \Rightarrow \qquad B = 1$$

$$\frac{x}{(x+1)(x+2)^2} \equiv -\frac{1}{x+1} + \frac{x+4}{(x+2)^2}$$

3

$$\frac{x^2+7x+5}{(x+2)^3} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

Compare numerators:

$$x^2 + 7x + 5 \equiv A(x+2)^2 + B(x+2) + C$$

Let: $x = -2$ $\quad 4 - 14 + 5 = C \qquad C = -5$

Compare coefficients: x^2 $\quad 1 = A$

Compare coefficients: constants $\quad 5 = 4A + 2B + C$

$$5 = 4 + 2B - 5$$

$$B = 3$$

$$\therefore \quad \frac{x^2+7x+5}{(x+2)^3} \equiv \frac{1}{x+2} + \frac{3}{(x+2)^2} - \frac{5}{(x+2)^3}$$

61.8 Solving by the Cover Up Method

One final method of solving partial fractions, which was first described by the scientist Oliver Heaviside, is the cover up method. The restriction with this method is that it can only be used on the highest power of any given linear factor. (This will make more sense after the second example). It makes a convenient way of finding the constants and is less prone to mistakes.

61.8.1 Example:

1

$$\frac{6x-8}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$$

To find A, we 'cover up' its corresponding factor $(x-1)$ and then set $x = 1$

$$\frac{6x-8}{(\square\square\square\square)(x-2)} = \frac{A}{(\square\square\square\square)}$$

$$\frac{6-8}{1-2} = \frac{-2}{-1} = 2 = A$$

Similarly, to find B, we 'cover up' its corresponding factor $(x-2)$ and then set $x = 2$

$$\frac{6x-8}{(x-1)(\square\square\square\square)} = \frac{B}{(\square\square\square\square)}$$

$$\frac{12-8}{2-1} = \frac{4}{1} = 4 = B$$

Hence:

$$\frac{6x-8}{(x-1)(x-2)} \equiv \frac{2}{x-1} + \frac{4}{x-2}$$

Why does this work? If we did it the long way by multiplying by one factor, say $(x-1)$, we get:

$$\frac{(6x-8)(x-1)}{(x-1)(x-2)} \equiv \frac{A(x-1)}{x-1} + \frac{B(x-1)}{x-2}$$

Cancelling terms we get:

$$\frac{(6x-8)}{(x-2)} \equiv A + \frac{B(x-1)}{x-2}$$

When $x = 1$ the B term becomes zero, so we have:

$$\frac{(6x-8)}{(x-2)} \equiv A$$

So the Cover Up Method is just a short cut method for multiplying out by one of the factors.

2 The cover up method can be used on the linear parts of other more complex partial fractions. This speeds up the process, and simplifies the subsequent calculations.
For example, in the problem earlier, we had this to solve:

$$\frac{4x}{(x+1)(x^2-3)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-3}$$

To find A: cover up $(x+1)$ and set $x = -1$

$$\frac{4x}{(\square\square\square\square)(x^2-3)} \equiv \frac{A}{(\square\square\square\square)}$$

$$\frac{-4}{(1-3)} \equiv A$$

$$\frac{-4}{-2} \equiv A \qquad \therefore A = 2$$

The other constants can now be found using the other methods.

3 The cover up method can also be used to partly solve problems with repeated linear factors. The proviso is that only the highest power of the repeated factor can be covered up.

$$\frac{6}{(x+2)(x-1)^2} \equiv \frac{A}{x+2} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

To find A: cover up $(x+2)$ and set $x = -2$

$$\frac{6}{(\square\square\square\square)(x-1)^2} \equiv \frac{A}{(\square\square\square\square)}$$

$$\frac{6}{(-2-1)^2} \equiv A$$

$$\frac{6}{9} \equiv A \qquad \therefore A = \frac{2}{3}$$

To find B: cover up $(x-1)^2$ and set $x = 1$

$$\frac{6}{(x+2)(\square\square\square\square)} \equiv \frac{B}{(\square\square\square\square)}$$

$$\frac{6}{3} \equiv B \qquad \therefore B = 2$$

The cover up method cannot be used to find C, so one of the other methods is required.

To find C set $x = 0$

$$\frac{6}{(x+2)(x-1)^2} \equiv \frac{2}{3(x+2)} + \frac{2}{(x-1)^2} + \frac{C}{x-1}$$

$$\frac{6}{(2)(-1)^2} \equiv \frac{2}{3(2)} + \frac{2}{(-1)^2} + \frac{C}{-1}$$

$$\frac{6}{2} = \frac{2}{6} + \frac{2}{1} - \frac{C}{1}$$

$$3 = \frac{1}{3} + 2 - C$$

$$C = -\frac{2}{3}$$

$$\frac{6}{(x+2)(x-1)^2} = \frac{2}{3(x+2)} + \frac{2}{(x-1)^2} - \frac{2}{3(x-1)}$$

61.9 Partial Fractions Worked Examples

61.9.1 Example:

1

$$\frac{16}{x^3 - 4x} \equiv \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

But $\frac{16}{x^3 - 4x} = \frac{16}{x(x^2-4)} = \frac{16}{x(x+2)(x-2)}$

$$\therefore \quad \frac{16}{x(x+2)(x-2)} \equiv \frac{A(x-2)(x+2) + Bx(x-2) + Cx(x+2)}{x(x+2)(x-2)}$$

$$16 = A(x-2)(x+2) + Bx(x-2) + Cx(x+2)$$

Let $x = 0$ $\quad 16 = A(-2)(2) = -4A \quad A = -4$

Let $x = -2$ $\quad 16 = B(-2)(-4) = +8B \quad B = 2$

Let $x = 2$ $\quad 16 = C(2)(4) = 8C \quad C = 2$

$$\therefore \quad \frac{16}{x^3 - 4x} = -\frac{4}{x} + \frac{2}{x+2} + \frac{2}{x-2}$$

2

Express $\frac{13x - 6}{x(3x-2)}$ as partial fractions.

$$\frac{13x-6}{x(3x-2)} = \frac{A}{x} + \frac{B}{3x-2} \quad \Rightarrow \quad \frac{A(3x-2) + B(x)}{x(3x-1)}$$

$$\therefore \quad 13x - 6 \equiv A(3x-2) + B(x)$$

Choose values of x

$x = 0 \quad \therefore -6 = -2A \quad \Rightarrow A = 3$

$x = \frac{2}{3} \quad \therefore \frac{26}{3} - 6 = \frac{2}{3}B \quad \Rightarrow B = 4$

Ans : $= \frac{3}{x} + \frac{4}{3x-2}$

3

$$\frac{12x}{(x+1)(2x+3)(x-3)} = \frac{A}{x+1} + \frac{B}{2x+3} + \frac{C}{x-3}$$

$$= \frac{A(2x+3)(x-3) + B(x+1)(x-3) + C(x+1)(2x+3)}{(x+1)(2x+3)(x-3)}$$

$$\therefore 12x \equiv A(2x+3)(x-3) + B(x+1)(x-3) + C(x+1)(2x+3)$$

Choose values of x

$x = 3 \quad \therefore 36 = C(3+1)(2 \times 3 + 3) \quad \Rightarrow 36C = 36 \quad \Rightarrow C = 1$

$x = -1 \quad \therefore -12 = A(-2+3)(-1-3) \quad \Rightarrow -4A = -12 \quad \Rightarrow A = 3$

$x = -\frac{3}{2} \quad \therefore -12 \times \frac{3}{2} = B\left(-\frac{3}{2} + 1\right)\left(-\frac{3}{2} - 3\right) \quad \Rightarrow \frac{9}{4}B = -18 \quad \Rightarrow B = -8$

Ans : $= \frac{3}{x+1} - \frac{8}{2x+3} + \frac{1}{x-3}$

61.10 Improper (Top Heavy) Fractions

An algebraic fraction is top heavy if the highest power of x in the numerator is greater to or equal to the highest power in the denominator. The examples below illustrate two methods of finding the unknowns. You can of course do a long division to find the whole number and remainder. Then work the partial fractions on the remainder.

61.10.1 Example:

1

$$\frac{x^2}{(x-1)(x+2)}$$

$$\equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

Note: A in not divided by another term because the fraction is a top heavy one and dividing out a top heavy fraction will give a whole number plus a remainder.

$$x^2 \equiv A(x-1)(x+2) + B(x+2) + C(x-1)$$

$x = 1 \qquad 1 = 3B \qquad \Rightarrow \quad B = \frac{1}{3}$

$x = -2 \qquad 4 = -3C \qquad \Rightarrow \quad C = -\frac{4}{3}$

$A = 1$ (coefficient of x^2)

$$\therefore \quad \frac{x^2}{(x-1)(x+2)} = 1 + \frac{1}{3(x-1)} - \frac{4}{3(x+2)}$$

2

e.g. $\dfrac{3x^2 + 6x + 2}{(2x+3)(x+2)^2}$ $\quad\leftarrow$ this is NOT top heavy

$$\equiv \frac{A}{2x+3} + \frac{B}{(x+2)^2} + \frac{C}{x+2}$$

$$\therefore \quad 3x^2 + 6x + 2 \equiv A(x+2)^2 + B(2x+3) + C(2x+3)(x+2)$$

$x = -2 \qquad 2 = -B \qquad \Rightarrow \quad B = -2$

etc…

3

e.g. $\dfrac{3x^2 + 6x + 2}{(2x+3)(x+2)}$ $\quad\leftarrow$ this IS top heavy

$$\equiv A + \frac{B}{2x+3} + \frac{C}{x+2}$$

$$\therefore \quad 3x^2 + 6x + 2 \equiv A(2x+3)(x+2) + B(x+2) + C(2x+3)$$

$x = -2 \qquad 2 = -C$

etc…

4 Here is an alternative method, which splits the numerator into parts that can be divided exactly by the denominator, giving the whole number part immediately.

$$\frac{x^2 + 3x - 11}{(x + 2)(x - 3)} = \frac{x^2 + 3x - 11}{x^2 - x - 6}$$

$$= \frac{x^2 - x - 6 + 4x - 5}{x^2 - x - 6}$$

$$\frac{x^2 + 3x - 11}{(x + 2)(x - 3)} = \frac{x^2 - x - 6}{x^2 - x - 6} + \frac{4x - 5}{x^2 - x - 6}$$

$$= 1 + \frac{4x - 5}{x^2 - x - 6}$$

$$= 1 + \frac{A}{x + 2} + \frac{B}{x - 3}$$

The partial fraction required is based on the remainder and is now:

$$\frac{4x - 5}{x^2 - x - 6} = \frac{A}{x + 2} + \frac{B}{x - 3}$$

which can be solved in the normal manner.

61.11 Using Partial Fractions

Some examples of using partial fractions for differentiation and integration. Partial fractions can also be used for series expansions.

61.11.1 Example:

1 Differentiate the following function: $f(x) = \dfrac{x+9}{2x^2+x-6}$

$$f(x) = \frac{x+9}{(2x-3)(x+2)} = \frac{A}{(2x-3)} + \frac{B}{(x+2)}$$

$$x + 9 = A(x+2) + B(2x-3)$$

Let $x = -2$: $\quad 7 = -7B \qquad B = -1$

Let $x = \dfrac{3}{2}$: $\quad \dfrac{3}{2} + 9 = A\left(\dfrac{3}{2} + 2\right) \Rightarrow 3 + 18 = A(3+4) \qquad A = 3$

$$\therefore \quad f(x) = \frac{3}{(2x-3)} - \frac{1}{(x+2)}$$

Recall: If $\quad y = [f(x)]^n \quad \Rightarrow \quad \dfrac{dy}{dx} = n f'(x)[f(x)]^{n-1}$

$$f(x) = 3(2x-3)^{-1} - (x+2)^{-1}$$

$$f'(x) = 3(-1)2(2x-3)^{-2} - (-1)(x+2)^{-2}$$

$$= -6(2x-3)^{-2} + (x+2)^{-2}$$

$$f'(x) = \frac{1}{(x+2)^2} - \frac{6}{(2x-3)^2}$$

We could have used the quotient rule, but this method is sometimes easier.

61.12 Topical Tips

- The number of unknown constants on the RHS should equal the degree of the polynomial in the denominator:
  e.g.

$$\frac{x^2+7x+5}{(x+2)^3} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

- The denominator on the LHS is a degree 3 polynomial, so the number of constants on the RHS = 3
- A rational function is one in which both numerator and denominator are both polynomials.

62 • C4 • Integration with Partial Fractions

62.1 Using Partial Fractions in Integration

The ideal format for integrating a fraction is :

$$\int \frac{1}{ax + b} dx = \frac{1}{a} = ln\,|ax + b| + c$$

Partial fractions gives us the tool to tackle fractions that are not in this ideal form.

62.2 Worked Examples in Integrating Partial Fractions

62.2.1 Example:

1

Find $\int \frac{1}{(x^2 - 1)} dx$

$$\frac{1}{(x^2 - 1)} = \frac{A}{(x + 1)} + \frac{B}{(x - 1)} = \frac{A(x - 1) + B(x + 1)}{(x + 1)(x - 1)}$$

$\therefore \quad 1 = A(x - 1) + B(x + 1)$

Let $x = 1 \quad \Rightarrow \quad 1 = 2B \quad \therefore B = \frac{1}{2}$

Let $x = -1 \quad \Rightarrow \quad 1 = -2A \quad \therefore A = -\frac{1}{2}$

$$\int \frac{1}{(x^2 - 1)} dx = \int \frac{1}{2(x - 1)} - \frac{1}{2(x + 1)} dx$$

$$= \frac{1}{2}\int \frac{1}{(x - 1)} dx - \frac{1}{2}\int \frac{1}{(x + 1)} dx$$

$$= \frac{1}{2} ln\,|x - 1| - \frac{1}{2} ln\,|x + 1| + c$$

$$= \frac{1}{2} ln \frac{|x - 1|}{|x + 1|} + c$$

2

Find $\int \frac{5(x + 1)}{(x - 1)(x + 4)} dx$

$$\frac{5(x + 1)}{(x - 1)(x + 4)} = \frac{A}{(x - 1)} + \frac{B}{(x + 4)} = \frac{A(x + 4) + B(x - 1)}{(x - 1)(x + 4)}$$

$\therefore \quad 5(x + 1) = A(x + 4) + B(x - 1)$

Let $x = -4 \quad \Rightarrow \quad -15 = -5B \quad \therefore B = 3$

Let $x = 1 \quad \Rightarrow \quad 10 = 5A \quad \therefore A = 2$

$$\int \frac{5(x + 1)}{(x - 1)(x + 4)} dx = \int \frac{2}{(x - 1)} dx + \int \frac{3}{(x + 4)} dx$$

$$= 2\int \frac{1}{(x - 1)} + 3\int \frac{1}{(x + 4)} dx$$

$$= 2\,ln\,|x - 1| + 3\,ln\,|x + 4| + c$$

3

Calculate the value of $\displaystyle\int_1^4 \frac{1}{x(x-5)}dx$

$$\frac{1}{x(x-5)} \equiv \frac{A}{x} + \frac{B}{x-5} = \frac{A(x-5)+B(x)}{x(x-5)}$$

$$\therefore \quad 1 \equiv A(x-5) + Bx$$

Let $x = 5$ $\quad 5B = 1 \quad \Rightarrow \quad B = \frac{1}{5}$

Let $x = 0$ $\quad -5A = 1 \quad \Rightarrow \quad A = -\frac{1}{5}$

$$\therefore \quad \frac{1}{x(x-5)} = -\frac{1}{5x} + \frac{1}{5(x-5)}$$

$$\int_1^4 \frac{1}{x(x-5)}dx = \frac{1}{5}\int_1^4 \left(-\frac{1}{x}\right) + \frac{1}{(x-5)}\,dx$$

$$= \frac{1}{5}\left[-\ln|x| + \ln|x-5|\right]_1^4$$

$$= \frac{1}{5}\left[\left(-\ln|4| + \ln|4-5|\right) - \left(-\ln|1| + \ln|1-5|\right)\right]$$

$$= \frac{1}{5}\left[\left(-\ln|4| + \ln|1|\right) + \ln|1| - \ln|4|\right] \qquad \text{(but: } \ln 1 = 0)$$

$$= \frac{1}{5}(-2\ln 4) = -\frac{2}{5}\ln 4 = \frac{2}{5}\ln\left(\frac{1}{4}\right) = \frac{1}{5}\ln\left(\frac{1}{16}\right)$$

4

Calculate the value of $\displaystyle\int_0^\infty \frac{1}{(x+1)(2x+3)}dx$

$$\frac{1}{(x+1)(2x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(2x+3)} \equiv \frac{A(2x+3)+B(x+1)}{(x+1)(2x+3)}$$

$$\therefore 1 \equiv A(2x+3) + B(x+1)$$

$x = -\frac{3}{2}$ $\quad -\frac{1}{2}B = 1 \quad \Rightarrow \quad B = -2$

$x = -1$ $\quad -2A + 3 = 1 \quad \Rightarrow \quad A = 1$

$$\therefore \quad = \frac{1}{(x+1)} - \frac{2}{(2x+3)}$$

$$\int_0^\infty \frac{1}{(x+1)(2x+3)}dx = \int_0^\infty \frac{1}{(x+1)} - \frac{2}{(2x+3)}dx$$

$$= \left[\ln(x+1) - \frac{2}{2}\ln(2x+3)\right]_0^\infty$$

$$= \left[\ln\left(\frac{x+1}{2x+3}\right)\right]_0^\infty = \left[\ln\left(\frac{x}{2x+3} + \frac{1}{2x+3}\right)\right]_0^\infty$$

Substitute 0 into this bit... $\displaystyle\left[\ln\left(\frac{x+1}{2x+3}\right)\right]_0^\infty = \ln\left(\frac{1}{3}\right)$

Rearrange & substitute ∞ into this bit... $\displaystyle = \left[\ln\left(\frac{1}{2+\frac{3}{x}} + \frac{1}{2x+3}\right)\right]_0^\infty = \ln\left(\frac{1}{2}\right)$

$$\therefore \quad = \left[\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{3}\right)\right] \qquad = \ln\left(\frac{3}{2}\right)$$

63 • C4 • Binomial Series

63.1 The General Binomial Theorem

In C2, the Binomial Theorem was used to expand $(a + b)^n$ for any +ve integer of n, and which gave a finite series that terminated after $n + 1$ terms. This was given as:

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n$$

The coefficient of each of the above terms can be found using a calculators ${}^nC_r$ button, however, this is only valid when n and r are positive integers.

So the formula ${}^nC_r = \dfrac{n!}{(n - r)!\, r!}$ cannot be used for fractional or negative values of n and r.

Because the expansion is finite, the RHS exactly equals the LHS of the equation. Plotting both sides of the equation as separate functions would give identical graphs.

Now we want to be able to use the Binomial Theorem, for any **rational** value of n.

In fact, restricting n to +ve integers is a just a special case of the general Binomial Theorem, in which n can take any **rational** value (which of course includes fractional and −ve values of n).

Rearranging the binomial $(a + b)^n$ into the form $(1 + x)^n$; the general Binomial Theorem now becomes:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$
$$+ \ldots \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

The big change here, is that the expansion has an infinite number of terms, (except for the special case mentioned above) and the RHS is now only an approximation of the function on the LHS (unless you can calculate all the infinite terms:-).

We must also determine if the expansion diverges or converges towards the value of the LHS.

63.2 Recall the Sum to Infinity of a Geometric Progression

Recall from C2, that the sum of a Geometric Progression (GP) is given by:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$$

The sum to infinity, S_∞, only has a meaning if the GP is a convergent series, (the sum to infinity of a divergent series is undefined).

The general formula for the sum of a GP is:

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

However, if r is small i.e. $-1 < r < 1$, then the term r^n tends to 0 as $n \to \infty$

Mathematically this is written:

$$\text{if } |r| < 1, \text{ then } \lim_{n \to \infty} r^n = 0$$

and the sum to infinity becomes:

$$S_\infty = \frac{a}{(1 - r)} \qquad |r| < 1$$

The GP is said to converge to the sum S_∞

63.3 Convergence and Validity of a Binomial Series

From our general binomial expansion:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

we can see the similarities to the Geometric Progression (GP) in the section above.

For a binomial expansion, the sum of all the terms to infinity only has a meaning if the binomial converges.

Thus: when $r \to \infty$, and if $x^r \to 0$ then the series will converge to the value of $(1 + x)^n$.

From the above equation, one can see that a binomial will converge only when $|x| < 1$.

We say the expansion is valid for $|x| < 1$. Valid just means convergence in this instance.

In the formula above, the role of x is a generic one. We can replace x with any variation of the term, so, for example, the binomial $(1 + bx)^n$ is only valid for $|bx| < 1$.

[Note: do not confuse the choice of variable letters used here with those used for a GP]

Another way of looking at the validity of the expansion is to plot the LHS and RHS of the equation as two separate functions.

The two graphs will only have a close match when $|x| < 1$.

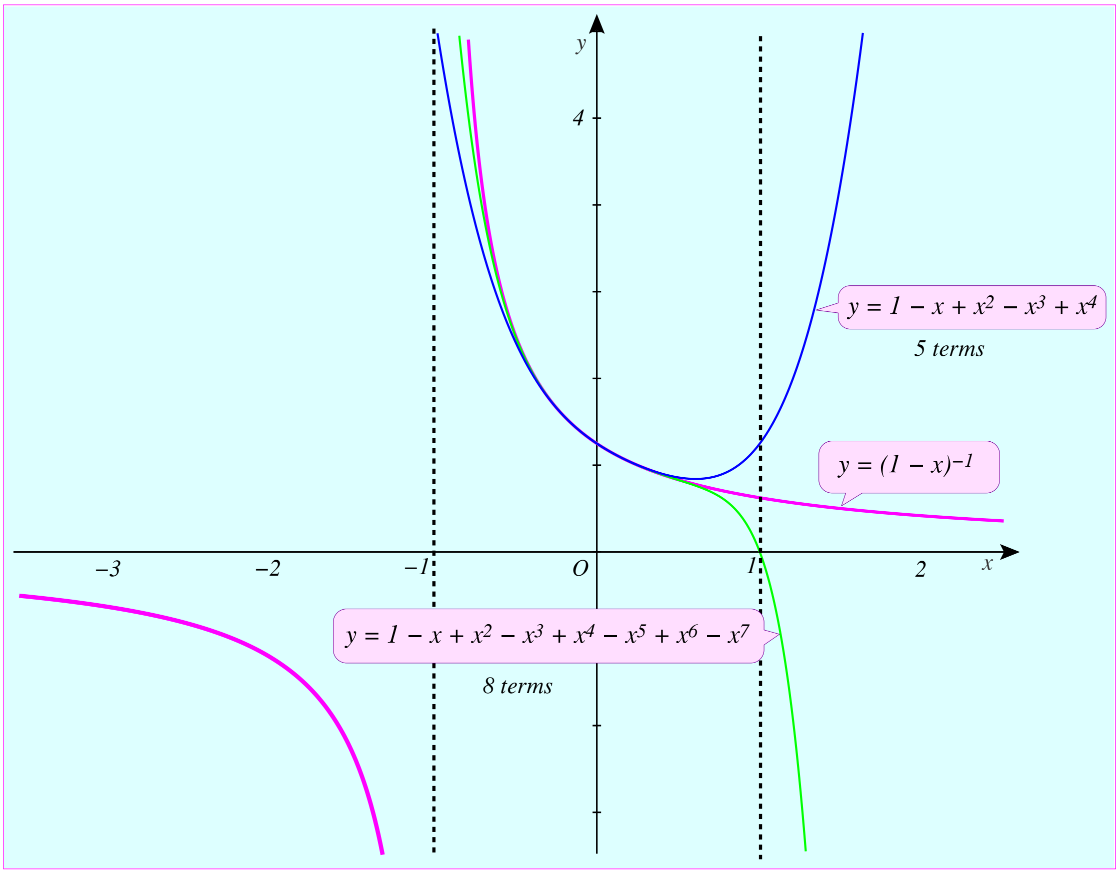
The example below shows how the expansion of $(1 + x)^{-1}$ compares when plotted on a graph.

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \ldots$$

The RHS matches the LHS most closely between the valid values of $-1 < x < 1$ (*i.e.* $|x| < 1$)

The best approximation is when x is small and close to 0. In this region the expansion converges quickly, with fewer terms required. When x is closer to ± 1, but still in the valid range, the convergence is slow, and many more terms are required.

Note the difference between the expansion to 5 terms and the one to 8 terms.



Binomial Expansions Compared

63.4 Handling Binomial Expansions

It is all too easy to get these expansions wrong, especially if a minus sign is involved.

Thus, for an expansion to the 5th term:

$$(1 - x)^n \approx 1 + n(-x) + \frac{n(n-1)}{2!}(-x)^2 + \frac{n(n-1)(n-2)}{3!}(-x)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}(-x)^4$$

In this case we get alternating signs for each term:

$$(1 - x)^n \approx 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4$$

Note that the signs will change again if n is −ve.

Some confusion can also be caused by the way the general theorem is stated with x as the variable and then being asked to evaluate something like $(1 - 3x)^n$.

The x term can take any coefficient b, which is also raised to the same power as the x term, thus:

$$(1 - bx)^n = 1 + n(-bx) + \frac{n(n-1)}{2!}(-bx)^2 + \frac{n(n-1)(n-2)}{3!}(-bx)^3 + \ldots$$

Stating the theorem with u as the variable, or even using a symbol may help in your understanding:

$$(1 + u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \frac{n(n-1)(n-2)(n-3)}{4!}u^4 + \ldots$$

$$(1 + \diamond)^n = 1 + n\diamond + \frac{n(n-1)}{2!}\diamond^2 + \frac{n(n-1)(n-2)}{3!}\diamond^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\diamond^4 + \ldots$$

Evaluating $(1 - 3x)^n$ becomes more obvious as u or $\diamond$ is replaced everywhere with $3x$.

In evaluating the coefficients, note the pattern that they form. Each succeeding value in the bracket is one less that the previous.

E.g. Assuming a value of $n = -2$, instead of writing down the 4th coefficients as:

$$\frac{-2(-2-1)(-2-2)}{3!}$$

write:

$$\frac{-2(-3)(-4)}{3!}$$

Once you see the pattern it is very easy to write down the next coefficients in turn. E.g:

$$\frac{-2(-3)(-4)(-5)}{4!}$$

63.4.2 Example:

1 Expand $\dfrac{1}{(1 + x)^2}$ up to the term in x^3

Solution:

$$(1 + x)^{-2} = 1 + (-2x) + \frac{-2(-2 - 1)}{2!}x^2 + \frac{-2(-3)(-4)}{3!}x^3 \ldots$$

$$(1 + x)^{-2} = 1 + (-2x) + \frac{-2(-3)}{2!}x^2 + \frac{-2(-3)(-4)}{3!}x^3 \ldots$$

$$(1 + x)^{-2} = 1 - 2x + \frac{-\cancel{2}(-3)}{\cancel{2}!}x^2 + \frac{(\cancel{2} \times \cancel{3})(-4)}{\cancel{3}!}x^3 \ldots$$

$$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots$$

Valid for $|x| < 1$

2 Expand $(1 + 3x)^{\frac{3}{2}}$ up to the term in x^3

Replace n with $\frac{3}{2}$ and replace all x's with $3x$

Solution:

$$(1 + 3x)^{\frac{3}{2}} = 1 + \frac{3}{2}(3x) + \frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2!}(3x)^2 + \frac{\frac{3}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(3x)^3 \ldots$$

$$(1 + 3x)^{\frac{3}{2}} = 1 + \frac{9}{2}x + \frac{\frac{3}{4}}{2!}9x^2 + \frac{-\frac{3}{8}}{3!}27x^3 \ldots$$

$$(1 + 3x)^{\frac{3}{2}} = 1 + \frac{9}{2}x + \frac{27}{8}x^2 - \frac{27}{16}x^3 \ldots$$

Valid for $|3x| < 1$ *or* $|x| < \dfrac{1}{3}$

3 Expand $\dfrac{5 + x}{1 - 2x}$ in ascending powers of x up to the term in x^3

Solution:

$$\frac{5 + x}{1 - 2x} = (5 + x)(1 - 2x)^{-1}$$

Expand: $(1 - 2x)^{-1} = 1 - 1(-2x) + \dfrac{-1(-2)}{2!}(-2x)^2 + \dfrac{-1(-2)(-3)}{3!}(-2x)^3 \ldots$

$$= 1 + 2x + 4x^2 + 8x^3 \ldots$$

$$\therefore (5 + x)(1 - 2x)^{-1} = (5 + x)(1 + 2x + 4x^2 + 8x^3)$$

$$= 5 + 10x + 20x^2 + 40x^3 + x + 2x^2 + 4x^3 + 8x^4$$

$$= 5 + 11x + 22x^2 + 44x^3 + 8x^4$$

Ans : $= 5 + 11x + 22x^2 + 44x^3$

Valid for $|2x| < 1$ *or* $|x| < \dfrac{1}{2}$

63.5 Using Binomial Expansions for Approximations

When $-1 < x < 1$ (*i.e.* $|x| < 1$) then the series will be a good approximation of $(1 + x)^n$.

For $(1 + bx)^n$ then the series is valid (or convergent) when $|bx| < 1$ or $|x| < \frac{1}{b}$.

63.5.1 Example:

1 Expand $\sqrt{(1 - 2x)}$ in ascending powers of x up to and including the term in x^3 and hence by choosing values for x, find an approximation for $\sqrt{2}$.

Solution:

$$(1 - 2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-2x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-2x)^3$$

$$(1 - 2x)^{\frac{1}{2}} = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \ldots$$

To find $\sqrt{2}$ let $(1 - 2x) = 2 \quad \therefore x = -\frac{1}{2}$

$$\therefore 2^{\frac{1}{2}} \cong 1 + \frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)^2 - \frac{1}{2}\left(-\frac{1}{2}\right)^3$$

$$\cong 1 + \frac{1}{2} - \frac{1}{2} \times \frac{1}{4} - \frac{1}{2} \times \left(-\frac{1}{8}\right)$$

$$\cong 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \ldots$$

$$\cong \frac{23}{16} \cong 1{\cdot}4375 \qquad (\sqrt{2} = 1{\cdot}41421 \text{ by calculator})$$

Valid for $|2x| < 1$ *or* $|x| < \frac{1}{2}$

2 Using the above expansion find the approximate value of $\sqrt{21}$ by substituting $x = 0.08$
Substituting & using the rules for surds:

Solution:

$$\sqrt{(1 - 2x)} = \sqrt{1 - 2 \times 0{\cdot}08} = \sqrt{0{\cdot}84}$$

$$= \sqrt{\frac{84}{100}} = \sqrt{\frac{4 \times 21}{100}} = \frac{2}{10}\sqrt{21}$$

$$\therefore \frac{2}{10}\sqrt{21} \cong 1 - 0{\cdot}08 - \frac{1}{2}(0{\cdot}08)^2 - \frac{1}{2}(0{\cdot}08)^3 + \ldots$$

$$\cong 0{\cdot}9165$$

$$\therefore \sqrt{21} \cong \frac{0{\cdot}9165 \times 10}{2}$$

$$\sqrt{21} \cong 4{\cdot}5827 \ (5\ sf)$$

$$\sqrt{21} = 4{\cdot}58257 \text{ (by calculator)}$$

Valid for $|2x| < 1$ *or* $|x| < \frac{1}{2}$

63.6 Expanding $(a + bx)^n$

This requires you to change the format from $(a + bx)^n$ to $(1 + kx)^n$ by taking out the factor a^n. Thus:

$$(a + bx)^n = \left[a\left(1 + \frac{bx}{a}\right)\right]^n = a^n\left(1 + \frac{bx}{a}\right)^n$$

$$= a^n\left[1 + n\frac{b}{a}x + \frac{n(n-1)}{2!}\left(\frac{b}{a}x\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}x\right)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{b}{a}x\right)^4\right]$$

Valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \frac{a}{b}$

63.6.1 Example:

1 Expand $\sqrt{(4 - 3x^2)}$ up to and including the term in x^4.

Solution:

$$\sqrt{(4 - 3x^2)} = (4 - 3x^2)^{\frac{1}{2}} \Rightarrow \left[4\left(1 - \frac{3}{4}x^2\right)\right]^{\frac{1}{2}} \Rightarrow 4^{\frac{1}{2}}\left(1 - \frac{3}{4}x^2\right)^{\frac{1}{2}} \Rightarrow 2\left(1 - \frac{3}{4}x^2\right)^{\frac{1}{2}}$$

Now: $\left(1 - \frac{3}{4}x^2\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\left(-\frac{3}{4}x^2\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-\frac{3}{4}x^2\right)^2 + \ldots$

$$\therefore \quad 2\left(1 - \frac{3}{4}x^2\right)^{\frac{1}{2}} = 2\left[1 - \frac{3}{8}x^2 - \frac{1}{8}\left(\frac{9}{16}x^4\right)\right]$$

$$= 2\left[1 - \frac{3}{8}x^2 - \frac{9}{128}x^4\right]$$

$$= 2 - \frac{3}{4}x^2 - \frac{9}{64}x^4$$

From $\left(1 - \frac{3}{4}x^2\right)$ expansion valid for $\left|\frac{3}{4}x^2\right| < 1$ *or* $|x| < \sqrt{\frac{4}{3}}$

2 Expand $\frac{4 - x}{(2 + x)^2}$ up to and including the term in x^3

Solution:

$$\frac{4 - x}{(2 + x)^2} = (4 - x)(2 + x)^{-2} = (4 - x) \, . \, 2^{-2} . \left(1 + \frac{1}{2}x\right)^{-2}$$

$$= \frac{1}{4}(4 - x)\left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \ldots\right)$$

$$= \frac{1}{4}\left(4 - 4x + 3x^2 - 2x^3 - x + x^2 - \frac{3}{4}x^3 + \ldots\right)$$

$$= \frac{1}{4}\left(4 - 3x + 4x^2 - \frac{11}{4}x^3 + \ldots\right)$$

$$= 1 - \frac{3}{4}x + x^2 - \frac{11}{16}x^3$$

From $\left(1 + \frac{1}{2}x\right)$ expansion valid for $\left|\frac{1}{2}x\right| < 1$ *or* $|x| < 2$

63.7 Simplifying with Partial Fractions

63.7.1 Example:

1 Use partial fractions to expand $\frac{1}{(1+x)(1-2x)}$ in ascending powers of x up to the term in x^3

Solution:

$$\frac{1}{(1+x)(1-2x)} \equiv \frac{A}{(1+x)} + \frac{B}{(1-2x)}$$

$$\equiv \frac{A(1-2x)}{(1+x)} + \frac{B(1+x)}{(1-2x)}$$

$$\therefore \quad 1 \equiv A(1-2x) + B(1+x)$$

Let $x = \frac{1}{2}$ $\quad 1 = A \times (0) + 1\frac{1}{2}B$

$$1 = \frac{3}{2}B \quad \Rightarrow \quad B = \frac{2}{3}$$

Let $x = -1$ $\quad 1 = A \times (1+2) + B \times (0)$

$$1 = 3A \quad \Rightarrow \quad A = \frac{1}{3}$$

$$\therefore \frac{1}{(1+x)(1-2x)} \equiv \frac{1}{3(1+x)} + \frac{2}{3(1-2x)}$$

Expand each term separately then add together:

$$\frac{1}{3(1+x)} = \frac{1}{3}(1+x)^{-1} = \frac{1}{3}\left[1 + (-1)x + \frac{(-1)(-1-1)}{2!}x^2 + \frac{(-1)(-1-1)(-12)}{3!}x^3\right]$$

$$= \frac{1}{3}\left[1 - x + \frac{(2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 \ldots\right]$$

$$= \frac{1}{3}[1 - x + x^2 - x^3 \ldots] \qquad = \frac{1}{3} - \frac{x}{3} + \frac{x^2}{3} - \frac{x^3}{3} \ldots$$

$$\frac{2}{3(1-2x)} = \frac{2}{3}(1-2x)^{-1}$$

$$= \frac{2}{3}\left[1 + (-1)(-2x) + \frac{(-1)(-1-1)}{2!}(-2x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-2x)^3\right]$$

$$= \frac{2}{3}[1 + 2x + 4x^2 + 8x^3 \ldots] \qquad = \frac{2}{3} + \frac{4x}{3} + \frac{8x^2}{3} + \frac{16x^3}{3} \ldots$$

Now combine the expansions:

$$\therefore \frac{1}{(1+x)(1-2x)} = \left[\frac{1}{3} - \frac{x}{3} + \frac{x^2}{3} - \frac{x^3}{3}\right] + \left[\frac{2}{3} + \frac{4x}{3} + \frac{8x^2}{3} + \frac{16x^3}{3}\right] \ldots$$

$$= 1 + x + 3x^2 + 5x^3 \ldots$$

Note that $\frac{2}{3}(1-2x)^{-1}$ is valid when $|-2x| < 1$ or when $|x| < \frac{1}{2}$

Note that $\frac{1}{3}(1+x)^{-1}$ is valid when $|x| < 1$

$\therefore$ combined expansion is valid when $|x| < \frac{1}{2}$

63.8 Binomial Theorem Digest:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \dots$$

$$+ \dots \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots \qquad \text{Valid for } |x| < 1$$

$$(a + bx)^n = \left[a\left(1 + \frac{bx}{a}\right)\right]^n = a^n\left(1 + \frac{bx}{a}\right)^n$$

$$= a^n\left[1 + n\frac{b}{a}x + \frac{n(n-1)}{2!}\left(\frac{b}{a}x\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}x\right)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{b}{a}x\right)^4\right]$$

Valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \frac{a}{b}$

- For the general Binomial Theorem any rational value of n can be used (i.e. fractional or negative values, and not just positive integers).
- For these expansions, the binomial must start with a 1 in the brackets. For binomials of the form $(a + bx)^n$, the a term must be factored out.

  Therefore, the binomial $(a + bx)^n$ must be changed to $a^n\left(1 + \frac{b}{a}x\right)^n$.
- When n is a positive integer the series is finite and gives an exact value of $(1 + x)^n$ and is valid for all values of x. The expansion terminates after $n + 1$ terms, because coefficients after this term are zero.
- When n is either a fractional and/or a negative value, the series will have an infinite number of terms. and the coefficients are never zero.
  - In these cases the series will either diverge and the value will become infinite or they will converge, with the value converging towards the value of binomial $(1 + x)^n$.
  - The general Binomial Theorem will converge when $|x| < 1$ ($i.e.\ -1 < x < 1$). This is the condition required for convergence and we say that the series is valid for this condition.
  - For binomials of the form $a^n\left(1 + \frac{b}{a}x\right)^n$, the series is only valid when $\left|\frac{b}{a}x\right| < 1$, or $|x| < \frac{a}{b}$
  - The range must always be stated.
  - When the series is convergent it will make a good approximation of $(1 + x)^n$ depending on the number of terms used, and the size of x. Small is better.

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots$$

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \dots$$

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

All valid for $|x| < 1$

64 • C4 • Parametric Equations

64.1 Intro to Parametric Equations

Some relationships between the variables x and y are so complicated that it is often convenient to express x and y in terms of a **third** variable called a **parameter**.

E.g. $x = t^4$ (1)
$y = t^3 - t$ (2)

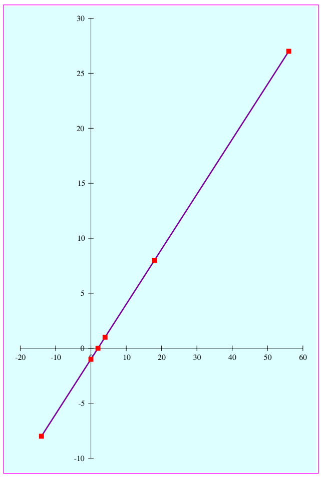
The equations (1) & (2) are called the **parametric equations** of the curve. By eliminating t from both equations it is possible to find a direct relationship between x and y, which is of course the **Cartesian** equation of the curve. (In this example we obtain $y = x^{\frac{3}{4}} - x^{\frac{1}{4}}$, a very tricky equation to deal with, which nicely illustrates the reason for using parametric equations).

64.1.2 Example:

1 If $y = t^3$, & $x = 2t^3 + 2$ sketch a graph to see how the curve is represented.

Solution:

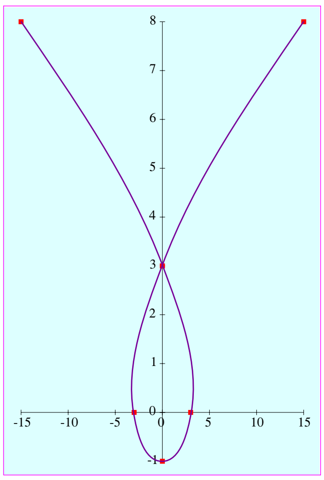
$t = -2 \quad x = -14 \quad y = -8$
$t = -1 \quad x = 0 \quad y = -1$
$t = 0 \quad x = 2 \quad y = 0$
$t = 1 \quad x = 4 \quad y = 1$
$t = 2 \quad x = 18 \quad y = 8$
$t = 3 \quad x = 56 \quad y = 27$



2 Sketch the curve for $x = t^3 - 4t$ & $y = t^2 - 1$

Solution:

t	x	y
−3	−15	8
−2	0	3
−1	3	0
0	0	−1
1	−3	0
2	0	3
3	15	8



64.2 Converting Parametric to Cartesian format

To convert to Cartesian equations:

- Rearrange the x-equation to get t on its own
- Substitute this into the y-equation.
- or visa versa! Choose the simpler of the two equations to find t = ?

64.2.1 Example:

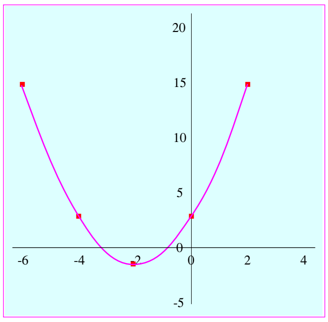
1 Express the parametric equation $x = t - 2$ & $y = t^2 - 1$ in cartesian form and sketch the curve.

Solution:

$$x = t - 2 \qquad \therefore t = x + 2$$

$$y = t^2 - 1 \qquad \therefore y = (x + 2)^2 - 1$$

x	y
−6	15
−4	3
−2	−1
0	3
2	15
4	35



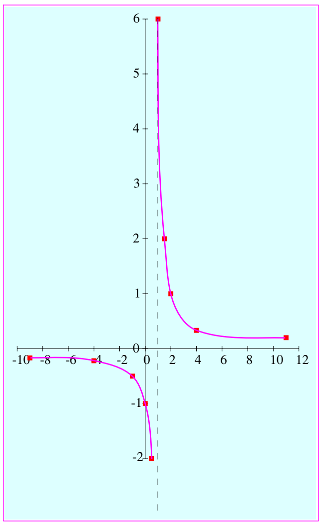
2 Sketch the curve of the parametric equation $x = t + 1$ & $y = \frac{1}{t}$

Solution:

$$x = t + 1 \qquad \therefore t = x - 1$$

$$y = \frac{1}{t} \qquad \therefore y = \frac{1}{x - 1}$$

x	y
−9	−0·1
−4	−0·2
−1	−0·5
−0·5	−0·666
0	−1
0·5	−2
1	∞
1·5	2
2	1
4	0·333
11	0·2



3

Express the parametric equation $x = \frac{1}{1 + t}$ & $y = t^2 + 4$ in cartesian form.

Solution:

$$x = \frac{1}{1 + t}$$

$$1 + t = \frac{1}{x}$$

$$t = \frac{1}{x} - 1$$

$$\therefore y = \left(\frac{1}{x} - 1\right)^2 + 4$$

$$y = \frac{1}{x^2} - \frac{2}{x} + 1 + 4$$

$$y = \frac{1}{x^2} - \frac{2}{x} + 5$$

4 Show that the parametric equation $x = at + \frac{1}{t^n}$ and $y = at - \frac{1}{t^n}$ can be given in the cartesian form as:

$$(x - y)(x + y)^n = 2^{n+1}a^n$$

Solution:
Substitute for x & y in the LHS of the above equation:

$$\left[\left(at + \frac{1}{t^n}\right) - \left(at - \frac{1}{t^n}\right)\right]\left[at + \frac{1}{t^n} + at - \frac{1}{t^n}\right]^n \Rightarrow$$

$$\left(\frac{2}{t^n}\right) \times (2at)^n \Rightarrow$$

$$\frac{2}{t^n} \times 2^n a^n t^n = 2^{n+1}a^n$$

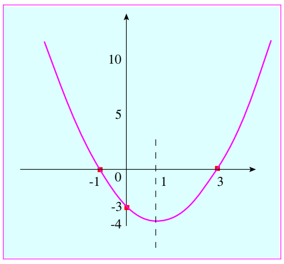
64.3 Sketching a Curve from a Parametric Equation

64.3.1 Example:

1 Sketch the curve $x = 1 - t$, $y = t^2 - 4$

Solution:
y-axis is cut at: $x = 0$ $\therefore 1 - t = 0$ $\Rightarrow$ $t = 1$ $\therefore y = -3$
Co-ordinate of y-axis cut at $(0, -3)$

x-axis is cut at: $y = 0$ $\therefore t^2 - 4 = 0$ $\Rightarrow$ $t^2 = 4$ $t = \pm 2$ $\therefore x = -1, 3,$
Co-ordinate of x-axis cut at $(-1, 0)$ and $(3, 0)$



Since t^2 is never −ve, the minimum value of y is −4
For all values of $y > -4$ there are 2 values of x in the form of $1 \pm k$.
Hence curve is symmetrical about the line $x = -1$.

64.4 Parametric Equation of a Circle

Circle centre (0, 0) radius r:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Circle centre (a, b) radius r:

$$x = a + r\cos\theta \qquad y = b + r\sin\theta$$

64.5 Differentiation of Parametric Equations

Two methods can be used:

- Eliminate the parameter and differentiate normally or
- Use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad \text{and} \qquad \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

64.5.1 Example:

1 Find the gradient of the curve $x = t^2$, $y = 2t$ at the point where $t = 3$

Solution: Method 1

$$t = \sqrt{x} \qquad \therefore \qquad y = 2\sqrt{x} = 2x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} \qquad = \frac{1}{\sqrt{x}}$$

When $t = 3$, $x = 3^2 = 9$

Gradient $\frac{dy}{dx} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

Method 2

$$x = t^2 \qquad \therefore \qquad \frac{dx}{dt} = 2t$$

$$y = 2t \qquad \therefore \qquad \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2 \times \frac{1}{2t} = \frac{1}{t}$$

When $t = 3$, $\frac{dy}{dx} = \frac{1}{t} = \frac{1}{3}$

2 Find the equation of the normal at the point (–8, 4) to the curve given parametrically by:

$$x = t^3, \qquad y = t^2$$

Solution:

$$t = x^{\frac{1}{3}} \qquad \Rightarrow \qquad y = x^{\frac{2}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$$

Gradient: $= \frac{2}{3}(-8)^{-\frac{1}{3}} = \frac{2}{3}.\frac{1}{(-2)} = -\frac{1}{3}$

Gradient of normal given by: $m_1 m_2 = -1 \qquad \therefore$ Gradient $= 3$

Equation of line given by $y - y_1 = m(x - x_1)$

$$\Rightarrow \qquad y - 4 = 3(x + 8)$$

$$\Rightarrow \qquad y = 3x + 28$$

3 Find the turning points on the curve given by $x = t, \; y = t^3 - 3t$

Solution:

$$\frac{dx}{dt} = 1, \qquad \frac{dt}{dx} = 1$$

$$\frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \left(3t^2 - 3\right) \times 1 = 3t^2 - 3$$

At the turning points $\frac{dy}{dx} = 0$

$$3t^2 - 3 = 0$$

$$3t^2 = 3$$

$$t^2 = 1$$

$$t = \sqrt{1} = \pm 1$$

From start equations:

When $t = 1 \;\Rightarrow\; x = 1$ & $y = -2$

When $t = -1 \;\Rightarrow\; x = -1$ & $y = 2$

Co-ordinates of turning points are $(1, -2)$, $(-1, 2)$

$x =$	$-1\frac{1}{2}$	-1	0		0	1	2
$t =$	$-1\frac{1}{2}$	-1	0		0	1	2
Sign $\frac{dy}{dx}$	+	0	−		−	0	+
Shape	/	—	\		\	—	/

$\therefore$ $(-1, 2)$ is a max $\qquad (1, -2)$ is a min

4 Find the equation of the general tangent to the curve given by $x = t, \; y = \frac{1}{t}$

Solution:

$$\frac{dx}{dt} = 1, \qquad \frac{dt}{dx} = 1$$

$$y = \frac{1}{t} = t^{-1} \qquad \Rightarrow \qquad \frac{dy}{dt} = -t^{-2} = -\frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{1}{t^2} \times 1 = -\frac{1}{t^2}$$

Need a general equation, so use point $\left(t, \; \frac{1}{t}\right)$ with a gradient of $-\frac{1}{t^2}$

$$\therefore \qquad y - y_1 = m(x - x_1)$$

$$y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$$

$\times$ by t^2 $\qquad t^2 y - t = -x + t$

$$t^2 y - t = -x + t$$

$$t^2 y + x - 2t = 0$$

5 Find the equation of the normal to the curve given by $x = t^2,\ y = t + \frac{1}{t}$

Solution:

$$x = t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2t, \quad \frac{dt}{dx} = \frac{1}{2t}$$

$$y = t + \frac{1}{t} = t + t^{-1} \quad \Rightarrow \quad \frac{dy}{dt} = 1 - t^{-2} = 1 - \frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \left(1 - \frac{1}{t^2}\right)\frac{1}{2t}$$

$$= \left(\frac{t^2 - 1}{t^2}\right)\frac{1}{2t}$$

$$= \frac{t^2 - 1}{2t^3}$$

When $t = 2$ $\quad \frac{dy}{dx} = \frac{2^2 - 1}{2 \times 2^3} = \frac{4 - 1}{16} = \frac{3}{16}$

When $t = 2$ $\quad x = 2^2 = 4, \quad y = 2 + \frac{1}{2} = \frac{5}{2}$

So we want equation of normal through point $\left(4,\ \frac{5}{2}\right)$

Gradient of tangent $= \frac{3}{16}$ $\qquad \therefore$ Gradient of normal $= -\frac{16}{3}$

Equation of normal is: $\quad y - y_1 = m(x - x_1)$

$$y - \frac{5}{2} = -\frac{16}{3}(x - 4)$$

$$y - \frac{5}{2} = -\frac{16x}{3} + \frac{64}{3}$$

$\times 6$ $\qquad 6y - 15 = -32x + 128$

$$6y + 32x - 143 = 0$$

6 If $x = t^2 - 3t$ $\quad$ & $\quad y = 4t^3 - 3t^2 - 18t + 5$

Find $\frac{dy}{dx}$ when $t = 2$

Solution:

$$\frac{dx}{dt} = 2t - 3, \quad \frac{dy}{dt} = 12t^2 - 6t - 18$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{12t^2 - 6t - 18}{2t - 3}$$

Ans : $\quad = \frac{48 - 12 - 18}{4 - 3} = 18$

7 Take the parametric curve defined by $x = 2t^2$ & $y = 4t$ with two points with the following co-ordinates, $P(2p^2,\ 4p)$ & $Q(2q^2,\ 4q)$.

a) Find the gradient of the normal to the curve at P

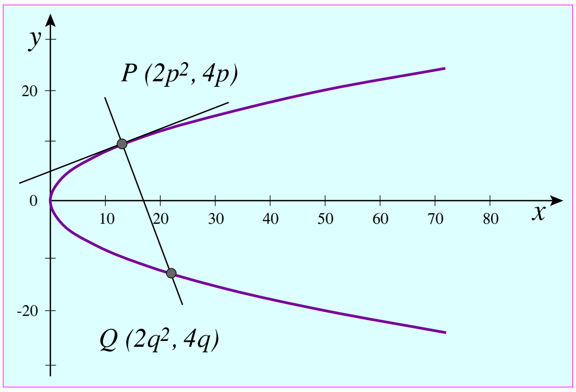
b) Find the gradient of the chord PQ

c) Show that $p^2 + pq + 2 = 0$ when chord PQ is normal to the curve at P

d) The normal to a point $U(8,\ 8)$ meets the curve again at point V. The normal to point V crosses the curve at point W. Find the co-ordinates of W.

Step 1 ----- Draw a sketch!!!!!!

$$t = \frac{y}{4} \quad \Rightarrow \quad x = 2\left(\frac{y}{4}\right)^2 \quad \Rightarrow \quad y^2 = 8x$$



a) Find the gradient at point P:

$$x = 2t^2 \ \&\ y = 4t$$

$$\therefore \frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 4$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 4 \times \frac{1}{4t} = \frac{1}{t}$$

The normal m_2 is given by: $m_2 \times \frac{1}{t} = -1$

$$\therefore m_2 = -t$$

At point $P(2p^2,\ 4p)$; $y = 4p \quad \Rightarrow \quad \therefore 4p = 4t \quad \Rightarrow \quad p = t$

The gradient of the normal at point $P = -p$

b) The gradient of a straight line is $\dfrac{y_1 - y_2}{x_1 - x_2}$

For the line PQ $\qquad m = \dfrac{4p - 4q}{2p^2 - 2q^2}$

Simplifying : $\qquad = \dfrac{2(p - q)}{(p - q)(p + q)} = \dfrac{2}{(p + q)}$

c) The line PQ is normal to the curve at P. Hence:

$$\frac{2}{(p + q)} = -p$$

$$2 = -p(p + q) = -p^2 - pq$$

$$\therefore p^2 + pq + 2 = 0$$

d) Consider the line UV as the same as PQ

For U (8, 8) $y \equiv 4p = 8 \quad \Rightarrow \quad p = 2$

Find the value of q using $p^2 + pq + 2 = 0$

$p = 2 : \quad 4 + 2q + 2 = 0$

$\therefore 2q = -6 \quad \Rightarrow \quad q = -3$

Co-ordinates of $V = (2q^2, 4q) = (18, -12)$

Now consider the line VW as the same as PQ.

So $4p = -12 \quad \Rightarrow \quad p = -3$

Find the value of q using $p^2 + pq + 2 = 0$

$p = -3 : \quad 9 - 3q + 2 = 0 \quad \therefore \; -3q = -11 \quad \Rightarrow \quad q = \dfrac{11}{3}$

$$\text{Co-ordinates of } W = \left(2 \times \left(\frac{11}{3}\right)^2, \; 4 \times \left(\frac{11}{3}\right)\right) = \left(\frac{242}{9}, \frac{44}{3}\right)$$

$$= \left(26\frac{8}{9}, \; 14\frac{2}{3}\right)$$

65 • C4 • Differentiation: Implicit Functions

65.1 Intro to Implicit Functions

For the most part we have dealt with '**explicit functions**' of x, where a value of y is defined only in terms of x.

Functions have been in the form $y = f(x)$, and the derivative $\frac{dy}{dx} = f'(x)$ is obtained by differentiating w.r.t x.

However, some functions cannot be rearranged into the simpler form of $y = f(x)$, or $x = f(y)$.

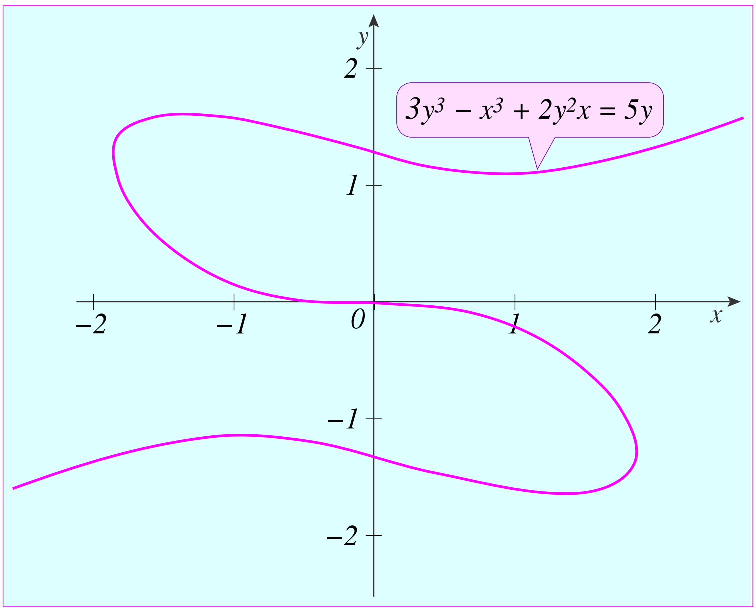
If we cannot express y solely in terms of x, we say y is given **implicitly** by x. Similarly, if we cannot express x solely in terms of y, we say x is given **implicitly** by y.

Even so, given a value of x, a value for y can still be found, after a bit of work.

E.g.	$y = 2x^2 - 3x + 4$	is expressed explicitly in terms of x.
	$x^2 + y^2 - 6x + 2y = 0$	is expressed implicitly.

Typical examples of implicit functions are found in the equations of circles, ellipses and hyperbolae.

An example implicit function showing the complex shape given by a cubic function.



65.2 Differentiating Implicit Functions

By now, differentiating an explicit function, such as $y = f(x)$, should have become second nature.
So much so, that without thinking, the first thing we write down when we see a differential is $\frac{dy}{dx} = \ldots$

In differentiating an implicit function, this blind technique won't work, since you cannot make y the subject of the equation first.

To differentiate an implicit function, we differentiate **both sides of the equation term by term** w.r.t x.

In fact, this is what we have always done, but we tend to forget that the differential of y wrt to x is $\frac{dy}{dx}$.

The difficult part of dealing with these functions is knowing what to do with terms such as y^2, x^3y^2 and this is where the chain and product rules come to the rescue.

We use the chain rule such that:

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y) \times \frac{dy}{dx}$$

Simply stated, the chain rule says take the differential of the outside function and multiply by the differential of the inside function.

The general rule for implicit functions becomes: differentiate the x bits as normal, and then the y bits w.r.t y and multiply by $\frac{dy}{dx}$.

Remember that any terms in y now differentiate to multiples of $\frac{dy}{dx}$.

65.2.1 Example:

Find $\frac{dy}{dx}$ if: $x^2 + 2y - y^2 = 5$

Differentiate both sides of the equation & consider each term:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2y) - \frac{d}{dx}(y^2) = 0$$

Term 1: $\frac{d}{dx}(x^2) = 2x$

Term 2: $\frac{d}{dx}(2y) = ?$ Use the chain rule to differentiate a y term w.r.t x:

$$\frac{d}{dx}(2y) = \frac{d}{dy}(2y) \times \frac{dy}{dx} \Rightarrow 2\frac{dy}{dx}$$

Term 3: $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx}$

Combining the resulting terms and rearrange to give $\frac{dy}{dx}$

$$2x + 2\frac{dy}{dx} - 2y\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad x + \frac{dy}{dx} - y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} - y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx}(1 - y) = -x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{-x}{(1 - y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{(y - 1)}$$

65.3 Differentiating Terms in y w.r.t x

Terms in y differentiate to multiples of $\frac{dy}{dx}$ using the chain rule.

65.3.1 Example:

1 Differentiate w.r.t to x: $x^2 + y^2 + 3y = 8$

Solution:

Differentiate both sides of the equation & consider each term:

$$2x + \frac{d}{dx}(y^2) + \frac{d}{dx}(3y) = 0$$

Assign chain rule to the y^2 term: $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx}$

$$\therefore \quad 2x + 2y\frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

Rearrange: $(2y + 2)\frac{dy}{dx} = -2x$

$$\therefore \quad \frac{dy}{dx} = \frac{-2x}{2y + 2}$$

2 Differentiate w.r.t to x: $x^2 + y^2 - 6x + 2y = 0$

Solution:

$$2x + 2y\frac{dy}{dx} - 6 + 2\frac{dy}{dx} = 0$$

Rearrange: $2x + (2y + 2)\frac{dy}{dx} - 6 = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{6 - 2x}{2y + 2}$$

3 Find an expression for the gradient of the curve: $3x^2 - 2y^3 = 1$

Solution:

$$6x - 6y^2\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{6x}{6y^2} = \frac{x}{y^2}$$

4 Differentiate w.r.t to x: $y = a^x$

Solution:

Take logs both sides: $\ln y = \ln a^x = x \ln a$

Differentiate w.r.t to x: $\frac{1}{y}\frac{dy}{dx} = \ln a$

$$\therefore \quad \frac{dy}{dx} = y \ln a$$

but: $y = a^x \quad \therefore \quad \frac{dy}{dx} = a^x \ln a$

Hence: $\frac{d}{dx}(a^x) = a^x \ln a$

5 Differentiate w.r.t to x: $\sin(x + y) = \cos 2y$

Solution:

Differentiate both sides of the equation & consider each term:

$$\frac{d}{dx}\left[\sin(x + y)\right] = \frac{d}{dx}(\cos 2y)$$

Assign chain rule to LHS:
$$\frac{d}{dx}\left[\sin(x + y)\right] = \frac{d}{dy}\left[\sin(x + y)\right] \times \frac{d}{dx}(x + y)$$
$$\frac{d}{dx}\left[\sin(x + y)\right] = \cos(x + y) \times \left(1 + \frac{dy}{dx}\right)$$

Use chain rule on the RHS:
$$\frac{d}{dx}(\cos 2y) = -\sin(2y) \times \frac{d}{dx}(2y)$$
$$\frac{d}{dx}(\cos 2y) = -\sin(2y) \times 2\frac{dy}{dx}$$

$$\therefore \qquad \left(1 + \frac{dy}{dx}\right)\cos(x + y) = -2\sin(2y)\frac{dy}{dx} \qquad \ldots (1)$$

$$\cos(x + y) + \cos(x + y)\frac{dy}{dx} = -2\sin(2y)\frac{dy}{dx}$$

$$2\sin(2y)\frac{dy}{dx} + \cos(x + y)\frac{dy}{dx} = -\cos(x + y)$$

$$\frac{dy}{dx} = \frac{-\cos(x + y)}{2\sin(2y) + \cos(x + y)}$$

It is not necessary to find the expression for gradient unless asked for. To find a gradient from given coordinates just substitute into equation (1), then rearrange for $\frac{dy}{dx}$.

65.4 Differentiating Terms with a Product of x and y

These need to be treated as a product of two functions, hence, we use the product and chain rules to differentiate them.

Recall:

$$\text{If} \quad y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The examples 1 & 3 below show the product and chain rule used in full. Once mastered, we can generally differentiate powers of y normally w.r.t y and append $\frac{dy}{dx}$.

65.4.1 Example:

1 Differentiate w.r.t to x: xy^2

Solution:

$$\text{Let } u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\text{Let } v = y^2 \quad \Rightarrow \quad \frac{dv}{dx} = \frac{dv}{dy} \times \frac{dy}{dx} = 2y\frac{dy}{dx}$$

Use chain rule

$$\therefore \quad \frac{d}{dx}(xy^2) = u\frac{dv}{dx} + v\frac{du}{dx} = x \,.\, 2y\frac{dy}{dx} + y^2 .\, 1$$

$$= 2xy\frac{dy}{dx} + y^2$$

2 Using the result from (1) above, differentiate w.r.t to x: $x^3 + xy^2 - y^3 = 5$

Solution:

$$3x^2 + \left[2xy\frac{dy}{dx} + y^2\right] - 3y^2\frac{dy}{dx} = 0$$

$$3x^2 + y^2 + \frac{dy}{dx}[2xy - 3y^2] = 0$$

$$\frac{dy}{dx}[2xy - 3y^2] = -3x^2 - y^2$$

$$\frac{dy}{dx} = \frac{-3x^2 - y^2}{[2xy - 3y^2]} = \frac{3x^2 + y^2}{[3y^2 - 2xy]}$$

3 Differentiate w.r.t to x: $y = xe^y$

Solution:

$$\text{Let } u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\text{Let } v = e^y \quad \Rightarrow \quad \frac{dv}{dx} = \frac{dv}{dy} \times \frac{dy}{dx} = e^y\frac{dy}{dx}$$

$$\frac{dy}{dx} = x\,e^y\frac{dy}{dx} + e^y$$

4 If $e^x y = \sin x$ show that: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

Solution:

Differentiate both sides of the equation w.r.t to x :

$$e^x \frac{dy}{dx} + ye^x = \cos x \qquad \text{Product rule: } u = e^x \qquad v = y$$

2nd Differentiation $$e^x \frac{d^2y}{dx^2} + e^x \frac{dy}{dx} + ye^x + \frac{dy}{dx}e^x = -\sin x$$

But $$\sin x = e^x y$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y + \frac{dy}{dx} = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

5 Differentiate w.r.t x: $5x^4 + x^2y^3 + 5y^2 = 0$

Solution:

Differentiate both sides of the equation w.r.t to x :

$$20x^3 + \frac{d}{dx}(x^2y^3) + \frac{d}{dx}(5y^2) = 0$$

Product rule: $u = x^2 \qquad v = y^3$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 3y^2\frac{dy}{dx}$$

Now: $$\frac{d}{dx}(x^2y^3) = \left[x^2\, 3y^2\frac{dy}{dx} + y^3\, 2x\right]$$

and $$\frac{d}{dx}(5y^2) = 10y\frac{dy}{dx}$$

$$20x^3 + \left[x^2\, 3y^2\frac{dy}{dx} + y^3\, 2x\right] + 10y\frac{dy}{dx} = 0$$

$$20x^3 + 3x^2y^2\frac{dy}{dx} + 2xy^3 + 10y\frac{dy}{dx} = 0$$

$$3x^2y^2\frac{dy}{dx} + 10y\frac{dy}{dx} = -20x^3 - 2xy^3$$

$$\frac{dy}{dx} = \frac{-20x^3 - 2xy^3}{3x^2y^2 + 10y}$$

6 Differentiate w.r.t x: $\sqrt{xy}$

Solution:

Differentiate both sides of the equation w.r.t to x (2 methods):

$$\sqrt{xy} \quad \Rightarrow \quad (xy)^{\frac{1}{2}} \quad \Rightarrow \quad x^{\frac{1}{2}}y^{\frac{1}{2}}$$

(1) $$\frac{d}{dx}\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) = x^{\frac{1}{2}}\,\frac{1}{2}\,y^{-\frac{1}{2}}\frac{dy}{dx} + y^{\frac{1}{2}}\,\frac{1}{2}\,x^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\frac{dy}{dx} + \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)$$

(2) $$\frac{d}{dx}(xy)^{\frac{1}{2}} = \frac{1}{2}(xy)^{-\frac{1}{2}}\left[\frac{d}{dx}(xy)\right] = \frac{1}{2}(xy)^{-\frac{1}{2}}\left[x\frac{dy}{dx} + y\right] = \frac{1}{2}\left[\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\frac{dy}{dx} + \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right]$$

65.5 Tangents and Normals of Implicit Functions

65.5.1 Example:

1 Find the equation of the tangent at (x_1, y_1) to the curve: $x^2 + 2y^2 - 6y = 0$

Solution:

Differentiate both sides of the equation w.r.t to x :

$$2x - 4y\frac{dy}{dx} - 6\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}(4y + 6) = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{4y + 6}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{2y + 3}$$

At (x_1, y_1) $\quad \frac{dy}{dx} = \frac{x_1}{2y_1 + 3}$

Equation of tangent is: $\quad y - y_1 = \frac{x_1}{2y_1 + 3}(x - x_1)$

2 Find the gradient for the curve $x^2 + 2xy - 2y^2 + x = 2$ at the point $(-4, 1)$.

Solution:

Differentiate both sides of the equation w.r.t to x :

$$2x + \frac{d}{dx}(2xy) - \frac{d}{dx}(2y^2) + 1 = 0$$

But: $\quad \frac{d}{dx}(2xy) = \left[2x\frac{dy}{dx} + 2y\right]$ $\qquad$ Product rule: $u = 2x \qquad v = y$

and $\quad \frac{d}{dx}(2y^2) = 4y\frac{dy}{dx}$

$$2x + \left[2x\frac{dy}{dx} + 2y\right] - 4y\frac{dy}{dx} + 1 = 0$$

$$2x + 2x\frac{dy}{dx} + 2y - 4y\frac{dy}{dx} + 1 = 0 \qquad \ldots (1)$$

$$2x\frac{dy}{dx} - 4y\frac{dy}{dx} = -2x - 2y - 1$$

$$\frac{dy}{dx}(2x - 4y) = -2x - 2y - 1$$

$$\frac{dy}{dx} = \frac{-2x - 2y - 1}{(2x - 4y)} = \frac{2x + 2y + 1}{4y - 2x}$$

When at $(-4, 1)$ $\quad \frac{dy}{dx} = \frac{2(-4) + 2(1) + 1}{4(1) - 2(-4)} = -\frac{5}{12}$

Alternatively, save time by substituting the given coordinates in the earlier equation (1)

$$-8 - 8\frac{dy}{dx} + 2 - 4\frac{dy}{dx} + 1 = 0$$

$$-12\frac{dy}{dx} = 5 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{5}{12}$$

3 Find the equation of the tangent to the curve $3x^2 - xy - 2y^2 + 12 = 0$ at the point (2, 3)

Solution:

Differentiate both sides of the equation w.r.t to x :

$$6x - x\frac{dy}{dx} - y - 4y\frac{dy}{dx} = 0$$

$$-x\frac{dy}{dx} - 4y\frac{dy}{dx} = y - 6x$$

$$\frac{dy}{dx} = \frac{6x - y}{x + 4y}$$

Gradient at point (2, 3) $\quad \frac{dy}{dx} = \frac{6(2) - 3}{2 + 4(3)} = \frac{9}{14}$

Equation of the tangent at point (2, 3)

$$y - 3 = \frac{9}{14}(x - 2)$$

$$14y = 9x + 24$$

4 Find the gradient of the curve $x^3y - 7 = \sin\left(\frac{\pi}{2}y\right)$ at the point where $y = 1$

Solution:

Find the x-coordinate to start with:

When $y = 1, \quad \Rightarrow \quad x^3 - 7 = \sin\left(\frac{\pi}{2}\right)$

$$\therefore \quad x^3 = 1 + 7$$

$$x = \sqrt[3]{8} = 2$$

Differentiate both sides of the equation w.r.t to x :

$$\frac{d}{dx}\left(x^3y\right) - 0 = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2}y\right)\right]$$

Now: $\frac{d}{dx}\left(x^3y\right) = \left[x^3\frac{dy}{dx} + y3x^2\right]$ $\qquad$ Product rule: $u = x^3 \qquad v = y$

and: $\frac{d}{dx}\left[\sin\left(\frac{\pi}{2}\right)\right] = \cos\left(\frac{\pi}{2}y\right) \times \frac{d}{dx}\left(\frac{\pi}{2}y\right) = \cos\left(\frac{\pi}{2}y\right) \times \frac{\pi}{2}\frac{dy}{dx}$ $\qquad$ Chain rule

$$\therefore \; x^3\frac{dy}{dx} + 3x^2y = \frac{\pi}{2}\cos\left(\frac{\pi}{2}y\right)\frac{dy}{dx}$$

When $x = 2, \; y = 1, \quad \Rightarrow \quad 8\frac{dy}{dx} + 3 \times 4 \times 1 = 0$

$$\therefore \quad 8\frac{dy}{dx} = -12$$

$$\frac{dy}{dx} = -\frac{12}{8} = -\frac{3}{2}$$

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65.6 Stationary Points in Implicit Functions

65.6.1 Example:

1 Find the gradient for the curve $y^2 - xy + 4x^2 = 6$ at the point where $x = 1$.

Solution:

Differentiate both sides of the equation w.r.t to x :

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(xy) + 8x = 0$$

Now: $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

and: $\frac{d}{dx}(xy) = \left[x\frac{dy}{dx} + y\right]$ Product rule: $u = x \quad v = y$

$$\therefore \quad 2y\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\right] + 8x = 0$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} - y + 8x = 0 \qquad \ldots (1)$$

$$\frac{dy}{dx}(2y - x) = y - 8x$$

$$\frac{dy}{dx} = \frac{y - 8x}{(2y - x)}$$

When $x = 1$ $\quad y^2 - y + 4 - 6 = 0 \quad \Rightarrow \quad y^2 - y - 2 = 0$

$\therefore \quad (y + 1)(y - 2) = 0 \quad \Rightarrow \quad y = -1 \qquad y = 2$

$(1, -1) \quad \frac{dy}{dx} = \frac{-1 - 8}{(-2 - 1)} = \frac{-9}{-3} = 3$

$(1, 2) \quad \frac{dy}{dx} = \frac{2 - 8}{(4 - 1)} = \frac{-6}{3} = -2$

Show that at the stationary points: $10x^2 - 1 = 0$

$$\frac{dy}{dx} = \frac{y - 8x}{(2y - x)} = 0$$

$$\therefore \; y - 8x = 0$$

$$y = 8x$$

Substitute into the original function:

$$(8x)^2 - x(8x) + 4x^2 = 6$$

$$64x^2 - 8x^2 + 4x^2 - 6 = 0$$

$$60x^2 - 6 = 0$$

$$10x^2 - 1 = 0$$

Alternatively, recognise that $\frac{dy}{dx} = 0$ and substitute into the differential at (1)

$$2y\frac{dy}{dx} - x\frac{dy}{dx} - y + 8x = 0$$

$$0 - 0 - y + 8x = 0$$

$\therefore \quad y = 8x \qquad$ *etc.*

2 Find an expression for the x-coordinates of the stationary points of the equation:
$ax^2y + by^3 = cx + 6$

Solution:

Differentiate both sides of the equation w.r.t to x :

$$\frac{d}{dx}(ax^2y) + \frac{d}{dx}(by^3) = c$$

Now: $\frac{d}{dx}(ax^2y) = \left[ax^2\frac{dy}{dx} + y2ax\right]$ Product rule: $u = ax^2 \quad v = y$

and: $\frac{d}{dx}(by^3) = 3by^2\frac{dy}{dx}$

$$\left[ax^2\frac{dy}{dx} + y\,2ax\right] + 3by^2\frac{dy}{dx} = c$$

$$ax^2\frac{dy}{dx} + 2axy + 3by^2\frac{dy}{dx} = c \qquad \ldots (1)$$

At the stationary point $\frac{dy}{dx} = 0$, so substitute this into the differential at (1).
Then find an expression for y and substitute that into the original equation.

$$\therefore \quad 0 + 2axy + 0 = c \quad \Rightarrow \quad 2axy = c$$

$$\therefore \quad y = \frac{c}{2ax} \qquad \ldots (2)$$

$$ax^2\frac{c}{2ax} + b\left(\frac{c}{2ax}\right)^3 = cx + 6 \qquad \text{Sub (2) into original equation}$$

$$\frac{c}{2}x + \frac{bc^3}{8a^3x^3} = cx + 6$$

$$\frac{bc^3}{8a^3x^3} = cx - \frac{c}{2}x + 6 \quad \Rightarrow \quad \frac{bc^3}{8a^3x^3} = \frac{c}{2}x + 6$$

The x-coordinate given by: $(cx + 3)x^3 = \frac{2bc^3}{8a^3}$

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65.7 Implicit Functions Digest

Function $f(y)$	Differential $\frac{dy}{dx} = f'(x)$
a	0
a^x	$a^x \ln a$
a^{kx}	$k\,a^{kx} \ln a$
xy	$x\frac{dy}{dx} + y$
x^2y	$x^2\frac{dy}{dx} + 2xy$

Function $f(y)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin(ky)$	$k\frac{dy}{dx}\cos(ky)$
$\cos(ky)$	$-k\frac{dy}{dx}\sin(ky)$
uv	$uv' + vu'$
$\frac{u}{v}$	$\frac{vu' - uv'}{v^2}$

66 • C4 • Differential Equations

66.1 Intro to Differential Equations

At last, after years of work, learning to differentiate and integrate various functions, we now get to put all this knowledge to work on practical problems.

A differential equation is one in which the variables x, y and one of the derivatives of y w.r.t x are connected in some way. For the purposes of this section we will only consider the first derivative of the function, $\frac{dy}{dx}$, although higher derivatives such as $\frac{d^2y}{dx^2}$ can be used. The first derivative leads to a **first order differential equation**.

The general form of a first order differential equation is:

$$f(y)\frac{dy}{dx} = g(x)$$

where f is a function of y only and g is a function of x only.

Typically, the differential will be w.r.t time t, such as a change of area with time, giving $\frac{dA}{dt}$.
Normally a differential equation is solved by eliminating the differential part by integration.

66.2 Solving by Separating the Variables

Differential equations are solved by separating the variables, which, in simple terms, means moving all the terms in y and dy to the LHS of the equation, and the terms in x and dx to the RHS. Both sides can then be integrated.

$$f(y)\frac{dy}{dx} = g(x)$$

$$\int f(y)\frac{dy}{dx}dx = \int g(x)\,dx \qquad \text{Integrate both sides w.r.t. } x$$

But: $$\frac{dy}{dx}dx = dy$$

$$\therefore \quad \int f(y)\,dy = \int g(x)\,dx$$

Technically $\frac{dy}{dx}$ is not a fraction, but can often be handled as if it were one.

Integrating both sides would normally give rise to a constant of integration on both sides, but convention has it that these are combined into one. This gives a general solution to the problem, with a whole family of curves being generated, depending on the value of the constant of integration. A particular solution is found when certain conditions are assumed, called the starting conditions, and the constant of integration can be calculated.

66.2.1 Example:

1

Solve: $\frac{dy}{dx} = xy^2 \quad \Rightarrow \quad \therefore \; \frac{1}{y^2}\,dy = x\,dx$

$$\int \frac{1}{y^2}\,dy = \int x\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + c$$

If $x = 0$, and $y = 0{\cdot}5$, $c = 0 - 2 = -2$

$$-\frac{1}{y} = \frac{x^2}{2} - 2$$

2 Find the general solution of $\frac{1}{x}\frac{dy}{dx} = \frac{2y}{x^2 + 1}$

Solution:

$$\frac{dy}{dx} = \frac{2xy}{x^2 + 1} \quad \Rightarrow \quad dy = \frac{2xy}{x^2 + 1}dx \quad \Rightarrow \quad \frac{1}{y}dy = \frac{2x}{x^2 + 1}dx$$

Tip: it is good practise to keep any constant (2 in the above case) in the numerator.

$$\therefore \quad \int \frac{1}{y}dy = \int \frac{2x}{x^2 + 1}dx \qquad \text{Recall: } \int \frac{f'(x)}{f(x)}\,dx = \ln |f(x)| + c$$

$$\therefore \quad \ln y = \ln (x^2 + 1) + c$$

Rearrange: $\ln y - \ln (x^2 + 1) = c \quad \Rightarrow \quad \ln\left(\frac{y}{x^2 + 1}\right) = c$

$$\therefore \quad \frac{y}{x^2 + 1} = e^c$$

$$y = e^c (x^2 + 1) \qquad \text{where } e^c \text{ is a constant, } k$$

$$\therefore \quad y = k(x^2 + 1)$$

3 A curve has an equation that satisfies the differential equation:

$$2\frac{dy}{dx} = \frac{\cos x}{y}$$

and which passes through the point (0, 2). Find the equation.

Solution:

$$2\,dy = \frac{\cos x}{y}\,dx \quad \Rightarrow \quad 2y\,dy = \cos x\,dx$$

$$\Rightarrow \int 2y\,dy = \int \cos x\,dx$$

$$\Rightarrow y^2 = \sin x + c \qquad \text{(general solution)}$$

Find c using (0, 2): $4 = \sin 0 + c \quad \therefore c = 4$

Ans : $y^2 = \sin x + 4$ (particular solution)

4 A curve is such that the gradient of the curve is proportional to the product of the x & y coordinates. If the curve passes through the points (2, 1) & $(4, e^2)$, find the equation.

Solution:

$$\frac{dy}{dx} \propto xy \quad \Rightarrow \quad \frac{dy}{dx} = kxy \quad \Rightarrow \quad \frac{1}{y}dy = kx\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}dy = \int kx\,dx \quad \Rightarrow \quad \ln y = \frac{kx^2}{2} + c$$

Find k & c using the given co-ordinates: $(4, e^2) \quad 2 = 8k + c$

$(2,\ 1) \quad 0 = 2k + c$

$$\therefore 6k = 2 \qquad k = \frac{1}{3} \qquad c = -\frac{2}{3}$$

substituting: $\ln y = \frac{x^2}{6} - \frac{2}{3} \quad \Rightarrow \ln y = \frac{x^2 - 4}{6}$

Ans : $y = e^{\frac{x^2 - 4}{6}}$

66.3 Rates of Change Connections

The key to doing these problems is to identify three components and write them down mathematically:

- What you are given
- What is required
- What is the connection between the two items above
  (Sometimes the chain rule must be used to establish a connection).

66.3.1 Example:

1 If $y = 4\cos 2\theta$ and θ increases at 5 radians per second, find the rate at which y is increasing when $\theta = 2\pi$.

Solution:

Given: $y = 4\cos 2\theta, \quad \frac{d\theta}{dt} = 5$

Required: $\frac{dy}{dt}$ when $\theta = 2\pi$

Connection (chain rule): $\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}$

$y = 4\cos 2\theta \quad \therefore \frac{dy}{d\theta} = -8\sin 2\theta$

$\frac{dy}{dt} = -8\sin 2\theta \times 5 = -40\sin 2\theta$

When $\theta = 2\pi$, $\frac{dy}{dt} = -40\sin 4\pi = 0$

In this case y is not increasing or decreasing.

2 A spherical balloon is inflated, such that its volume is increasing at a steady rate of 20 cm^3 per second. Find the rate of change of the surface area when the radius is 10 cm.

Solution:

Given that the volume increases: $\frac{dV}{dt} = 20$

Volume of a sphere is: $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dr} = 4\pi r^2$

Surface area of a sphere is: $A = 4\pi r^2 \quad \Rightarrow \quad \frac{dA}{dr} = 8\pi r$

We require the rate of change of area: $\frac{dA}{dt}$

From the chain rule: $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$

$\therefore \quad \frac{dA}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times 20$

$\therefore \quad \frac{dA}{dt} = \frac{40}{r}$

When $r = 10$

$$\frac{dA}{dt} = \frac{40}{10} = 4\ cm^2\ sec^{-1}$$

66.4 Exponential Growth and Decay

The general form of exponential growth is:

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\frac{dy}{dt} = k$$

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int k\,dt \qquad \text{Integrate both sides wrt ‘}t\text{’}$$

$$\int \frac{1}{y}dy = \int k\,dt$$

$$\ln y = kt + c$$

$$y = e^{kt+c}$$

$$y = e^{kt}e^{c}$$

$$\therefore \quad y = Ae^{kt} \qquad \text{where} \qquad A = e^{c}$$

Similarly, the general form of exponential decay is:

$$\frac{dy}{dt} = -ky$$

$$\therefore \quad y = Ae^{-kt}$$

66.4.1 Example:

1 A wonder worm experiment has found that the number of wonder worms, N, increases at a rate that is proportional to the number of worms present at the time.

Solution:

The rate of change in population is $\frac{dN}{dt}$

$$\frac{dN}{dt} \propto N$$

$$\therefore \quad \frac{dN}{dt} = kN \qquad \text{where } k \text{ is a positive constant}$$

$$\therefore \quad \frac{1}{N}\frac{dN}{dt} = k$$

Integrate both sides w.r.t t etc.

2 A chemical reaction produces two chemicals A and B. During the reaction, x grams of chemical A is produced during the same time as y grams of chemical B. The rate at which chemical A is produced is proportional to e^x, whilst the production rate for chemical B is proportional to e^y. Show how A & B change w.r.t each other.

Solution:

Given: $\frac{dx}{dt} \propto e^{x} \qquad \frac{dy}{dt} \propto e^{y}$

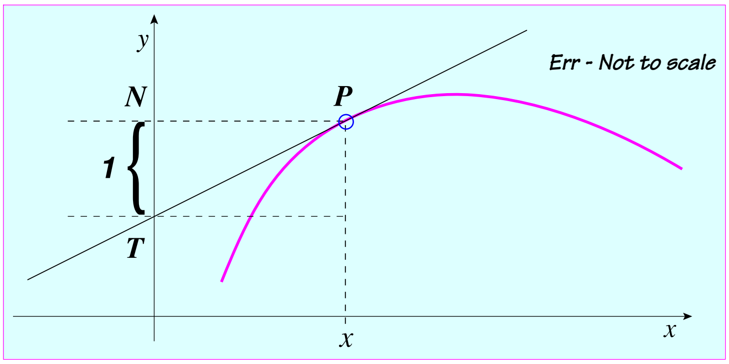
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = k_y e^{y} \times \frac{1}{k_x e^{x}} = ke^{y-x}$$

66.5 Worked Examples for Rates of Change

66.5.1 Example:

1 At each point P of a curve for which $x > 0$, the tangent cuts the y-axis at T. N is the foot of the perpendicular from P to the y-axis. If T is always 1 unit below N, find the equation of the curve.



Solution:

Gradient of tangent = $\frac{1}{x}$

Gradient of tangent $\quad \frac{dy}{dx} = \frac{1}{x}$

$\therefore y = \ln x + c$

2 A rat has a mass of 30gms at birth. It reaches maturity in 3 months. The rate of growth is modelled by the differential equation:

$$\frac{dm}{dt} = 120(t - 3)^2$$

where m = mass of the rat, t months after birth. Find the mass of the rat when fully grown.

Solution:

Given: $\frac{dm}{dt} = 120(t - 3)^2$

$m = 120 \int (t - 3)^2 \, dt$

$m = 120\left[\frac{(t-3)^3}{3}\right] + c$

$m = 40 (t - 3)^3 + c$

Evaluate c:

$t = 0, \qquad m = 30$

$30 = 40(0 - 3)^3 + c$

$c = 1110$

$\therefore \qquad m = 40 (t - 3)^3 + 1110$

When $t = 3, \qquad m = 40 (3 - 3)^3 + 1110$

$m = 1110 \; gm$

3 A farmer thinks that the rate of growth of his weeds is proportional to the amount of daylight that they receive. If t = the time in years after the shortest day of the year, the length of effective daylight, on any given day, is given by:

$$12 - 4\cos(2\pi t) \text{ hours}$$

On the shortest day of one year, the height of the plant is 120cm. 73 days later the weed has grown to 130 cm. What will the height be on the longest day of the following year?

Solution:

Given: Daylight hours: $D(t) = 12 - 4\cos(2\pi t)$

Let: h = height, & $\frac{dh}{dt}$ = growth rate

Given: growth rate: $\frac{dh}{dt} \propto 12 - 4\cos(2\pi t)$

$$\frac{dh}{dt} = k[12 - 4\cos(2\pi t)]$$

$$\therefore \quad h = k\int 12 - 4\cos(2\pi t)\, dt$$

$$\Rightarrow \quad h = 4k\int 3 - \cos(2\pi t)\, dt$$

$$= 4k\left[3t - \frac{\sin(2\pi t)}{2\pi}\right] + c$$

Substitute to find c

When $t = 0$ (shortest day), $h = 120$

$$120 = 4k(0 - 0) + c$$

Hence $c = 120$

Substitute to find k

When $t = \frac{73}{365} = \frac{1}{5}$, $h = 130$

$$130 = 4k\left[3t - \frac{\sin(2\pi t)}{2\pi}\right] + 120$$

$$130 = 4k\left[\frac{3}{5} - \frac{\sin(\frac{2\pi}{5})}{2\pi}\right] + 120$$

$$130 = 4k(0{\cdot}600 - 0{\cdot}151) + 120$$

$$130 - 120 = 1{\cdot}795k$$

$$k = \frac{10}{0{\cdot}449} = 5{\cdot}572$$

Assume that $t = 0.5$ on the longest day.

$$h = 4 \times 5{\cdot}572\left[3 \times 0{\cdot}5 - \frac{\sin(2\pi \times 0{\cdot}5)}{2\pi}\right] + 120$$

$$h = 22{\cdot}290[1{\cdot}5 - 0] + 120$$

$$h = 33{\cdot}435 + 120$$

$$h = 153\ cm \text{ (3 sf)}$$

4 A spherical balloon is inflated and when the **diameter** of the balloon is 10cm its volume is increasing at a rate of 200 cm³/sec. Find the rate at which its surface area is increasing at that time.

Solution:

Given: volume of sphere: $V = \frac{4}{3}\pi r^3$

Required: rate of change of volume: $\frac{dV}{dt}$

Connection: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\frac{dV}{dr} = \frac{3 \times 4}{3}\pi r^2 = 4\pi r^2$

$\therefore \frac{dV}{dt} = 4\pi r^2.\frac{dr}{dt}$

Now $\frac{dV}{dt} = 200$ when $2r = 10$

Hence: $200 = 100\pi\frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{2}{\pi}$ $\Rightarrow$ rate of increase of radius at this particular time.

Given: sfc area of sphere: $S = 4\pi r^2$

Connection: $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt} = 8\pi r.\frac{dr}{dt}$

When $2r = 10$, $\frac{dr}{dt} = \frac{2}{\pi}$ so that:

$\frac{dS}{dt} = 40\pi.\frac{2}{\pi} = 80\ cm^2/sec$

5 A culture of bacteria grows at a rate proportional to the number of bacteria in the culture. The number of bacteria in the culture is 1000 at lunch time. After 1 hour the number of bacteria is 3300. What is the number of bacteria after 3 hours and 24 hours?

Solution:

Given: $\frac{dP}{dt} \propto P$

$\therefore \frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt$

$\int \frac{dP}{P} = \int k\,dt$

$ln\,P = kt + c$

Find c: At lunchtime $t = 0$ and population $P = 1000$

$ln\,1000 = c \Rightarrow c = 6{\cdot}9$

Find k:

$ln\,3300 = k + 6{\cdot}9$

$k = 8{\cdot}1 - 6{\cdot}9 = 1{\cdot}2$

After 3 hours: $ln\,P = 1{\cdot}2 \times 3 + 6.9 \Rightarrow ln\,P = 10{\cdot}5 \Rightarrow P \approx 36315$

After 24 hours: $ln\,P = 1{\cdot}2 \times 24 + 6.9 \Rightarrow ln\,P = 35{\cdot}7 \Rightarrow P \approx 3{\cdot}2 \times 10^{15}$

6 A single super cell starts to divide and grow and after t hours the population has grown to P. At any given time the population of bacteria increases at a rate proportional to P^2.
Find how many hours it takes for the population to reach 10,000, given that after 1 hour the population is 1000, and after 2 hours the population is 2000.

Solution:

Given: $\frac{dP}{dt} \propto P^2$

$$\therefore \frac{dP}{dt} = kP^2 \quad \Rightarrow \quad \frac{dP}{P^2} = k\,dt$$

$$\int \frac{dP}{P^2} = \int k\,dt \quad \Rightarrow \quad \int P^{-2}\,dP = \int k\,dt$$

$$-P^{-1} = kt + c \quad \Rightarrow \quad \frac{1}{P} = -(kt + c)$$

$$P = -\frac{1}{(kt + c)}$$

Find c:

At time $t = 1$, $P = 1000$

$$1000 = -\frac{1}{(k + c)} \quad \Rightarrow \quad 1000\,(k + c) = -1$$

At time $t = 2$, $P = 2000$

$$2000 = -\frac{1}{(2k + c)} \quad \Rightarrow \quad 2000\,(2k + c) = -1$$

Use simultaneous equations

$$1000k + 1000c = -1 \qquad (1)$$

$$4000k + 2000c = -1 \qquad (2)$$

$$2000k + 2000c = -2 \qquad (3) = (1) \times 2$$

$$2000k = 1 \qquad (4) = (3) - (2)$$

$$\therefore \quad k = \frac{1}{2000}$$

$$\therefore \frac{1000}{2000} + 1000c = -1 \qquad \text{Substitute } k \text{ into (1)}$$

$$1000c = -\frac{3}{2} \quad \Rightarrow \quad c = -\frac{3}{2000}$$

When population $P = 10{,}000$

$$P = -\frac{1}{(kt + c)}$$

$$kt + c = -\frac{1}{P} \quad \Rightarrow \quad kt = -\frac{1}{P} - c$$

$$t = \frac{1}{k}\left(-\frac{1}{P} - c\right)$$

$$t = 2000\left(-\frac{1}{10000} + \frac{3}{2000}\right) \quad \Rightarrow \quad t = \left(-\frac{2000}{10000} + \frac{6000}{2000}\right)$$

$$t = 3 - \frac{2}{10} = 2{\cdot}8\ hrs$$

7 The population of a small village is 1097 in the year 1566. Assuming the population, P, grows according to the differential equation below, and where t is the number of years after 1566:

$$\frac{dP}{dt} = 0.3Pe^{-0.3t}$$

1) Find the population of the village in 1576, correct to 3 significant figures.

2) Find the maximum population the village will grow to, in the long term.

Solution:

$$\frac{dP}{dt} = 0.03Pe^{-0.03t}$$

$$\frac{dP}{P} = 0.03e^{-0.03t}dt$$

$$\int \frac{dP}{P} = \int 0.03e^{-0.03t}dt$$

$$ln(P) = -\frac{0.03}{0.03}e^{-0.03t} + c$$

$$ln(P) = -e^{-0.03t} + c$$

To find c: $P = 1097$ & $t = 0$

$$ln(1097) = -e^{0} + c = -1 + c$$

$$7 = -1 + c$$

$$c = 8$$

$$ln(P) = -e^{-0.03t} + 8$$

$$= 8 - e^{-0.03t}$$

To find the population in 10 years time:

$$ln(P) = 8 - e^{-0.03 \times 10} = 8 - e^{-0\cdot3} = 8 - 0\cdot7408 = 7\cdot2592$$

$$P = 1420 \text{ (3 sf)}$$

To find the limiting population in the long term:

$$ln(P) = 8 - e^{-0.03t} = 8 - \frac{1}{e^{0.03t}}$$

Note that as time increases, the term $\frac{1}{e^{0.03t}} \rightarrow 0$

Therefore, in the long term:

$$ln(P) = 8 - 0$$

$$P = 2980 \text{ (3 sf)}$$

8 Solve $\frac{dy}{dt} = y \sin t$ and assume the starting conditions to be $y = 50$ when $t = \pi$ *secs*

Solution:

$$\frac{dy}{dt} = y \sin t$$

$$\frac{1}{y}\, dy = \sin t \, dt$$

$$\int \frac{1}{y}\, dy = \int \sin t \, dt$$

$$\ln y = -\cos t + c$$

$$y = e^{-\cos t + c} \quad \Rightarrow \quad y = e^{c}\, e^{-\cos t}$$

Let the constant $e^{c} = A$

$$\therefore \quad y = A\, e^{-\cos t}$$

To find A:

$$50 = A\, e^{-\cos \pi}$$

$$50 = A\, e^{-(-1)}$$

$$\therefore A = \frac{50}{e}$$

$$y = \frac{50}{e}\, e^{-\cos t}$$

$$y = 50\, e^{-1}\, e^{-\cos t}$$

$$y = 50\, e^{-(1 - \cos t)}$$

9 A system is modelled by the equation:

$$p = 60\left(1 - e^{-\frac{t}{4}}\right)$$

After T hours, p is 48 cms. Show that:

$T = a \ln b$ where a & b are integers

Find a & b.

Solution:

$$48 = 60 - 60e^{-\frac{t}{4}}$$

$$48 - 60 = -60e^{-\frac{t}{4}}$$

$$\frac{-12}{60} = -e^{-\frac{t}{4}}$$

$$\frac{1}{5} = e^{-\frac{t}{4}}$$

$$\ln\left(\frac{1}{5}\right) = -\frac{t}{4}$$

$$t = -4 \ln\left(\frac{1}{5}\right)$$

$t = 4 \ln 5$ where $a = 4$ & $b = 5$ Note the change of sign here!!!

10 From Q 9 above, show that:

$$\frac{dp}{dt} = 15 - \frac{p}{4}$$

Find p, when it is growing at a rate of 13 cm per hour.

Solution:

$$p = 60 - 60e^{-\frac{t}{4}}$$

$$\frac{dp}{dt} = -60 \times \left(-\frac{1}{4}\right)e^{-\frac{t}{4}}$$

$$= 15\,e^{-\frac{t}{4}}$$

But $\quad 60e^{-\frac{t}{4}} = 60 - p$

$$\therefore \quad e^{-\frac{t}{4}} = \frac{60 - p}{60} = 1 - \frac{p}{60}$$

$$\frac{dp}{dt} = 15\left(1 - \frac{p}{60}\right) \quad \Rightarrow \quad 15 - \frac{15p}{60}$$

$$\frac{dp}{dt} = 15 - \frac{p}{4}$$

If the system is growing at a rate of 13 cms per hour, find p:

$$\frac{dp}{dt} = 15 - \frac{p}{4} = 13 \quad \Rightarrow \quad \frac{p}{4} = 15 - 13 = 2$$

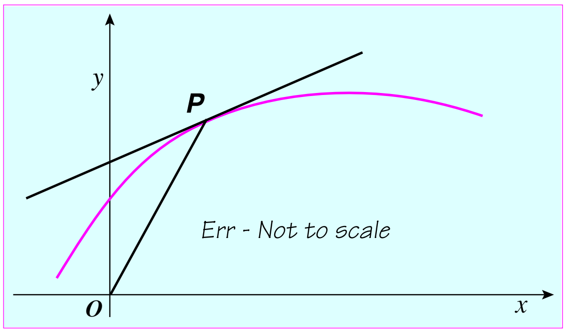
$$\therefore \quad p = 8$$

11 The gradient of the tangent at each point P of a curve is equal to the square of the gradient OP. Find the equation of the curve.

Solution:

Gradient of line $OP = \frac{y}{x}$

Gradient of tangent at $P = \frac{dy}{dx}$



$$\text{Now } \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 = \frac{y^2}{x^2}$$

$$\therefore \quad \frac{1}{y^2}dy = \frac{1}{x^2}dx \quad \Rightarrow \quad y^{-2}dy = x^{-2}dx$$

$$\int y^{-2}dy = \int x^{-2}dx$$

$$-\frac{1}{y} = -\frac{1}{x} + c$$

$$\frac{1}{y} = \frac{1}{x} - c \quad \Rightarrow \quad \frac{1}{y} = \frac{1 - cx}{x}$$

$$\therefore \quad y = \frac{x}{1 - cx}$$

12 From the equation $x = 15 - 12e^{-\frac{t}{14}}$ show that $t = 14\,ln\left(\frac{a}{b}\right)$ when $x = 10$ and where a & b are integers.

Solution:

$$10 = 15 - 12e^{-\frac{t}{14}}$$

$$10 = 15 - 12e^{-\frac{t}{14}}$$

$$\frac{10 - 15}{12} = e^{-\frac{t}{14}} \quad \Rightarrow \quad \frac{5}{12} = e^{-\frac{t}{14}}$$

$$ln\left(\frac{5}{12}\right) = -\frac{t}{14}$$

$$t = -14\,ln\left(\frac{5}{12}\right) \quad \Rightarrow \quad t = 14\,ln\left(\frac{12}{5}\right)$$ Note the change of sign here!!!

Show that $\frac{dx}{dt} = \frac{1}{14}(15 - x)$

$$\frac{dx}{dt} = -12\left(-\frac{1}{14}\right)e^{-\frac{t}{14}} \quad \Rightarrow \quad \frac{12}{14}e^{-\frac{t}{14}}$$

But $e^{-\frac{t}{14}} = \frac{15 - x}{12}$

$$\therefore \quad \frac{dx}{dt} = \frac{12}{14}\left(\frac{15 - x}{12}\right) = \frac{1}{14}(15 - x)$$

66.6 Heinous Howlers

Handling logs causes many problems, here are a few to avoid.

1 $ln(y + 2) = ln(4x - 5) + ln\,3$

You cannot just remove all the ln's so: $(y + 2) \neq (4x - 5) + 3$

To solve, put the RHS into the form of a single log first: $ln(y + 2) = ln\left[3(4x - 5)\right]$

$\therefore \quad (y + 2) = 3(4x - 5)$

2 $ln(y + 2) = 2\,ln\,x$

You cannot just remove all the ln's so: $(y + 2) \neq 2x$

To solve, put the RHS into the form of a single log first: $ln(y + 2) = ln\,x^2$

$\therefore \quad (y + 2) = x^2$

3 $ln(y + 2) = x^2 + 3x$

You cannot convert to exponential form this way: $(y + 2) \neq e^{x^2} + e^{3x}$

To solve, raise e to the whole of the RHS : $(y + 2) = e^{x^2 + 3x}$

67 • C4 • Vectors

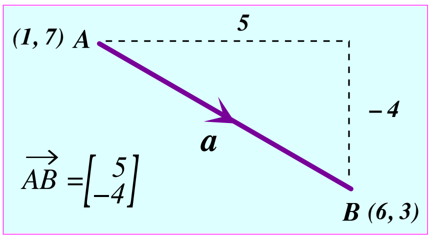
67.1 Vector Representation

A **scalar** has magnitude only, e.g. length or distance, speed, area, volumes.

A **vector** has magnitude AND direction. e.g. velocity, acceleration, momentum. i.e. A journey from one point to another. Moving from point *A* to *B* is called a **translation**, and the vector a **translation vector**.

Notation: Three ways of expressing vectors:

- $\overrightarrow{AB}$ = from A to B
- $\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \equiv \begin{pmatrix}5\\ -4\end{pmatrix}$ = 5 across; 4 down
  where 5 & −4 are the components in the x & y direction.
- ***a*** (bold print) or in hand writing ***a*** or ***a***



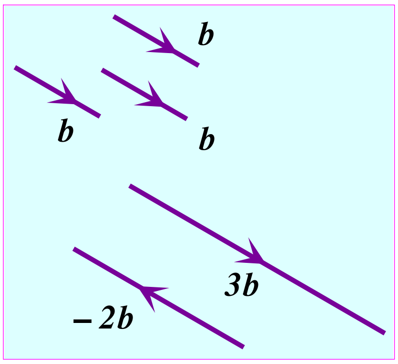
The translation vector can be calculated from the co-ordinates *A* (1, 7), *B* (6, 3):

$$\overrightarrow{AB} = \boldsymbol{a} = \begin{pmatrix}B_x - A_x\\ B_y - A_y\end{pmatrix} = \begin{pmatrix}6 - 1\\ 3 - 7\end{pmatrix} = \begin{pmatrix}5\\ -4\end{pmatrix}$$

The length of the line in the diagram represents the magnitude of the vector and vectors are equal if the magnitude and direction are the same.

Vectors are parallel if they have the same direction and are scalar multiples of the original vector. e.g. the vector ***3b*** is parallel to the vector ***b***, and three times longer.

The vector **−2*b*** is 2 times the magnitude of ***b*** and in the opposite direction.



Be aware that the notation only tells you in which direction to move a point and nothing about its position in space. In effect the vector carries two pieces of information, its magnitude and the inverse of its gradient. Hence these are sometimes called ‘free’ vectors.

67.2 Scaler Multiplication of a Vector

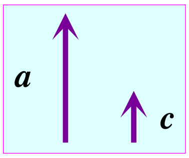
If $\overrightarrow{AB} = \boldsymbol{a} = \begin{pmatrix}x\\ y\end{pmatrix}$ and k is a constant number then:

$$k\boldsymbol{a} = \begin{pmatrix}kx\\ ky\end{pmatrix}$$

The constant k is called a scalar because it ‘scales up’ the length of the vector

67.3 Parallel Vectors

If $\boldsymbol{a} = 3\boldsymbol{c}$ then the two vectors will look like this:



Vectors are parallel if one is a scalar multiple of the other.

If $\boldsymbol{a} = \begin{pmatrix}0\\ 15\end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix}0\\ 5\end{pmatrix}$ then $\boldsymbol{a}$ and $\boldsymbol{c}$ are parallel because $\boldsymbol{a} = 3\begin{pmatrix}0\\ 5\end{pmatrix} = 3\boldsymbol{c}$

67.4 Inverse Vector

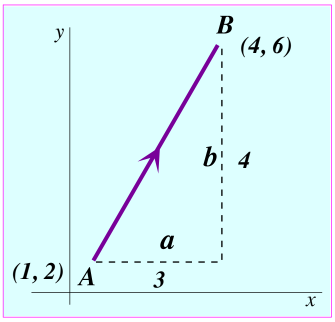
If $\overrightarrow{AB} = \boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $\overrightarrow{BA} = -\boldsymbol{a} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

67.5 Vector Length or Magnitude

The magnitude or length of a vector, (also called its modulus) is written:

$$\left|\overrightarrow{OP}\right| = \left|\begin{pmatrix} x \\ y \end{pmatrix}\right| \qquad \text{or for 3-D} \qquad \left|\overrightarrow{OQ}\right| = \left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right|$$

Calculate the length using Pythagoras's theorem:



$$\left|\overrightarrow{AB}\right|^2 = \left|\begin{pmatrix} a \\ b \end{pmatrix}\right|^2 = a^2 + b^2 \qquad \text{If } \overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{25} = 5.$$

Similarly for 3-D vectors $\left|\overrightarrow{OQ}\right| = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{a^2 + b^2 + c^2}$

67.5.1 Example:

1 A line is drawn between two points A (1, 4, 2) and B (2, –1, 3). Find the distance between the two points.

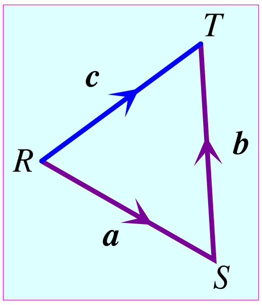
$$\left|\overrightarrow{AB}\right| = \sqrt{(2 - 1)^2 + (-1 - 4)^2 + (3 - 2)^2} = \sqrt{1 + 25 + 1} = \sqrt{27}$$

$$\left|\overrightarrow{AB}\right| = 3\sqrt{3}$$

67.6 Addition of Vectors

You should know that adding two vectors means finding the shortcut of their journeys. This is the same as making one translation followed by another.

e.g. $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -9 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$



$$\overrightarrow{RT} = \overrightarrow{RS} + \overrightarrow{ST}$$

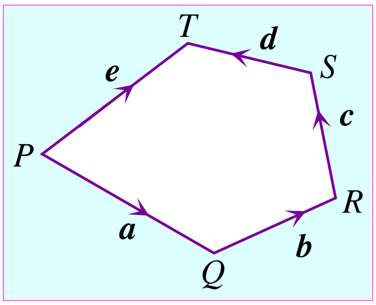
$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$

The vector $\overrightarrow{RT}$ is called the resultant of the vectors $\overrightarrow{RS}$ and $\overrightarrow{ST}$

It can be shown that if $\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ then: $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ (commutative rule)

If $\overrightarrow{RS} = \boldsymbol{a}$ then $\overrightarrow{SR} = -\boldsymbol{a}$ (same magnitude but opposite direction).

In a similar manner, larger paths can be added or subtracted.



For example:

$$\overrightarrow{PT} = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST}$$

$$\boldsymbol{e} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}$$

67.6.1 Example:

1

Find the values of s and t given that $s\begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}5\\13\end{pmatrix}$

$2s - t = 5$

$3s + 4t = 13$

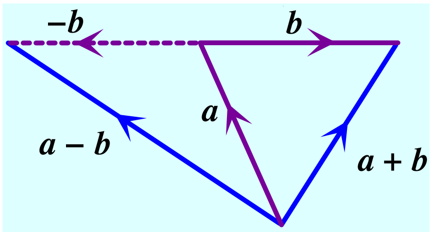
$\therefore s = 3$ & $t = 1$

67.7 Subtraction of Vectors

Note that a vector subtraction can be written:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$

This is the same as saying: move along vector $\boldsymbol{a}$, followed by a move along vector $-\boldsymbol{b}$

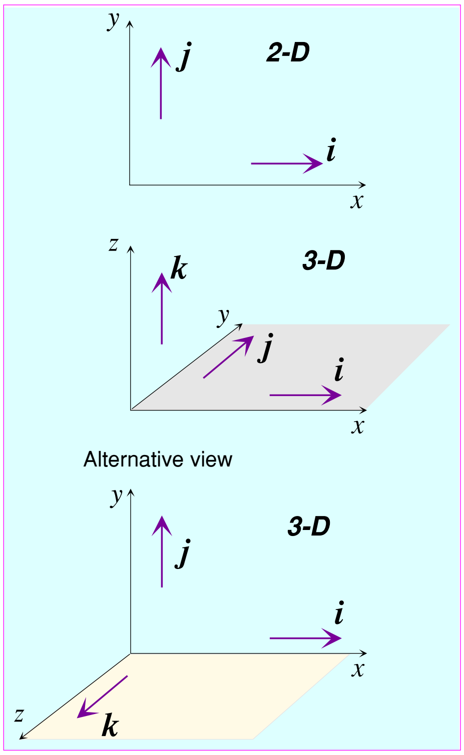


67.8 The Unit Vectors

A unit vector is a vector with length or magnitude of 1.

Any vector can be given as a multiple of $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

$$e.g. \quad \overrightarrow{AB} = \begin{pmatrix}4\\5\end{pmatrix} = 4\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix}$$



In 2-D, the unit vectors are ***i*** & ***j*** where:

$$\boldsymbol{i} = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \boldsymbol{j} = \begin{pmatrix}0\\1\end{pmatrix}$$

This enables us to write vectors in a more compact format.

$$e.g. \quad \overrightarrow{AB} = \begin{pmatrix}4\\5\end{pmatrix} = 4\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix} = 4\boldsymbol{i} + 5\boldsymbol{j}$$

In 3-D, the unit vectors are ***i***, ***j*** & ***k***, where:

$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\text{e.g.} \quad \overrightarrow{AB} = \begin{pmatrix}4\\5\\6\end{pmatrix} = 4\begin{pmatrix}1\\0\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\\0\end{pmatrix} + 6\begin{pmatrix}0\\0\\1\end{pmatrix} = 4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}$$

Any vector can be expressed in terms of ***i***, ***j*** & ***k***

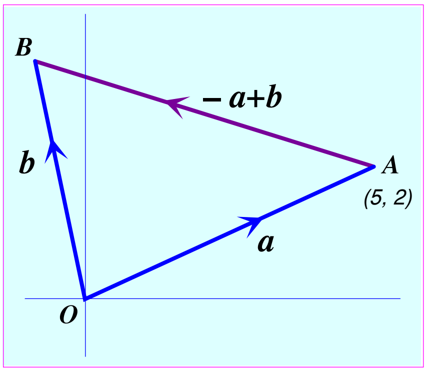
67.9 Position Vectors

A translation vector, on its own, has no frame of reference, it just hangs in space. It only tells you how to go from point A to point B, not where point A is.

Position vectors on the other hand, are the vector equivalent of a set of co-ordinates. The position vector allows a translation vector to be fixed in space, using the origin as its fixed point of reference.

The position vectors of a point A, with co-ordinates (5, 2), is the translation vector which takes you from the origin to the point (5, 2). So the co-ordinates of point A are the same as the translation vector from point O to A.

$$\therefore \text{ position vector of point } A = \overrightarrow{OA} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 5\boldsymbol{i} + 2\boldsymbol{j}$$



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$$
$$= -\boldsymbol{a} + \boldsymbol{b}$$

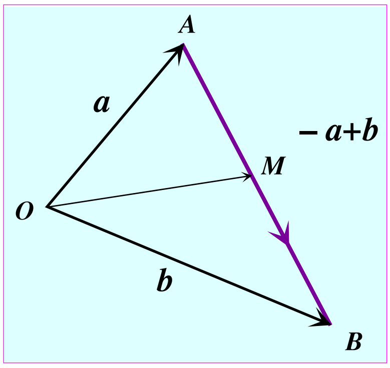
> The vector co-ordinates of point A are the same as the translation vector from the origin, point O to point A, *i.e.* $\overrightarrow{OA}$.

67.9.1 Example:

1 Points A & B have position vector $\boldsymbol{a}$ and $\boldsymbol{b}$. Find the position vectors of
a) the midpoint M of AB and
b) the point of trisection T such that $AT = \frac{2}{3}AB$

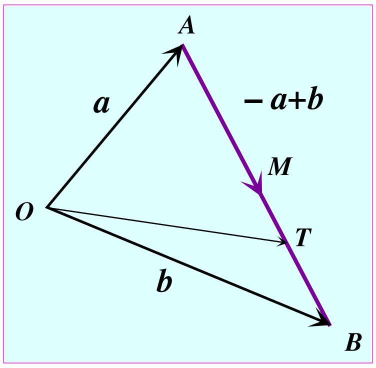
Solution:

a)



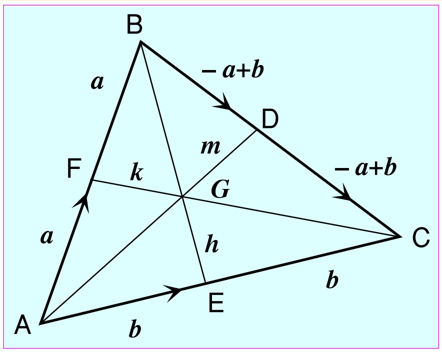
$$\overrightarrow{AM} = \tfrac{1}{2}(-\boldsymbol{a} + \boldsymbol{b})$$
$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$= \boldsymbol{a} + \tfrac{1}{2}(-\boldsymbol{a} + \boldsymbol{b})$$
$$= \tfrac{1}{2}\boldsymbol{a} + \tfrac{1}{2}\boldsymbol{b}$$
$$= \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

b)



$$\overrightarrow{OT} = \overrightarrow{OA} + \overrightarrow{AT}$$
$$\overrightarrow{OT} = \boldsymbol{a} + \tfrac{2}{3}(-\boldsymbol{a} + \boldsymbol{b})$$
$$= \tfrac{1}{3}\boldsymbol{a} + \tfrac{2}{3}\boldsymbol{b}$$
$$= \tfrac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$$

2



In a triangle ABC, the midpoints of BC, CA, and AB are D, E, and F respectively. Prove that the lines AD, BE and CF meet at a point G, which is the point of trisection of each of the medians.

AG via 3 different medians:

i) via AD:

$$\overrightarrow{AG} = \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DG}$$

$$= 2\boldsymbol{b} + (\boldsymbol{a} - \boldsymbol{b}) + m.\overrightarrow{DA}$$

$$\text{Where } \overrightarrow{DA} = -\boldsymbol{a} + \boldsymbol{b} - 2\boldsymbol{b}$$

$$= -\boldsymbol{a} - \boldsymbol{b}$$

$$\overrightarrow{AG} = 2\boldsymbol{b} + \boldsymbol{a} - \boldsymbol{b} + m(-\boldsymbol{a} - \boldsymbol{b})$$

$$= \boldsymbol{b} + \boldsymbol{a} + m(-\boldsymbol{a} - \boldsymbol{b})$$

$$= \boldsymbol{a}(1 - m) + \boldsymbol{b}(1 - m)$$

ii) via BE:

$$\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{EG}$$

$$= \boldsymbol{b} + h.\overrightarrow{EB}$$

$$\text{Where } \overrightarrow{EB} = -\boldsymbol{b} + 2\boldsymbol{a}$$

$$\overrightarrow{AG} = \boldsymbol{b} + h(-\boldsymbol{b} + 2\boldsymbol{a})$$

$$= \boldsymbol{a}(2h) + \boldsymbol{b}(1 - h)$$

iii) via FC:

$$\overrightarrow{AG} = \overrightarrow{AF} + \overrightarrow{FG}$$

$$= \boldsymbol{a} + k.\overrightarrow{FC}$$

$$= \boldsymbol{a} + k(-\boldsymbol{a} + 2\boldsymbol{b})$$

$$= \boldsymbol{a}(1 - k) + \boldsymbol{b}(2k)$$

We can say that all the above vectors are the same, and therefore equal.
Hence coefficients of $\boldsymbol{a}$ are equal and coefficients of $\boldsymbol{b}$ are equal:

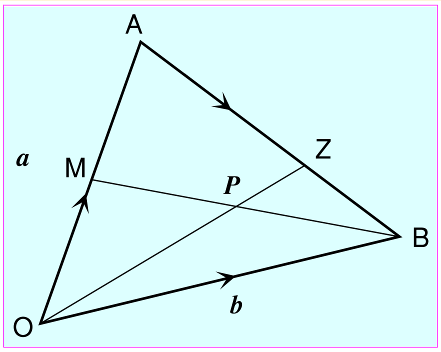
Coefficients of $\boldsymbol{a}$ $\quad (1 - m) = 2h = (1 - k) \quad \Rightarrow m = k \quad (1)$

Coefficients of $\boldsymbol{b}$ $\quad (1 - m) = (1 - h) = 2k \quad (2)$

Subs (1) into (2) $\quad (1 - m) = 2k \quad \Rightarrow (1 - k) = 2k \quad \Rightarrow k = \frac{1}{3} \quad \therefore m = \frac{1}{3}$

$$2h = (1 - k) \quad \Rightarrow 2h = \left(1 - \frac{1}{3}\right) \quad \Rightarrow h = \frac{1}{3}$$

3



In a triangle OAB, O is the origin with A having a position vector of $\boldsymbol{a}$ and B having a position vector of $\boldsymbol{b}$. M is the midpoint of OA. Z is a point on AB such that $AZ = 2ZB$. P is a point on OZ such that $OP = 3PZ$.

a) Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ the position vectors of M & Z.
b) Prove that the $MP = \frac{1}{4}(2\boldsymbol{b} - \boldsymbol{a})$
c) Hence or otherwise show that $M, P, \& B$ are on the same line, i.e. colinear.

a)

$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\boldsymbol{a}$$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\boldsymbol{a} + \boldsymbol{b}$$

$$\overrightarrow{ZB} = \frac{1}{3}(-\boldsymbol{a} + \boldsymbol{b})$$

$$\overrightarrow{OZ} = \boldsymbol{b} - \frac{1}{3}(-\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{b} + \frac{1}{3}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b} = \frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$

b)

$$\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OP}$$

$$\overrightarrow{OP} = \frac{3}{4}\overrightarrow{OZ} = \frac{3}{4}\left(\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}\right) = \frac{1}{4}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$

$$\overrightarrow{MP} = -\frac{1}{2}\boldsymbol{a} + \frac{1}{4}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} = \frac{1}{2}\boldsymbol{b} - \frac{1}{4}\boldsymbol{a}$$

$$\overrightarrow{MP} = \frac{1}{4}(2\boldsymbol{b} - \boldsymbol{a})$$

c)

$$\overrightarrow{MB} = \overrightarrow{MO} + \overrightarrow{OB}$$

$$\overrightarrow{MB} = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$$

$$\overrightarrow{MB} = \frac{1}{2}(2\boldsymbol{b} - \boldsymbol{a})$$

$$\overrightarrow{MP} = \frac{1}{2}\overrightarrow{MB}$$

The vectors $\overrightarrow{MB}$ & $\overrightarrow{MP}$ are parallel [same vector part $(2\boldsymbol{b} - \boldsymbol{a})$ with different scalar part] and both lines have a common point M. Therefore, the points $M, P, \& B$ are on the same line, i.e. colinear.

67.10 The Scalar (Dot) Product of Two Vectors

This is where two vectors are multiplied together. One form of multiplication is the **scaler product**. The answer is interpreted as a single number, which is a **scalar**. This is also known as the 'DOT' product, where a dot is used instead of a multiplication sign.

(Not to be confused with the vector product which is called the 'CROSS' product in vector terminology, hence the careful selection of the names).

N.B. You can't have a dot product of three or more vectors, as it has no meaning.

There are two main uses for the DOT product:

- Calculating the angle between two vectors
- Proving that two vectors are either parallel or perpendicular

The dot product comes in two forms. The component form of the dot product is shown below:

$$\boldsymbol{a} \bullet \boldsymbol{b} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} \bullet \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} a_x b_x \\ a_y b_y \end{pmatrix} = (a_x \times b_x) + (a_y \times b_y) = a_x b_x + a_y b_y$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \end{pmatrix} = (3 \times 2) + (4 \times 5) = 26$$

$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 8 \\ -3 \end{pmatrix} = (2 \times 0) + (-1 \times 8) + (4 \times -3) = -20$$

67.10.1 Example:

1 If $\boldsymbol{p} = (2\boldsymbol{i} + 3\boldsymbol{j})$ and $\boldsymbol{q} = (5\boldsymbol{i} - 9\boldsymbol{j})$, find $\boldsymbol{p} \bullet \boldsymbol{q}$

$(2 \times 5) + (3 \times -9) = 10 - 27 = -17$

The other definition of the dot product uses the angle between vectors directly and is:

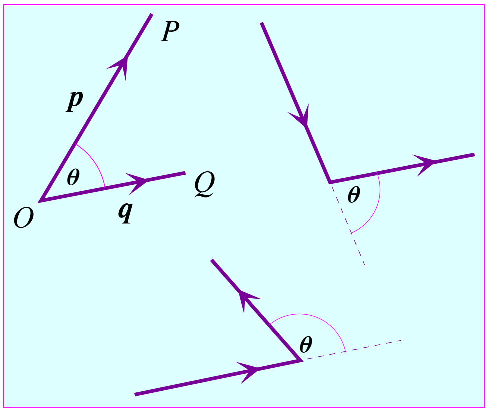
$$\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}| \cos\theta$$

$$\therefore \quad \cos\theta = \frac{\boldsymbol{p} \bullet \boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$$

where θ is the angle between the two vectors and $|\boldsymbol{p}|$ & $|\boldsymbol{q}|$ are the scalar lengths or magnitudes of the vectors.

Note that θ is the angle between the two **direction** vectors of the line (more later).

Observe that the RHS of the equation is made up of scalar quantities, since $|\boldsymbol{p}|$ & $|\boldsymbol{q}|$ are scalars, as is $\cos\theta$. Hence the dot product is a scalar quantity. In addition, because $|\boldsymbol{p}|$ & $|\boldsymbol{q}|$ are always +ve values, the dot product takes the sign of $\cos\theta$.



It is important that the vectors are put 'tail to tail' to get a true idea of the angle between them.

The inclusion of $\cos\theta$ in the equation brings some useful results:

- If $\boldsymbol{p}$ and $\boldsymbol{q}$ are parallel then $\theta = 0$, $\therefore \cos\theta = 1$ and $\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|$
- If $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular then $\theta = 90$, $\therefore \cos\theta = 0$ and $\boldsymbol{p} \bullet \boldsymbol{q} = 0$

- If the angle θ is acute then $\cos\theta > 0$ and $\boldsymbol{p} \bullet \boldsymbol{q} > 0$
- If the angle θ is between 90° & 180° then $\cos\theta < 0$ and $\boldsymbol{p} \bullet \boldsymbol{q} < 0$

- If $\boldsymbol{p} \cdot \boldsymbol{q} = 0$, then either $|\boldsymbol{p}| = 0$, $|\boldsymbol{q}| = 0$ or $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular
- Recall that $\cos\theta = -\cos(180 - \theta)$ (2nd quadrant)

Note also that:

$$\boldsymbol{i} \bullet \boldsymbol{j} = 0 \quad \boldsymbol{i} \bullet \boldsymbol{k} = 0 \quad \boldsymbol{j} \bullet \boldsymbol{k} = 0 \qquad \text{(unit vectors perpendicular)}$$

$$\boldsymbol{i} \bullet \boldsymbol{i} = 1 \quad \boldsymbol{j} \bullet \boldsymbol{j} = 1 \quad \boldsymbol{k} \bullet \boldsymbol{k} = 1 \qquad \text{(unit vectors parallel)}$$

$$\boldsymbol{p} \bullet \boldsymbol{q} = \boldsymbol{q} \bullet \boldsymbol{p} \qquad \text{(commutative law)}$$

$$\boldsymbol{s} \bullet (\boldsymbol{p} + \boldsymbol{q}) = \boldsymbol{s} \bullet \boldsymbol{p} + \boldsymbol{s} \bullet \boldsymbol{q} \qquad \text{(distributive over vector addition)}$$

$$\boldsymbol{p} \bullet (k\boldsymbol{q}) = (k\boldsymbol{p}) \bullet \boldsymbol{q} = k(\boldsymbol{p} \bullet \boldsymbol{q}) \qquad (k \text{ is a scalar})$$

67.11 Proving Vectors are Perpendicular

If two lines or vectors are perpendicular, then $\theta = 90°$, hence $\cos\theta = 0$, $\therefore \boldsymbol{p} \cdot \boldsymbol{q} = 0$

67.11.1 Example:

1 Prove that the vectors $\boldsymbol{p} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} -4 \\ -8 \\ -1 \end{pmatrix}$ are perpendicular.

Solution:

$$\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos\varphi$$

If $\varphi = 90° \quad \Rightarrow \quad \cos\varphi = 0$

$\therefore \quad \boldsymbol{p} \bullet \boldsymbol{q} = 0$ if 2 vectors are perpendicular.

$$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ -8 \\ -1 \end{pmatrix} = (3 \times -4) + (-2 \times -8) + (4 \times -1)$$

$$= -12 + 16 - 4 = 0 \qquad \therefore \text{ perpendicular}$$

67.12 Finding the Angle Between Two Vectors

Recall:

$$\cos\theta = \frac{\boldsymbol{p} \bullet \boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$$

where $\boldsymbol{p} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$; $\boldsymbol{q} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$ and $\boldsymbol{p} \bullet \boldsymbol{q} = a_x b_x + a_y b_y$ and $|\boldsymbol{p}| = \sqrt{(a_x)^2 + (a_y)^2}$, $|\boldsymbol{q}| = \sqrt{(b_x)^2 + (b_y)^2}$

Find the value of $\boldsymbol{p} \cdot \boldsymbol{q}$ first, as if this is 0, then the lines are perpendicular.

67.12.1 Example:

1 Find the angle between the two vectors $\boldsymbol{a} = 3i + 4j$ and $\boldsymbol{b} = 5i - 12j$.

Solution:

$$\cos\theta = \frac{\boldsymbol{p} \bullet \boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$$

$$\cos\theta = \frac{(3 \times 5) + (4 \times -12)}{\sqrt{3^2 + 4^2} \times \sqrt{5^2 + (-12)^2}} = \frac{15 - 48}{5 \times 13}$$

$$\cos\theta = \frac{-33}{65}$$

$$\theta = \cos^{-1}\left(\frac{-33}{65}\right) = 120{\cdot}5° \ (1dp)$$

2 Find the angle between the two vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7 \end{pmatrix}$

Solution:

$$\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos\varphi$$

$$(2 \times -1) + (3 \times 7) = \sqrt{2^2 + 3^2} \, . \, \sqrt{(-1)^2 + 7^2} \cos\varphi$$

$$19 = \sqrt{13} \, . \, \sqrt{50} \cos\varphi$$

$$\cos\varphi = \frac{19}{\sqrt{13} \, . \, \sqrt{50}} = 0{\cdot}745$$

$$\varphi = 41{\cdot}8°$$

67.13 Vector Equation of a Straight Line

The **Vector Equation** tells you how to get to any point on a line if you start at the origin. So we define $\boldsymbol{r}$ as the position vector of any point on the line, (i.e. the vector co-ordinates of some point R).

What we do is to move from the origin (O) to a known point on the line (A), then move in the direction of the slope to a point B.

Since the line AB is parallel with a vector $\boldsymbol{p}$, then $\overrightarrow{AB} = t\boldsymbol{p}$, where t is a scalar, and $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

Therefore, the general vector equation of a straight line is:

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{p}$$

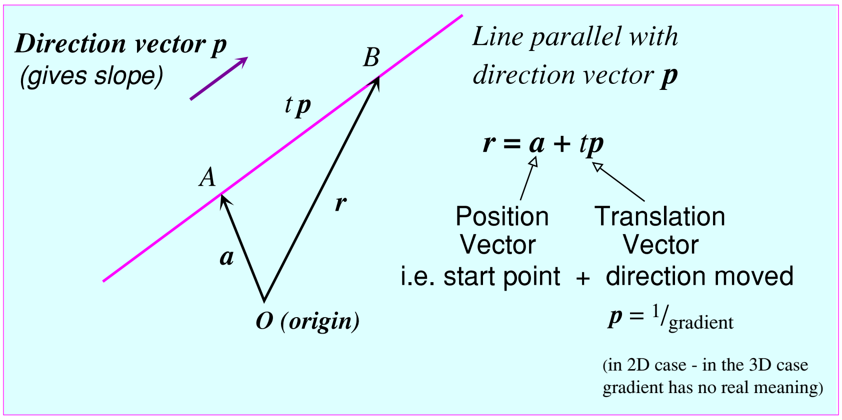
$$\text{or} \quad \boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{p}$$

Where: $\boldsymbol{a}$ = the position vector of a given point on the line, (e.g. point A)

t = an ordinary number which is a variable (i.e. a scalar). Sometimes this is labelled λ *or* μ

$\boldsymbol{p}$ = '**direction vector**' of the line which defines the '**slope**', (strictly speaking the inverse gradient).

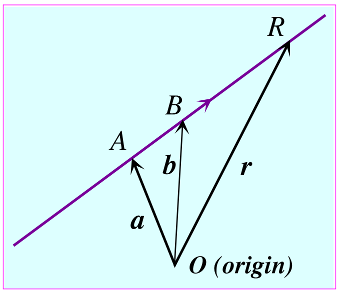
Think of $t\boldsymbol{p}$ as the translation vector part of the line.
The direction vector is the translation vector when $t = 1$



An alternative form of the vector equation of a straight line can be written in component form. If $\boldsymbol{a} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$ and $\boldsymbol{p} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ then:

$$\boldsymbol{r} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The vector equation of a line that passes through two points A & B can be found thus:



The vector $\quad \boldsymbol{r} = \overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

but $\quad \overrightarrow{AR} = t \times \overrightarrow{AB}$

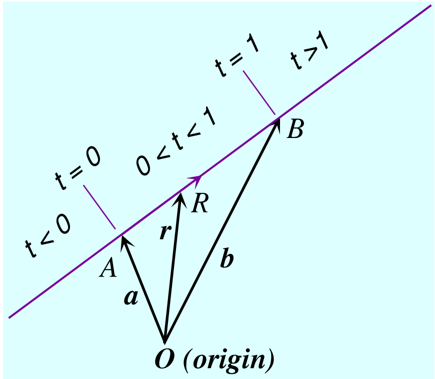
$$\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$$

$$\therefore \quad \boldsymbol{r} = \overrightarrow{OR} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$$

In this case the vector $(\boldsymbol{b} - \boldsymbol{a})$ is the direction vector of the line.

The value of t varies according to its position on the line:

- If $t < 0$, point R is on the line BA produced
- If $t = 0$, point $R = A$ and $\boldsymbol{r} = \boldsymbol{a}$
- If $0 < t < 1$, point R is between A and B
- If $t = 1$, point $R = B$ and $\boldsymbol{r} = \boldsymbol{b}$
- If $t > 0$, point R is on the line AB produced



67.13.1 Example:

1 Draw the line with the vector equation: $\boldsymbol{r} = \begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}$

Method 1

t	0	1	2	3
point	$\begin{pmatrix}2\\3\end{pmatrix}$	$\begin{pmatrix}4\\4\end{pmatrix}$	$\begin{pmatrix}6\\5\end{pmatrix}$	$\begin{pmatrix}8\\6\end{pmatrix}$

then plot the graph.

Method 2

$\begin{pmatrix}2\\1\end{pmatrix}$ = the gradient, so plot $\begin{pmatrix}2\\3\end{pmatrix}$ and work out the other points according to the gradient.

2 Find a vector equation for the line which passes through (3, 1) and which has the gradient $\frac{1}{2}$

$$\boldsymbol{r} = \begin{pmatrix}3\\1\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}$$

3 Find a vector equation for the line which passes through (3, 5) and (9, –2)

$$\boldsymbol{r} = \begin{pmatrix}3\\5\end{pmatrix} + t\begin{pmatrix}9 - 3\\-2 - 5\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix} + t\begin{pmatrix}6\\-7\end{pmatrix}$$

4 Find the vector equation for the line which passes through the point A (4, –1, 3) and parallel to the vector $2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{p}$$

$$\therefore \quad \boldsymbol{r} = (4i - \boldsymbol{j} + 3\boldsymbol{k}) + t(2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})$$

5 Find the vector equation for the line which passes through the point with position vector $3\boldsymbol{i} + 2\boldsymbol{j}$ and parallel to the vector $2\boldsymbol{i} - \boldsymbol{j}$

$$\therefore \quad \boldsymbol{r} = (3i + 2j) + t(2\boldsymbol{i} - j)$$

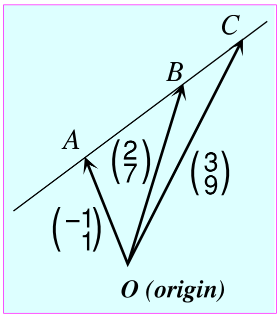
6 Find the vector equation for the line parallel to the vector $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ and which passes through the point with position vector $5\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}$

$$\boldsymbol{a} = 5\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k} \quad \Rightarrow \begin{pmatrix}5\\-2\\7\end{pmatrix} \qquad \boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k} \quad \Rightarrow \begin{pmatrix}3\\4\\-1\end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\-2\\7\end{pmatrix} + t\begin{pmatrix}3\\4\\-1\end{pmatrix}$$

7 Find the vector equation for the straight line which passes through the points A, B and C, given that the position vectors of A, B and C, are $\begin{pmatrix}-1\\1\end{pmatrix}$, $\begin{pmatrix}2\\7\end{pmatrix}$ and $\begin{pmatrix}3\\9\end{pmatrix}$ respectively.

Solution:



To obtain the equation we need a direction vector parallel to the line, say BC (or it could be AB, BA etc.) and a position vector, say A (could be B or C).

Position vector $\quad \boldsymbol{a} = \begin{pmatrix}-1\\1\end{pmatrix}$

Direction vector $\quad \overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} \quad \Rightarrow \quad \overrightarrow{BC} = -\begin{pmatrix}2\\7\end{pmatrix} + \begin{pmatrix}3\\9\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$$\boldsymbol{r} = \overrightarrow{OA} + \lambda\overrightarrow{BC}$$

$$\therefore \quad \boldsymbol{r} = \begin{pmatrix}-1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\end{pmatrix}$$

Observe that the equation of the line can be calculated in several different ways such as:

$$\boldsymbol{r} = \overrightarrow{OB} + \lambda\overrightarrow{BC} \quad \text{or} \quad \boldsymbol{r} = \overrightarrow{OC} + \lambda\overrightarrow{BA} \quad \text{or} \quad \boldsymbol{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} \quad \text{etc.}$$

Although this would give different equations all would be valid, and give the position of any point on the line for a suitable value of λ.

67.14 To Show a Point Lies on a Line

67.14.1 Example:

1 Show that the point with position vector $\boldsymbol{i} + 2\boldsymbol{j}$ lies on the line L, with vector equation $\boldsymbol{r} = 4\boldsymbol{i} - \boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j})$

Solution:

If on the line, the point must satisfy the equation of the line.

$$\boldsymbol{i} + 2\boldsymbol{j} = 4\boldsymbol{i} - \boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j})$$

$$\boldsymbol{i} + 2\boldsymbol{j} = 4\boldsymbol{i} - \boldsymbol{j} + \lambda\boldsymbol{i} - \lambda\boldsymbol{j}$$

Matching term coefficients:

$\boldsymbol{i}$ term $\quad 1 = 4 + \lambda \quad \Rightarrow \quad \lambda = -3$

$\boldsymbol{j}$ term $\quad 2 = -1 - \lambda \quad \Rightarrow \quad \lambda = -3$

As $\lambda = -3$ in both cases, the point with position vector $\boldsymbol{i} + 2\boldsymbol{j}$ lies on the line L.
If λ had not matched, then the point would not have been on the line. For a 3-D example coefficients of ALL three unit vectors must be equal for the point to be on the line.

67.15 Intersection of Two Lines

Two lines intersect if the position vector of both lines satisfy both equations. Two lines such as

$$\boldsymbol{r}_1 = \boldsymbol{p} + s\boldsymbol{q} \qquad \& \qquad \boldsymbol{r}_2 = \boldsymbol{a} + t\boldsymbol{p}$$

intersect when $\boldsymbol{r}_1 = \boldsymbol{r}_2$

$$\text{i.e.} \qquad \boldsymbol{p} + s\boldsymbol{q} = \boldsymbol{a} + t\boldsymbol{p}$$

Note that in a 2-D world, individual lines either intersect or are parallel. In a 3-D world, individual lines may also intersect or be parallel, but they may not do either, (think of a railway line crossing over a road via a bridge). In this case they are called **skew**.

67.15.1 Example:

1 Find the co-ordinate where these two lines meet:

$$\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \textit{ and } \boldsymbol{r}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Solution:

Equate x components $\Rightarrow \quad 1 + t = 3 + s \qquad (1)$

Equate y components $\Rightarrow \quad 2 + t = -2 + 4s \qquad (2)$

Subtract and resolve simultaneous equations:

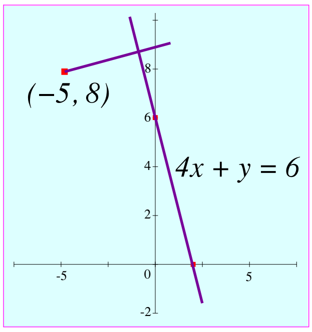
$$\therefore \quad -1 = 5 - 3s \quad \Rightarrow \quad s = 2 \quad \Rightarrow \quad t = 4$$

$$\therefore \text{ Intersection } \Rightarrow \boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\therefore \text{ Intersection } \Rightarrow \boldsymbol{r}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$\therefore$ Co-ordinate is (5, 6)

2



Find the co-ordinates of the foot of the perpendicular from (−5, 8) to the line $4x + y = 6$ (using the vector method).

Soultion:

From the equation, the gradient of the line is – 4, hence, gradient of perpendicular is $\frac{1}{4}$

Equation of perpendicular line is: $\boldsymbol{r} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

The x component is $\Rightarrow \quad x = -5 + 4t$

The y component is $\Rightarrow \quad y = 8 + t$

Substitute components into: $4x + y = 6$

$$4(-5 + 4t) + (8 + t) = 6 \quad \Rightarrow 17t - 12 = 6 \quad \Rightarrow \quad t = \frac{18}{17}$$

$$\boldsymbol{r} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} + \frac{18}{17}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \therefore \quad \text{Co-ordinates} = \left(-\frac{13}{17}, 9\frac{1}{17}\right)$$

3 Write down, in parametric form, the co-ordinates of any point on the line which passes through (5, 4) in the direction of $\begin{pmatrix}3\\5\end{pmatrix}$. Use these equations to find where this line meets $x + y = 8$

Line is expressed as: $\mathbf{r} = \begin{pmatrix}5\\4\end{pmatrix} + t\begin{pmatrix}3\\5\end{pmatrix}$

$x = 5 + 3t \qquad y = 4 + 5t$

Substitute into : $x + y = 8$

$5 + 3t + 4 + 5t = 8 \quad \Rightarrow 8t = -1 \quad \Rightarrow \quad t = -\frac{1}{8}$

$\therefore \mathbf{r} = \begin{pmatrix}5\\4\end{pmatrix} - \frac{1}{8}\begin{pmatrix}3\\5\end{pmatrix} \Rightarrow \begin{pmatrix}5\\4\end{pmatrix} - \begin{pmatrix}\frac{3}{8}\\\frac{5}{8}\end{pmatrix} \Rightarrow \begin{pmatrix}4\frac{5}{8}\\3\frac{3}{8}\end{pmatrix}$ Co-ordinates $= \left(4\frac{5}{8}, 3\frac{3}{8}\right)$

67.16 Angle Between Two Lines

In the previous examples on angles we took the simple case of finding the angle between vectors. This time we need the angle between two lines, expressed with a vector equation. In this case we need to consider the two **direction** vectors of the lines.

Recall the dot product of two lines is defined by:

$$\cos\theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$$

Note that θ is the angle between the two **direction** vectors of the lines. Where $\mathbf{p}$ is the **direction** vector of the line $\mathbf{r} = \mathbf{a} + s\mathbf{p}$ etc.

67.16.1 Example:

1 Find the angle between $\mathbf{r}_1 = (4, -1, 2) + s(2, 2, -5)$ and $\mathbf{r}_2 = (3, -5, 6) + t(1, -2, -1)$.

Solution:

$\mathbf{r}_1 = \begin{pmatrix}4\\-1\\2\end{pmatrix} + s\begin{pmatrix}2\\2\\-5\end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix}3\\-5\\6\end{pmatrix} + s\begin{pmatrix}1\\-2\\-1\end{pmatrix}$

Direction vectors are used to find the angle:

$\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix}2\\2\\-5\end{pmatrix} \bullet \begin{pmatrix}1\\-2\\-1\end{pmatrix} = 2 - 4 + 5 = -3$

$|\mathbf{p}| = \sqrt{2^2 + 2^2 + (-5)^2} = \sqrt{4 + 4 + 25} = \sqrt{33} = 5{\cdot}74$

$|\mathbf{q}| = \sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6} = 2{\cdot}45$

$|\mathbf{p}||\mathbf{q}| = 3\sqrt{22} = 14{\cdot}07$

$\cos\theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = -\frac{3}{3\sqrt{22}} = -0{\cdot}213$

$\theta = 102{\cdot}3°$

Recall that if lines are perpendicular, $\theta = 90°$, hence $\cos\theta = 0$ and therefore $\mathbf{p} \bullet \mathbf{q} = 0$

Similarly, if lines are parallel, $\theta = 0°$, hence $\cos\theta = 1$ and therefore $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}||\mathbf{q}|$

67.17 Co-ordinates of a Point on a Line

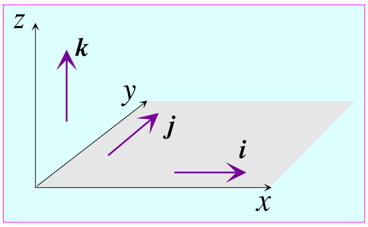
If we have an equation of a line, say, $\boldsymbol{r} = (4, -1, 2) + s(2, 3, -5)$ then the co-ordinates of any point on a line are given by adding the parts of the equation together:

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \qquad \text{Cco-ordinates of a point Q:} = \begin{pmatrix} 4 + 2s \\ -1 + 3s \\ 2 - 5s \end{pmatrix}$$

Note that when $s = 0$ then point Q coincides with the start point $(4, -1, 2)$.

67.18 3D Vectors

Note the convention that z is 'up'.



$\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$ means 5 in the x-direction, 4 in the y-direction, and 3 in the z-direction and can also be written as $5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$.

- The equation of a 3-D line still works the same way as a 2-D line.
- Now have the concept of planes. The horizontal plane is defined by the x-y axes and z will be zero. Vertical planes are defined by the z-y axes ($x = 0$), and the z-x axes ($y = 0$).
- Lines in 3-D can be parallel, but non parallel lines do not necessarily intersect. (Think of railway lines crossing a road). Lines which are not parallel & do not meet are called 'skew'
- In 2-D, a vector direction can be thought of in terms of gradient. This does not follow in 3-D.

67.18.1 Example:

1 The co-ordinates of $A = (-6, 3, 4)$ & $B = (-4, 9, 5)$. The line AB meets the xy plane at C. Find the co-ordinates of C.

Solution:

The translation vector $= \overrightarrow{AB} = \boldsymbol{b} \quad \boldsymbol{a} = \begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} - \begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$

This becomes the direction vector of a straight line such that:

$$\boldsymbol{r} = \begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$$

This cuts the plane where $z = 0 \quad \therefore 4 + t = 0 \quad \Rightarrow \quad t = -4$

$$\boldsymbol{r} = \begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix} + (-4)\begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -8 \\ -24 \\ -4 \end{pmatrix} = \begin{pmatrix} -14 \\ -21 \\ 0 \end{pmatrix} \Rightarrow (-14, -21, 0)$$

The co-ordinates of $C = (-14, -21, 0)$

2 Find the point of intersection of these two lines:

$$\boldsymbol{r}_1 = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad \& \quad \boldsymbol{r}_2 = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

Solution:

$a)$ Equate x components: $6 + 2t = -1 - s \quad \Rightarrow \quad s = -7 - 2t$

$b)$ Equate y components: $9 - 3t = -3 + 4s \quad \Rightarrow \quad 12 - 3t = 4s$

Substitute for s $\quad \therefore 12 - 3t = 4(-7 - 2t) \Rightarrow \quad 12 - 3t = -28 - 8t$

$5t = -40 \quad \Rightarrow \quad t = -8, \; s = 9$

Compare co-ords: first line $(-10, 33, -5)$

In second line $(-10, 33, 44)$

$\therefore$ Lines do not meet, they are skew.

3 Find the value of u for which the lines $\boldsymbol{r} = (\boldsymbol{j} - \boldsymbol{k}) + s(\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$ and $\boldsymbol{r} = (\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}) + t(\boldsymbol{i} + u\boldsymbol{k})$ intersect.

Solution:

$$(1) \quad \boldsymbol{r}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \& \quad (2) \quad \boldsymbol{r}_2 = \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ u \end{pmatrix}$$

x component: $0 + s = 1 + t$

y component: $1 + 2s = 7 \qquad \therefore s = 3$

$\therefore \quad 3 = 1 + t \quad \Rightarrow \quad t = 2$

$-1 + s = -4 + tu \quad \Rightarrow \quad 2u = 3 + s \quad \Rightarrow u = 3$

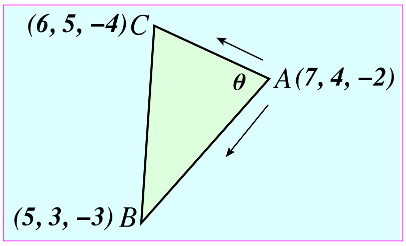
$$(1) \quad \boldsymbol{r}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

$$(2) \quad \boldsymbol{r}_2 = \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ u \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

4 The points A, B, & C have position vectors $\boldsymbol{a} = 7\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 5\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{c} = 6\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$

a) Find angle BAC

b) Find the area of the triangle ABC



Arrange tail to tail contact for measuring angles

Solution:

$$(a) \quad \overrightarrow{AC} = \begin{pmatrix} 6 \\ 5 \\ -4 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \qquad \overrightarrow{AB} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$$

Recall: $\quad \boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos\theta$

$$\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = \sqrt{6} \times \sqrt{6} \times \cos\theta$$

$$2 - 1 + 2 = 6\cos\theta$$

$$\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°$$

$$(b) \quad Area = \frac{1}{2}ab\sin c$$

$$= \frac{1}{2}\sqrt{6} \times \sqrt{6}\ \sin 60 = 3\sin 60 = \frac{3\sqrt{3}}{2}$$

5 The points A, B, & C have position vectors

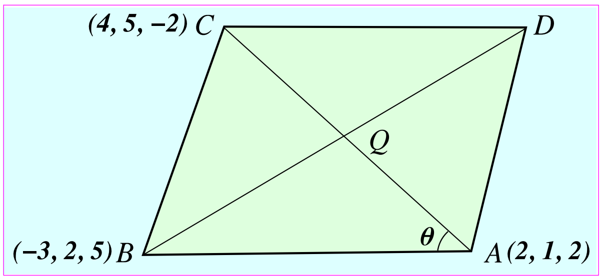
$$\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}; \quad \boldsymbol{c} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$$

The point D is such that $ABCD$ forms a parallelogram.

a) Find the position vector of D

b) Find the position vector of the point of intersection, Q, of the diagonals of the parallelogram

c) Find angle BAC



Solution:

(a) $\quad \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} \quad$ and $\quad \overrightarrow{BC} = \overrightarrow{AD}$

$$\overrightarrow{BC} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -7 \end{pmatrix}$$

$$\therefore \quad \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} \Rightarrow \quad \overrightarrow{OD} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 7 \\ 3 \\ -7 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ -5 \end{pmatrix}$$

Hence: $\quad D = \begin{pmatrix} 9 \\ 4 \\ -5 \end{pmatrix}$

(b) $\quad \overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AC}$

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix} \qquad \therefore \overrightarrow{AQ} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$$

(c) $$\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \sqrt{35} \times \sqrt{9} \times \cos\theta$$

$$-5 + 2 - 6 = \sqrt{35} \times 3\cos\theta$$

$$\cos\theta = \frac{-9}{3\sqrt{35}} = -\frac{3}{\sqrt{35}}$$

$$\theta = 120°$$

6 Two vectors are perpendicular to each other.

$$\boldsymbol{r}_1 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \qquad \boldsymbol{r}_2 = \begin{pmatrix} -3 \\ 1 \\ -6 \end{pmatrix} + t\begin{pmatrix} 3 \\ a \\ b \end{pmatrix}$$

a) Find the linear relationship between a & b.

As the vectors are perpendicular then the dot product of the vector must be zero, viz:

$$\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos\varphi$$

$$\text{If } \varphi = 90° \quad \Rightarrow \quad \cos\varphi = 0 \quad \therefore \quad \boldsymbol{p} \bullet \boldsymbol{q} = 0$$

In this case we only consider the directional vector part so:

$$\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ a \\ b \end{pmatrix} = 0$$

$$(1 \times 3) + (4 \times a) + (5 \times b) = 0 \quad \Rightarrow \quad 3 + 4a + 5b = 0$$

$$4a + 5b = -3$$

b) If the lines also intersect, then find the values s & t as well as a & b.

Rewriting the vectors:

$$\boldsymbol{r}_1 = \begin{pmatrix} 4 + s \\ 1 + 4s \\ 1 + 5s \end{pmatrix} \qquad \boldsymbol{r}_2 = \begin{pmatrix} -3 + 3t \\ 1 + at \\ -6 + bt \end{pmatrix}$$

Since they intersect then:

$$\begin{pmatrix} 4 + s \\ 1 + 4s \\ 1 + 5s \end{pmatrix} = \begin{pmatrix} -3 + 3t \\ 1 + at \\ -6 + bt \end{pmatrix}$$

Equate x components: $4 + s = -3 + 3t \quad \Rightarrow \quad 7 + s = 3t$ (1)

Equate y components: $1 + 4s = 1 + at \quad \Rightarrow \quad a = 4\frac{s}{t}$ (2)

Equate z components: $1 + 5s = -6 + bt \quad \Rightarrow \quad b = \frac{7 + 5s}{t}$ (3)

4 unknowns require 4 equations to solve the problem.

And from above we have: $4a + 5b = -3$ (4)

Substituting (2) & (3) into (4)

$$4\left(4\frac{s}{t}\right) + 5\left(\frac{7 + 5s}{t}\right) = -3 \quad \Rightarrow \quad \frac{16s}{t} + \frac{35 + 25s}{t} = -3$$

$$41s + 35 = -3t$$

From (1) $\quad 41s + 35 = -(7 + s) \quad \Rightarrow \quad 42s = -42 \quad \Rightarrow s = -1$

$$\therefore t = 2,\ a = -2,\ b = 1$$

c) Find the co-ordinates of the intersection.

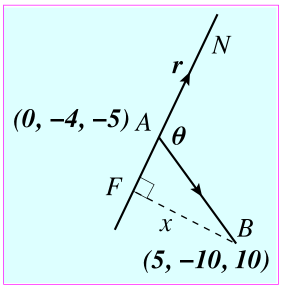
Substitute the values for the variables into the vectors and compare LHS & RHS:

$$\begin{pmatrix} 4 + s \\ 1 + 4s \\ 1 + 5s \end{pmatrix} = \begin{pmatrix} -3 + 3t \\ 1 + at \\ -6 + bt \end{pmatrix}$$

Intersect at these co-ordinates: $\begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix}$

7 A line, N, has a vector equation

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -4 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$$



The point B has the co-ordinates (5, −10, 10).
The point A, co-ordinates (0, − 4, −5) lies on the line N.

a) Find the angle between $\overrightarrow{AB}$ and the line N.

b) Find the distance from the foot of the perpendicular, F, on line N, to the point B.

Solution:

(*a*)

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -10 \\ 10 \end{pmatrix} - \begin{pmatrix} 0 \\ -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 - 0 \\ -10 - (-4) \\ 10 - (-5) \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 15 \end{pmatrix}$$

$$\boldsymbol{p} \bullet \boldsymbol{q} = \begin{pmatrix} 5 \\ -6 \\ 15 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \qquad \left[\overrightarrow{AB} \bullet \text{ direction vector of } \boldsymbol{r}\right]$$

$$= (5 \times 2) + (-6 \times 3) + (15 \times -6) = 10 - 18 - 90 = -98$$

$$|\boldsymbol{p}| = \sqrt{25 + 36 + 225} = \sqrt{286}$$

$$|\boldsymbol{q}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

$$\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos\theta \quad \Rightarrow \quad -98 = \sqrt{286} \times 7\cos\theta$$

$$\therefore \quad \cos\theta = \frac{-98}{\sqrt{286} \times 7} = -0{\cdot}8278 \quad \Rightarrow \quad \theta = 145{\cdot}9°$$

(*b*)

$$\sin(180 - 145{\cdot}9) = \sin 34{\cdot}1 = \frac{x}{\sqrt{286}} \quad \Rightarrow \quad x = \sqrt{286} \times \sin 34{\cdot}1 = 9{\cdot}49$$

8 Two lines have equations of $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + \lambda(\boldsymbol{i} + 3\boldsymbol{j})$ and $\boldsymbol{r} = 6\boldsymbol{i} - \boldsymbol{j} + \mu(\boldsymbol{i} - 4\boldsymbol{j})$. Find the position vector of the point of intersection.

Solution:

If the required point has position vector $\boldsymbol{p}$, then this must satisfy the vector equation of both lines.

$$\boldsymbol{p} = 2\boldsymbol{i} + \boldsymbol{j} + \lambda(\boldsymbol{i} + 3\boldsymbol{j})$$

$$\boldsymbol{p} = 6\boldsymbol{i} - \boldsymbol{j} + \mu(\boldsymbol{i} - 4\boldsymbol{j})$$

$$\therefore \quad 2\boldsymbol{i} + \boldsymbol{j} + \lambda(\boldsymbol{i} + 3\boldsymbol{j}) = 6\boldsymbol{i} - \boldsymbol{j} + \mu(\boldsymbol{i} - 4\boldsymbol{j})$$

Equating coefficients of terms:

$\boldsymbol{i}$ term: $\quad 2 + \lambda = 6 + \mu \quad \Rightarrow \quad \lambda - \mu = 4 \qquad (1)$

$\boldsymbol{j}$ term: $\quad 1 + 3\lambda = -1 - 4\mu \quad \Rightarrow \quad 3\lambda + 4\mu = -2 \qquad (2)$

Solving (1) & (2) gives: $\quad \lambda = 2, \ \mu = -2$

Substitute $\lambda = 2$ or $\mu = -2$ into the appropriate equation:

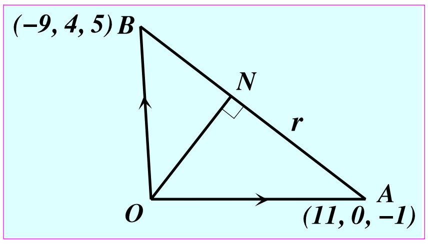
$$\boldsymbol{p} = 2\boldsymbol{i} + \boldsymbol{j} + \lambda(\boldsymbol{i} + 3\boldsymbol{j})$$

$$= 2\boldsymbol{i} + \boldsymbol{j} + 2(\boldsymbol{i} + 3\boldsymbol{j})$$

$$= 4\boldsymbol{i} + 7\boldsymbol{j}$$

9 A vector passes through two points A & B with the following co-ordinates:

$$A = \begin{pmatrix} 11 \\ 0 \\ -1 \end{pmatrix} \qquad B = \begin{pmatrix} -9 \\ 4 \\ 5 \end{pmatrix}$$



Find the vector equation of the line AB and the co-ordinates of point N if the vector $\overrightarrow{ON}$ is perpendicular to the vector $\overrightarrow{AB}$. Hence, find the length of $\overrightarrow{ON}$, and the area of the triangle ABO.

Solution:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \begin{pmatrix} -11 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -20 \\ 4 \\ 6 \end{pmatrix}$$

$$\therefore \text{ Equation of the line } AB \text{ is: } \boldsymbol{r} = \begin{pmatrix} 11 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} -20 \\ 4 \\ 6 \end{pmatrix}$$

The vector $\overrightarrow{ON}$, and the co-ordinates of N, are both given by the equation of the line $\boldsymbol{r}$, and all that is required is to find the appropriate value of s. Let t be the value of s at point N.

As the lines are perpendicular then the scalar or dot product is zero. Using the direction vectors of both vectors we have:

$$\overrightarrow{AB} \bullet \overrightarrow{ON} = 0$$

$$\begin{pmatrix} -20 \\ 4 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 11 - 20t \\ 4t \\ -1 + 6t \end{pmatrix} = -20(11 - 20t) + 4(4t) + 6(-1 + 6t) = 0$$

$$\Rightarrow -220 + 400t + 16t - 6 + 36t = 0$$

$$\Rightarrow t = \frac{226}{452} = \frac{1}{2}$$

$$\therefore \text{ Co-ordinates of point } N: \begin{pmatrix} 11 - 10 \\ 2 \\ -1 + 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

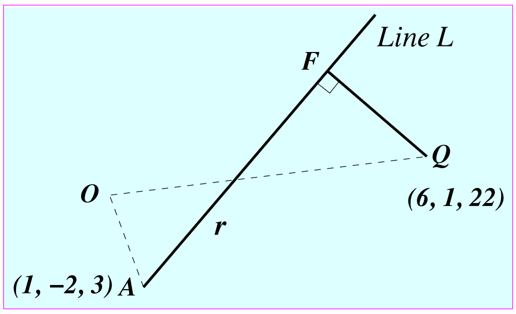
Length of $\overrightarrow{ON}$ is: $\left|\overrightarrow{ON}\right| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

Length of $\overrightarrow{AB}$ is: $\left|\overrightarrow{AB}\right| = \sqrt{(-20)^2 + 4^2 + 6^2} = \sqrt{400 + 16 + 36} = 21{\cdot}26$

Area of triangle ABO is: $\frac{1}{2} \times 3 \times 21{\cdot}26 = 31{\cdot}89$ *sq units*

10 A line L has a vector equation of

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$



A point Q has the co-ordinates (6, 1, 22).

Find the co-ordinates of the foot, F, of the perpendicular from the line L to the point Q and find the distance from Q to the line L.

Solution:

If F represents the foot of the perpendicular from Q, then the vector $\overrightarrow{OF}$ is:

$$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$$

$$\overrightarrow{OF} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 + 3\mu \\ -2 + 3\mu \\ 3 + 3\mu \end{pmatrix}$$ where μ is the value of s that defines F.

As the lines are perpendicular then the scalar or dot product is zero. Using the direction vectors of both vectors we have:

$\overrightarrow{AF} \bullet \overrightarrow{FQ} = 0$ Measure angles 'tail to tail'

The vector $\overrightarrow{QF}$ is:

$$\overrightarrow{FQ} = \overrightarrow{OQ} - \overrightarrow{OF}$$

$$\overrightarrow{FQ} = \begin{pmatrix} 6 \\ 1 \\ 22 \end{pmatrix} - \begin{pmatrix} 1 + 3\mu \\ -2 + 3\mu \\ 3 + 3\mu \end{pmatrix} = \begin{pmatrix} 5 - 3\mu \\ 3 - 3\mu \\ 19 - 3\mu \end{pmatrix}$$

$$\begin{pmatrix} 5 - 3\mu \\ 3 - 3\mu \\ 19 - 3\mu \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3(5 - 3\mu) + 3(3 - 3\mu) + 3(19 - 3\mu) = 0$$

$$\Rightarrow 15 - 9\mu + 9 - 9\mu + 57 - 9\mu = 0$$

$$\Rightarrow 81 - 27\mu = 0$$

$$\therefore \quad \mu = 3$$

$\therefore$ the co-ordinates of $F = \begin{pmatrix} 1 + 9 \\ -2 + 9 \\ 3 + 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ 12 \end{pmatrix}$ and $\overrightarrow{FQ} = \begin{pmatrix} 5 - 9 \\ 3 - 9 \\ 19 - 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 10 \end{pmatrix}$

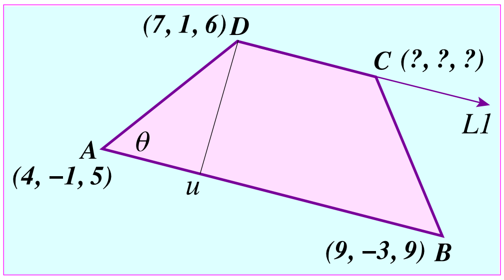
The distance of $\overrightarrow{QF} \Rightarrow$ $\left|\overrightarrow{QF}\right| = \sqrt{4^2 + 6^2 + 10^2} = \sqrt{16 + 36 + 100}$

$$\left|\overrightarrow{QF}\right| = \sqrt{152}$$

11 A trapezium has the co-ordinates for points A, B, and D as shown.

Find the angle θ, the equation of the line $L1$, and the co-ordinate of point C.

$\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel, and the magnitudes of $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are equal.



(a) Find the angle θ:

$$\cos\theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{\overrightarrow{AB} \bullet \overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|}$$

$$\overrightarrow{AB} = \begin{pmatrix} 9 \\ -3 \\ 9 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} \qquad \& \qquad \overrightarrow{AD} = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AB} \bullet \overrightarrow{AD} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = (5 \times 3) + (-2 \times 2) + (4 \times 1) = 15 - 4 + 4 = 15$$

$$|\overrightarrow{AB}| = \sqrt{25 + 4 + 16} = \sqrt{45}$$

$$|\overrightarrow{AD}| = \sqrt{9 + 4 + 1} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{15}{\sqrt{45}\,\sqrt{14}}$$

(b) The equation of L1:

$$L1 = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} \qquad \text{The co-ordinate of point } C = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} + s \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$$

where s is the scalar that satisfies the point C.

***(c)* The co-ordinate of point *C*.** There are several ways to tackle this, but one of the easiest ways is to compare the magnitudes of the parallel lines in the trapezium:

From the equation $L1$
$$\overrightarrow{DC} = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} + s \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} = s \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$$

$$|\overrightarrow{DC}| = \sqrt{25s^2 + 4s^2 + 16s^2} = s\sqrt{45} \qquad \ldots (1)$$

From the diagram
$$|\overrightarrow{DC}| = |\overrightarrow{AB}| - 2|\overrightarrow{Au}|$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{|\overrightarrow{Au}|}{|\overrightarrow{AD}|} \qquad \therefore \; |\overrightarrow{Au}| = |\overrightarrow{AD}| \cos\theta$$

$$|\overrightarrow{Au}| = |\overrightarrow{AD}| \cos\theta = \sqrt{14} \times \frac{15}{\sqrt{45}\,\sqrt{14}} = \frac{15}{\sqrt{45}}$$

$$|\overrightarrow{DC}| = |\overrightarrow{AB}| - 2|\overrightarrow{Au}| \quad \Rightarrow \quad \sqrt{45} - 2\frac{15}{\sqrt{45}} = \frac{15}{\sqrt{45}} \qquad \ldots (2)$$

From (1) & (2) $s\sqrt{45} = \frac{15}{\sqrt{45}} \quad \Rightarrow \quad s = \frac{15}{\sqrt{45}\sqrt{45}} = \frac{15}{45} = \frac{1}{3}$

$$\therefore \text{The co-ordinate of point } C = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 + \frac{5}{3} \\ 1 - \frac{2}{3} \\ 6 + \frac{4}{3} \end{pmatrix} = \begin{pmatrix} 8\frac{2}{3} \\ \frac{1}{3} \\ 7\frac{1}{3} \end{pmatrix}$$

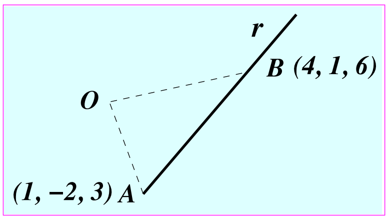
67.19 Topical Tips

When doing vector problems, it pays to draw a sketch. For 3-D work, just plot the x & y axes and let the z axis hang in space. Although this might not work out in every case, it does give a very good sense of how the vectors are laid out.

In plotting a line for a given vector equation, say:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

just plot the starting point A (1, −2, 3) and give the scalar s an easy value like 1 and plot B (4, 1, 6).



67.20 Vector Digest

$$\overrightarrow{AB} = (B\text{ co-ords}) - (A\text{ co-ords})$$

$$\overrightarrow{AB} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} - \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \text{translation vector} \equiv \text{‘slope’}$$

$$\left|\overrightarrow{OQ}\right| = \left\|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\| = \sqrt{a^2 + b^2 + c^2}$$

Equation of a line $= \boldsymbol{r} =$ Start point co-ords + (scalar × Direction vector)

$$\boldsymbol{r} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} + \lambda\begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix}$$

Scalar Product (n.b. the answer is a scalar)

$$\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos\theta$$

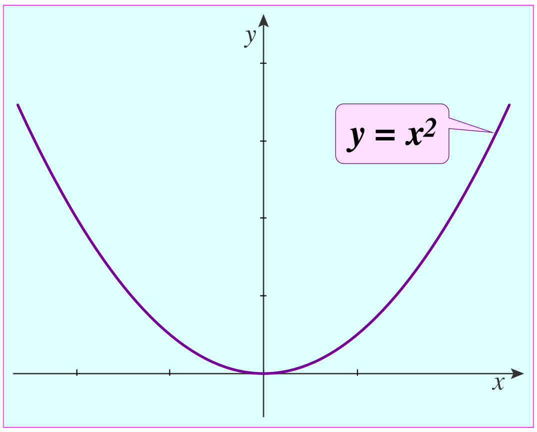
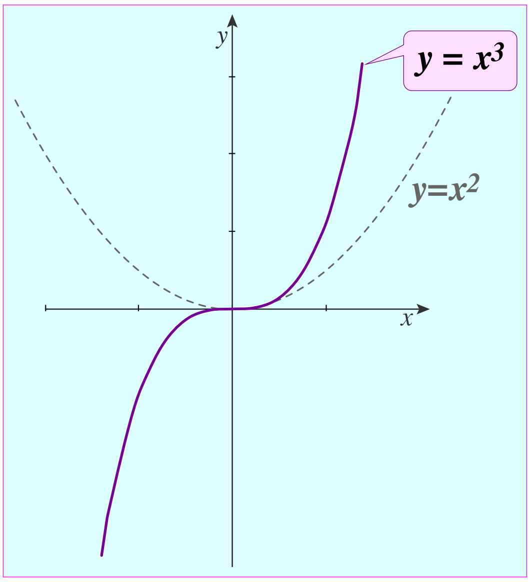
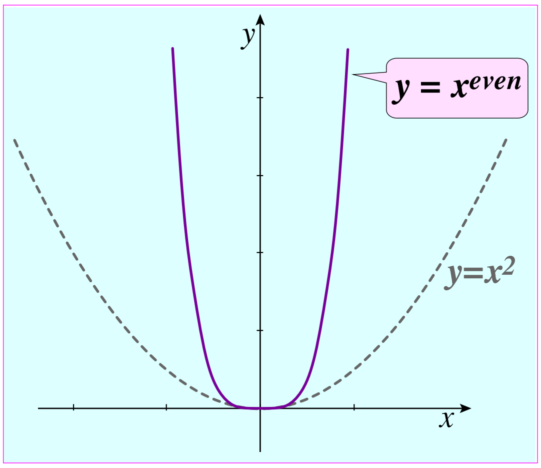
$$\therefore \quad \cos\theta = \frac{\boldsymbol{p} \bullet \boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$$

$$\boldsymbol{a} \bullet \boldsymbol{b} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} \bullet \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} a_x b_x \\ a_y b_y \end{pmatrix} = (a_x \times b_x) + (a_y \times b_y) = a_x b_x + a_y b_y$$

When $90 < \theta < 180 \qquad \boldsymbol{p} \bullet \boldsymbol{q} < 0$

When lines are perpendicular $\qquad \theta = 90°, \qquad \cos 90° = 0 \qquad \boldsymbol{p} \bullet \boldsymbol{q} = 0$

When lines are parallel $\qquad \theta = 0°, \qquad \cos 0° = 1 \qquad \boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|$

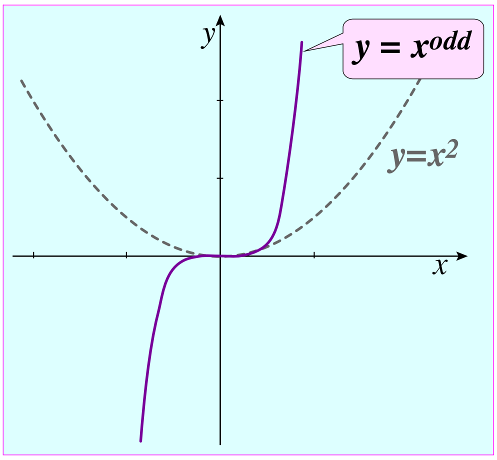
Function	Properties	Illustration
$y = kx^2$ $k > 0$	**Quadratic Function:** y is proportional to the square of x. As x doubles, y increases 4 fold. Function is even. $[f(x) = f(-x)]$ Domain: $x \in \mathbb{R}$ Range: $f(x) \geqslant 0$ Intercept (0, 0) Line symmetry about the y-axis. Decreasing function for $x < 0$ Increasing function for $x > 0$	
$y = kx^3$ $k > 0$	**Cubic Function:** y is proportional to the cube of x. As x doubles, y increases 8 fold. Function is odd. $[-f(x) = f(-x)]$ Domain: $x \in \mathbb{R}$ Range: $f(x) \in \mathbb{R}$ Intercept (0, 0) Rotational symmetry about the origin - order 2. Increasing function	
$y = kx^{even}$ $k > 0$	**Even Power Function:** Function is even Domain: $x \in \mathbb{R}$ Range: $f(x) \geqslant 0$ Intercept (0, 0) Passes points (−1, 1) and (1, 1) Line symmetry about the y-axis. Decreasing function for $x < 0$ Increasing function for $x > 0$	

$y = kx^{odd}$

$(k > 0)$

Odd Power Function:

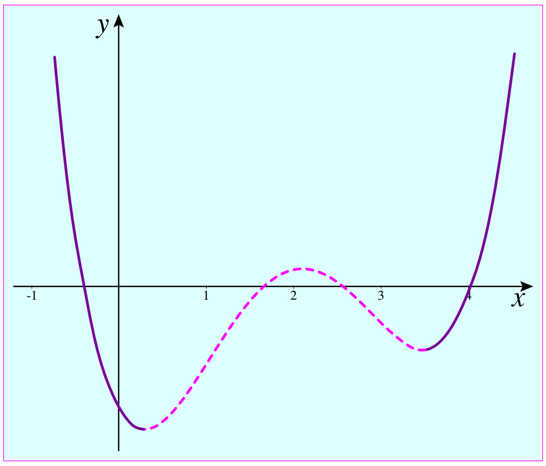
Function is odd
Domain: $x \in \mathbb{R}$
Range: $f(x) \in \mathbb{R}$
Intercept (0, 0)
Passes points $(-1, -1)$ and $(1, 1)$
Rotational symmetry about the origin - order 2.
Increasing function



$y = kx^n + x^{n-1} \ldots + x^{n-2} + \ldots + c$
$n =$ even
$(k > 0)$

Even Order Polynomial Function:

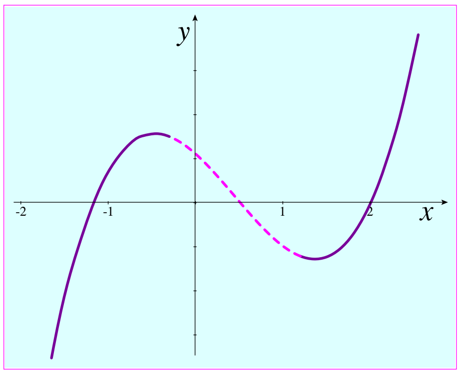
Domain: $x \in \mathbb{R}$
Range: $f(x) \geqslant$ *min vertex*
Intercepts yes
No of turning points: $n - 1$



$y = kx^n + x^{n-1} \ldots + x^{n-2} + \ldots + c$
$n =$ odd
$(k > 0)$

Odd Order Polynomial Function:

Domain: $x \in \mathbb{R}$
Range: $f(x) \in \mathbb{R}$
Intercepts yes
No of turning points: $n - 1$

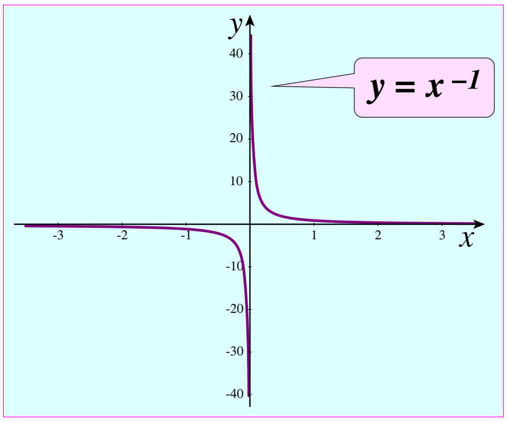


$y = \frac{k}{x} = kx^{-1}$

$(k > 0)$

Reciprocal Function:

Curve call a Hyperbola.
y is inversely proportional to x.
As x doubles, y decreases 2 fold.
Function is odd
Domain: $x \in \mathbb{R}, \; x \neq 0$
Range: $f(x) \in \mathbb{R}, \; f(x) \neq 0$
No intercepts
Asymptotes are x-axis and y-axis
Decreasing function
Rotational symmetry about the origin - order 2.

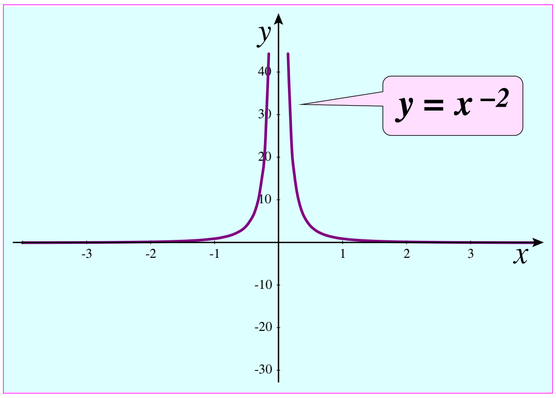


$y = \frac{k}{x^2} = kx^{-2}$

$(k > 0$

Inverse Square Function:

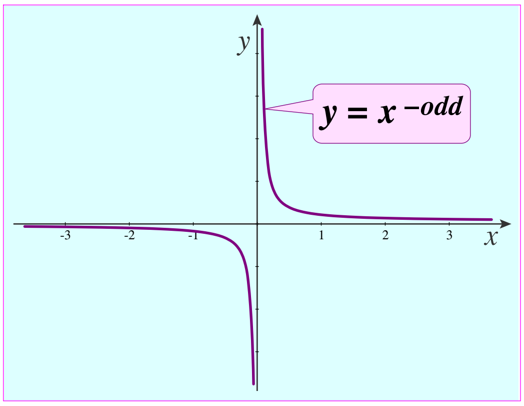
y is inversely proportional to the square of x.
As x doubles, y decreases 4 fold
Function is even
Domain: $x \in \mathbb{R},\ x \neq 0$
Range: $f(x) > 0$
No intercepts
Asymptotes are x-axis and y-axis
Symmetric about the y-axis.



$y = \frac{k}{x^{odd}} = kx^{-odd}$

$(k > 0)$

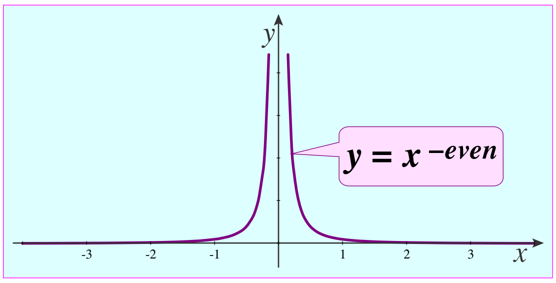
Inverse Odd Power Function:
Generic shape for this type of graph.
Function is odd
Domain: $x \in \mathbb{R},\ x \neq 0$
Range: $f(x) \in \mathbb{R},\ f(x) \neq 0$
No intercepts
Asymptotes: x-axis and y-axis
Decreasing function
Rotational symmetry about the origin - order 2.



$y = \frac{k}{x^{even}} = kx^{-even}$

$(k > 0)$

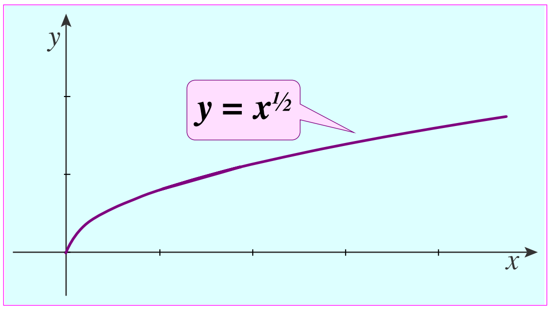
Inverse Even Power Function:
Generic shape for this type of graph.
Function is even
Domain: $x \in \mathbb{R},\ x \neq 0$
Range: $f(x) > 0$
No intercepts
Decreasing function for $x < 0$
Increasing function for $x > 0$



$y = k\sqrt{x} = kx^{\frac{1}{2}}$

$(k > 0)$

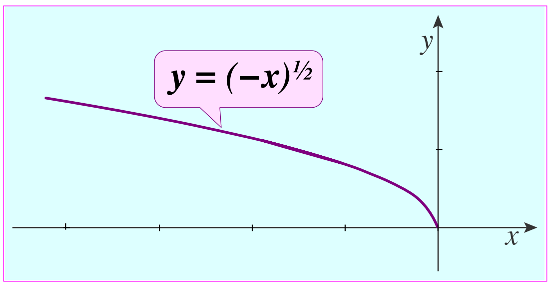
Square Root Function:
y is inversely proportional to the square root of x.
As x increases 4 fold, y increases 2 fold.
Domain: $x \in \mathbb{R},\ x \geqslant 0$
Range: $f(x) \geqslant 0$
Intercept (0, 0)
Increasing function from $x \geqslant 0$



$y = k\sqrt{-x} = k(-x)^{\frac{1}{2}}$

Square Root (–ve x) Function:

Domain: $x \in \mathbb{R},\ x \leqslant 0$
Range: $f(x) \geqslant 0$
Intercept (0, 0)
Decreasing function

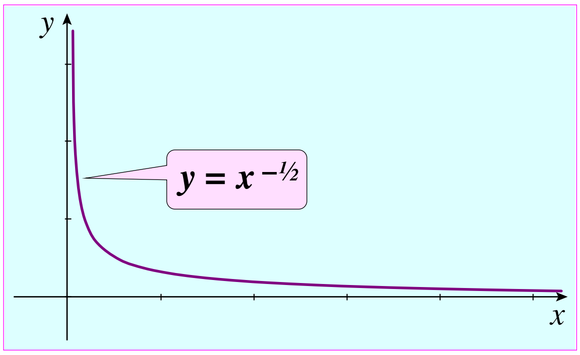


$y = \frac{k}{\sqrt{x}} = kx^{-\frac{1}{2}}$

$(k > 0)$

Inverse Square Root Function:

y is inversely proportional to the square root of x.
As x increases 4 fold, y decreases 2 fold.
Domain: $x \in \mathbb{R},\ x > 0$
Range: $f(x) > 0$
No Intercepts
Asymptotes: x-axis and y-axis

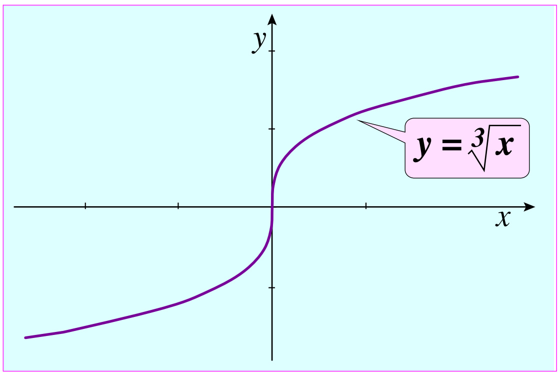


$y = \sqrt[3]{x}$

Cube Root Function:

Odd function
Domain: $x \in \mathbb{R}$
Range: $f(x) \in \mathbb{R}$

Intercept (0, 0)
Passes points (1, 1), (0, 0), (–1, –1)
Rotational symmetry about the origin - order 2.
No domain contraints on odd numbered roots, as can take 5th, 7th, etc roots of a negative number



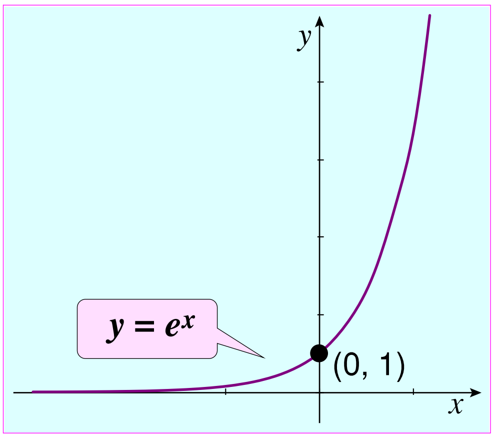
$y = ke^x$

$(k > 0)$

Exponential Function:

y is proportional to a number raised to the power x.
As x increases, y increases exponentially.

Domain: $x \in \mathbb{R}$,
Range: $f(x) > 0$
Intercept (0, 1)
Asymptote: x-axis
Increasing function for +ve x

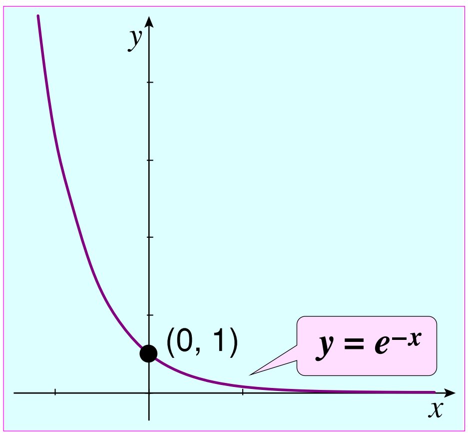


$y = \frac{k}{e^x} = ke^{-x}$

$(k > 0)$

Decaying Exponential Function:

y is inversely proportional to a number raised to the power $-x$.
As x increases, y decreases exponentially.
Domain: $x \in \mathbb{R}$
Range: $f(x) > 0$
Intercept (0, 1)
Horizontal asymptote: x-axis
Decreasing function for +ve x



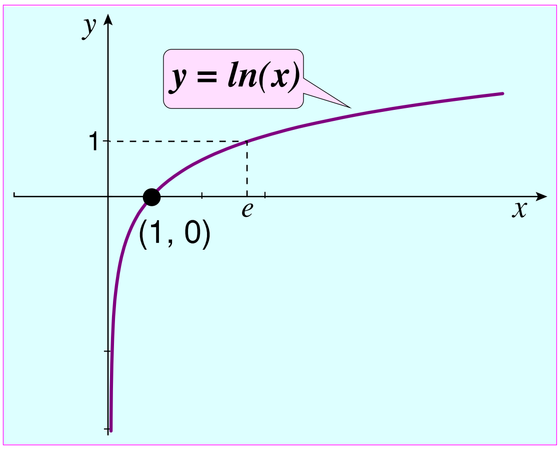
$y = k\,ln(x)$

$y = k\,log_b(x)$

$k > 0$

Log (*ln*) Function:

Domain: $x \in \mathbb{R},\ x > 0$
Range: $f(x) \in \mathbb{R}$
Intercept (1, 0)
Asymptote: y-axis
Increasing function for +ve x
Reflection of $f(x) = e^x$ in the line $y = x$, hence inverse of e^x



$y = k\,ln(-x)$

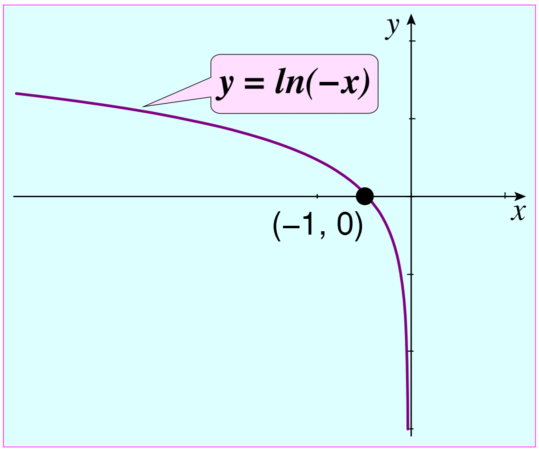
$y = k\,log_b(-x)$

$k > 0$

Log Function ($-x$):

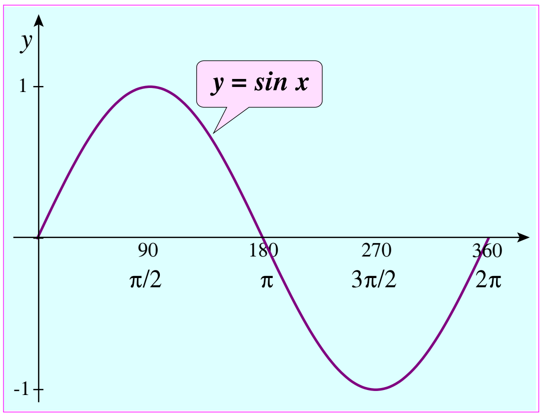
Domain: $x \in \mathbb{R},\ x < 0$
Range: $f(x) \in \mathbb{R}$
Intercept (−1, 0)
Vertical asymptote: y-axis

Decreasing function



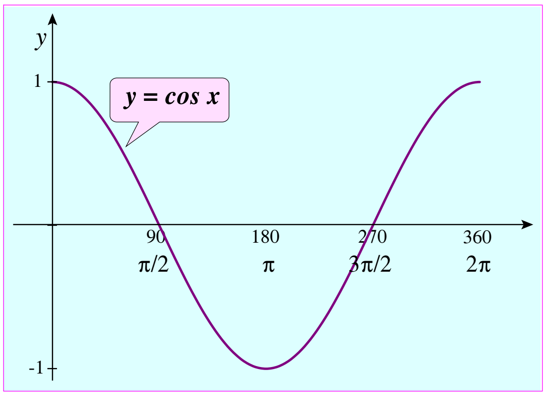
$y = \sin x$

Sine Function:
Odd function
Domain: $x \in \mathbb{R}$
Range: $-1 \leqslant f(x) \leqslant 1$
Periodic function, period 2π
$\sin(\theta + 2\pi) = \sin\theta$
$\sin(-\theta) = -\sin\theta$
x-intercept $(n\pi, 0)$
y-intercept $(0, 0)$
Rotational symmetry, order 2, about the origin and also at every point it crosses the x-axis.
Line symmetry about every vertical line passing through each vertex.



$y = \cos x$

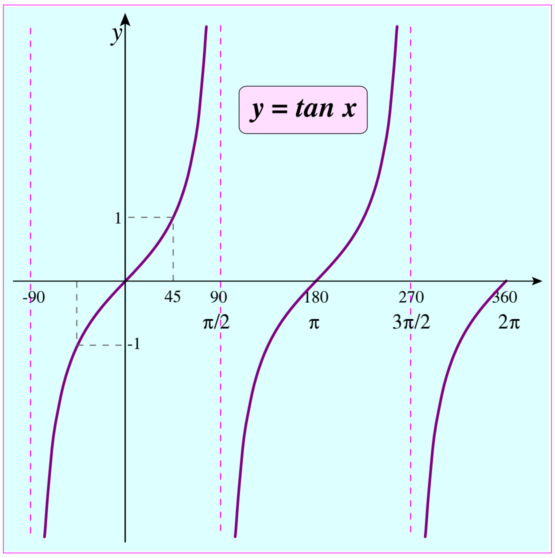
Cosine Function:
Even function
Domain: $x \in \mathbb{R}$
Range: $-1 \leqslant f(x) \leqslant 1$
Periodic function, period 2π
$\cos(\theta + 2\pi) = \cos\theta$
$\cos(-\theta) = \cos\theta$
x-intercept $\left(\frac{\pi}{2} + n\pi, \ 0\right)$
y-intercept $(0, 1)$
Rotational symmetry, order 2, about the origin and also at every point it crosses the x-axis.
Line symmetry about every vertical line passing through each vertex.



$y = \tan x$

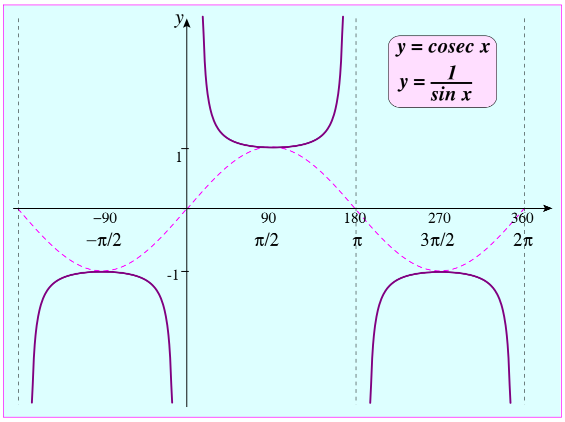
Tangent Function:
Odd function
Domain: $x \in \mathbb{R}, x \neq \frac{\pi}{2} + n\pi$
Range: $f(x) \in \mathbb{R}$
Periodic function, period π
x-intercept $(n\pi, \ 0)$
y-intercept $(0, 0)$
Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Rotational symmetry, order 2, about the origin and also about $\pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \ldots$



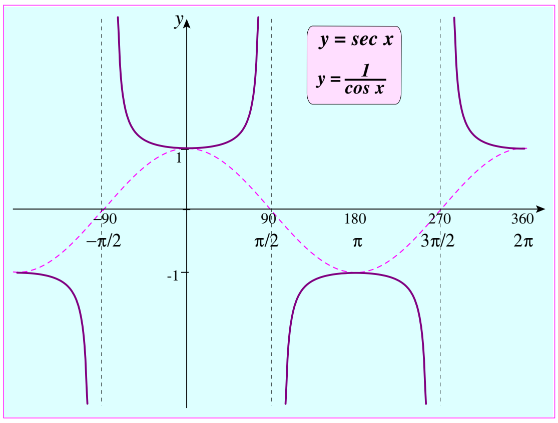
$y = cosec\,x$

Cosecant Function:
Odd function
Domain: $x \in \mathbb{R},\ x \neq n\pi$
Range: $-1 \geqslant f(x) \geqslant 1$
$|cosec\,x| \geqslant 1$
Periodic function, period 2π
No x or y intercepts
Vertical asymptotes: $x = n\pi$
where $sin\,x$ crosses the x-axis at any multiple of π ($sin\,x = 0$)
Rotational symmetry about the origin - order 2.
Line symmetry about every vertical line passing through each vertex.



$y = sec\,x$

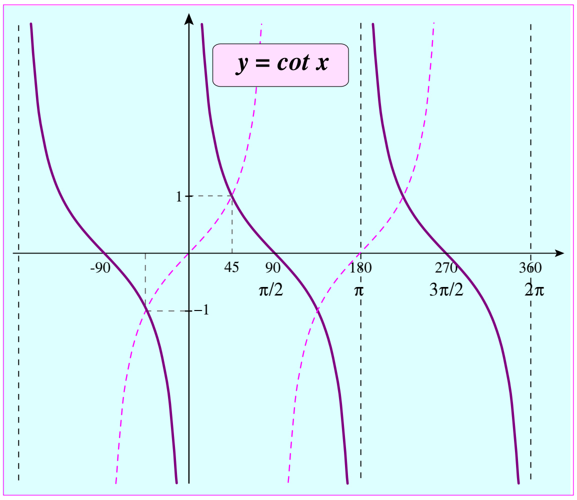
Secant Function:
Even function
Domain: $x \in \mathbb{R},\ x \neq \dfrac{\pi}{2} + n\pi$
Range: $-1 \geqslant f(x) \geqslant 1$
$|sec\,x| \geqslant 1$
Periodic function, period 2π
y-intercept: $(0, 1)$
Vertical asymptotes: $x = \dfrac{\pi}{2} + n\pi$
where $cos\,x$ crosses the x-axis at odd multiples of ½π ($cos\,x = 0$)
Line symmetry about the y-axis and every vertical line passing through each vertex.



$y = cot\,x$

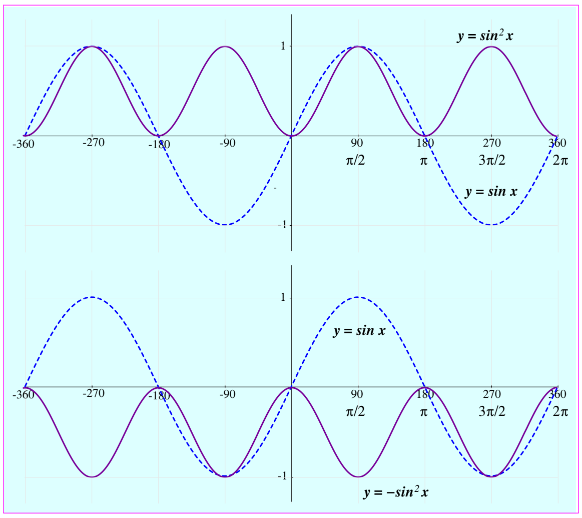
Cotangent Function:
Odd function
Domain: $x \in \mathbb{R},\ x \neq n\pi$
Range: $f(x) \in \mathbb{R}$

Periodic function, period π
x-intercepts: $\left(\dfrac{\pi}{2} + n\pi,\ 0\right)$ where $tan\,x$ has asymptotes
Vertical asymptotes: $x = n\pi$
where $tan\,x$ crosses the x-axis at any multiple of π ($tan\,x = 0$)
Rotational symmetry about the origin - order 2.



$y = sin^2 x$

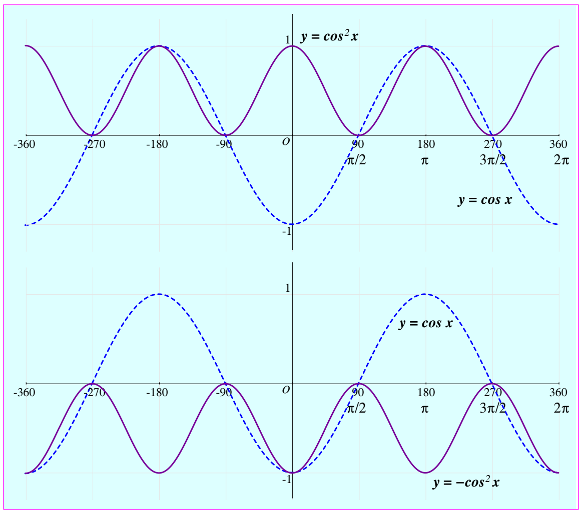
Squared Sine Function:



$y = -sin^2 x$

$y = cos^2 x$

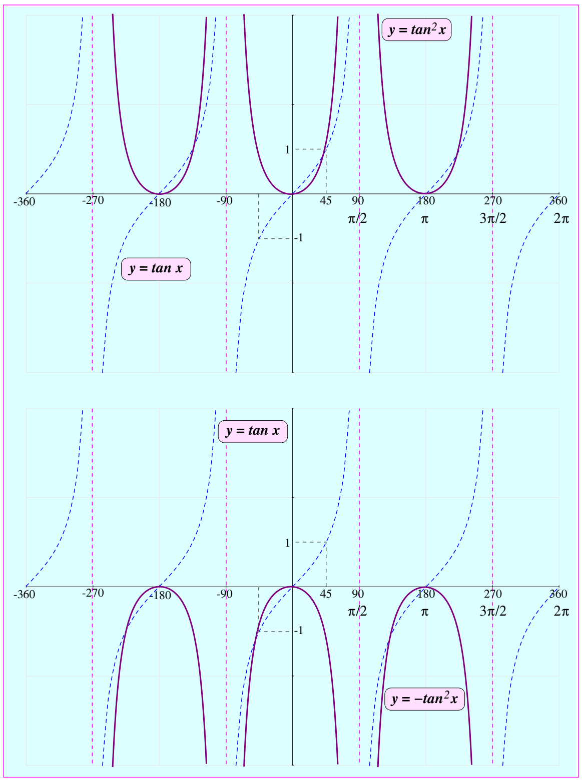
Squared Cosine Function:



$y = -cos^2 x$

$y = tan^2 x$

Squared Tangent Function:



$y = -tan^2 x$

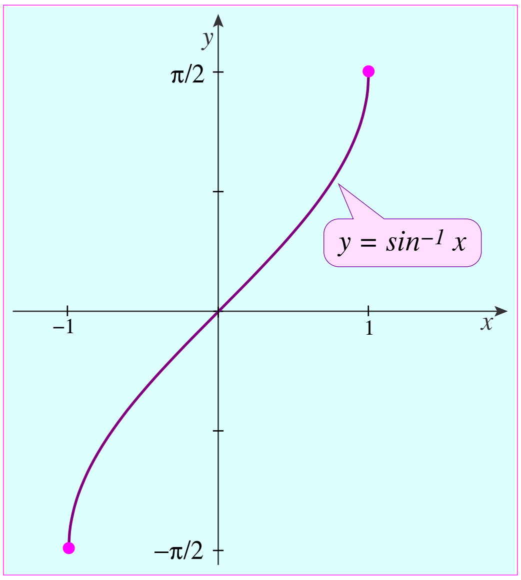
$y = sin^{-1} x$

Inverse Sine Function:

Odd function
Restricted Domain: $-1 \leqslant x \leqslant 1$
Range: $-\frac{\pi}{2} \leqslant sin^{-1}x \leqslant \frac{\pi}{2}$

Intercept: (0, 0)

Rotational symmetry about the origin - order 2.
Increasing function



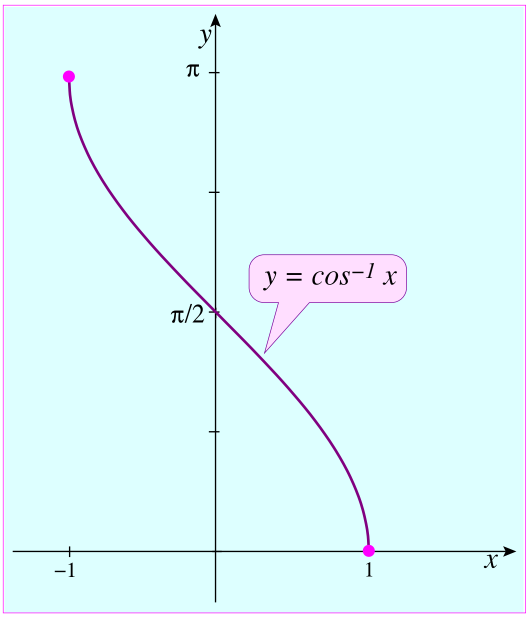
$y = cos^{-1} x$

Inverse Cosine Function:

Restricted Domain: $-1 \leqslant x \leqslant 1$
Range: $0 \leqslant cos^{-1}x \leqslant \pi$

y-intercept $\left(0, \frac{\pi}{2}\right)$

Decreasing function

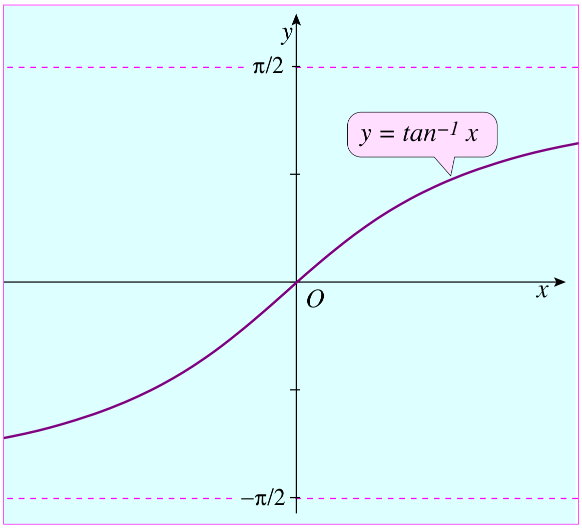


$y = tan^{-1} x$

Inverse Tangent Function:

Odd function
Domain: $x \in \mathbb{R}$
Range: $-\frac{\pi}{2} \leqslant tan^{-1}x \leqslant \frac{\pi}{2}$

Intercept (0, 0)
Horizontal asymptotes: $y = \pm\frac{\pi}{2}$
Rotational symmetry about the origin - order 2.
Increasing function



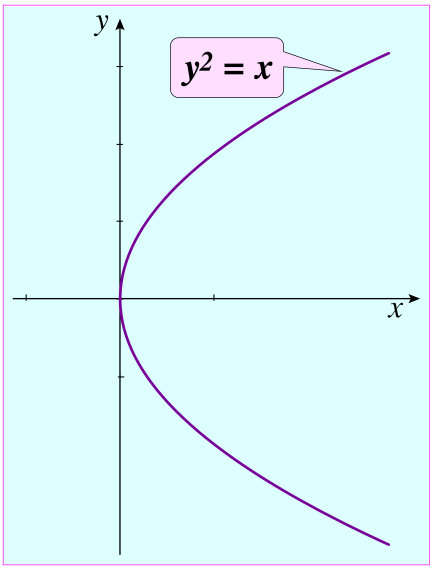
$y^2 = x$

$y = \sqrt{x}$
+
$y = -\sqrt{x}$

Square Root Relation:

Domain: $x \in \mathbb{R}, \quad x \geqslant 0$
Range: $f(x) \in \mathbb{R}$

Intercept (0, 0)
Passes points (4, 2), (0, 0), (4, −2) plus others.
Not a true function. Can be made up of two functions: $y = \sqrt{x}$ (top half)
$y = -\sqrt{x}$ (bottom half)

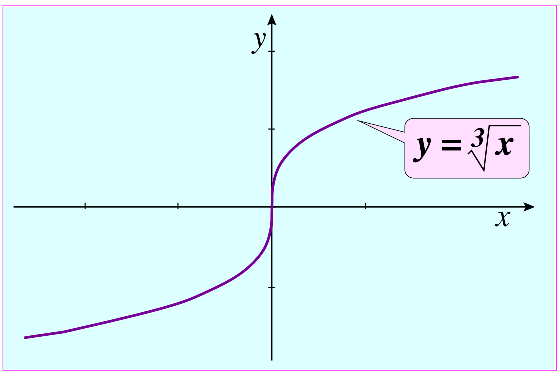


$y = \sqrt[3]{x}$

Cube Root Function:

Odd function
Domain: $x \in \mathbb{R}$
Range: $f(x) \in \mathbb{R}$

Intercept (0, 0)
Passes points (1, 1), (0, 0), (−1, −1)
Rotational symmetry about the origin - order 2.
No domain contraints on odd numbered roots, as can take 5th, 7th, etc roots of a negative number.



69 • Apdx • Facts, Figures & Formulæ

69.1 Quadratics

69.1.1 Completing the Square

Standard solution:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

$$x^2 - bx + c = \left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

For a quadratic of the form $a(x + k)^2 + q$

$$y = a(x + k)^2 + q$$

Co-ordinates of vertex $(k,\ q)$

Axis of symmetry $x = k$

If $a > 0$, graph is ∪ shaped, vertex is a minimum point

If $a < 0$, graph is ∩ shaped, vertex is a maximum point

For a quadratic of the form $ax^2 + bx + c$

Turning point is when $x = -\frac{b}{2a}$; $y = -\frac{b^2}{4a} + c$

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

69.1.2 Quadratic Formula

The roots of a quadratic are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression "$b^2 - 4ac$" is known as the **discriminant**.

If…	Then…	Roots or solutions	Notes
Discriminant > 0	Graph intersects the x-axis twice	2 distinct real solutions	If the **discriminant** is a perfect square, the solution is rational and can be factorised.
Discriminant = 0	Graph intersects the x-axis once	1 real solution $= -\frac{b}{2a}$	Sometimes called repeated or coincident roots. The **quadratic** is a perfect square.
Discriminant < 0	Graph does not intersect the x-axis	No real solutions	Only complex roots, which involve imaginary numbers ($\sqrt{-1}$).

69.2 Series

69.2.1 Sigma Notation

The sigma notation can be handled according to these rules:

$$\sum_{r=1}^{n}(a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$$

$$\sum_{r=1}^{k} a_r + \sum_{r=k+1}^{n} a_r = \sum_{r=1}^{n} a_r \qquad r < k < n$$

$$\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$$

$$\sum_{1}^{n} c = nc \qquad \text{where } c \text{ is a constant}$$

$$\sum_{1}^{n} 1 = n$$

69.2.2 Standard Sigma Results

Certain standard sums exist such as:

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left[\frac{1}{2}n(n+1)\right]^2 = \left[\sum_{r=1}^{n} r\right]^2$$

These standard results can be used to derive more complicated series.

69.2.3 Arithmetic Progression

$$U_n = a + (n-1)d$$

$$S_n = n\left[\frac{a+l}{2}\right] \qquad or \qquad S_n = \frac{n}{2}[a+l]$$

where $l = a + (n-1)d$

Sum to Infinity of a Arithmetic Progression

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$U_n = a + U_{n-1}d$$

$$U_{n+1} = a + U_n d$$

69.2.4 Geometric Progression

$$U_n = ar^{(n-1)}$$

$$U_n = U_0 r^n$$

Sum of a Geometric Progression:

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

$$S_n = \frac{a(r^n - 1)}{(r - 1)} \qquad r > 1$$

Sum to Infinity of a Geometric Progression:

$$S_\infty = \frac{a}{(1 - r)} \qquad |r| < 1$$

69.2.5 Binomial Expansion (Positive integers)

The Binomial theorem, where n is a positive integer:

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$$

$$(n \in \mathbb{N})$$

$$(a + b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + {}^nC_{n-1} ab^{n-1} + b^n$$

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$$

$$(a + b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \qquad or \qquad = \sum_{r=0}^{n}{}^nC_r a^{n-r}b^r$$

Where:

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$${}^nC_r = {}^nC_{n-r}$$

$${}^nC_2 = \binom{n}{2} = \frac{n(n-1)}{2 \times 1} \qquad {}^nC_3 = \binom{n}{3} = \frac{n(n-1)(n-2)}{3 \times 2 \times 1}$$

The k-th term:

$$= {}^nC_{k-1} a^{n-k+1}b^{k-1} \qquad or \qquad \binom{n}{k-1}a^{n-k+1}b^{k-1}$$

For the term in b^r

$$= {}^nC_r a^{n-r}b^r \qquad or \qquad \binom{n}{r}a^{n-r}b^r$$

Note: the combination format will only work if n is a positive integer. For $n < 1$ then the full version of the Binomial theorem is required.
Where n is a positive integer, the expansion terminates after $n + 1$ terms, and is valid for all values of x.
The use of the ${}^nC_r$ form of combination symbol, is simply that this is the symbology used on calculators.

69.2.6 Binomial Expansion (Rational or negative Index)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$
$$+ \ldots \frac{n(n-1)\ldots(n-r+1)}{r!}x^r \ldots \qquad (|x| < 1, n \in \mathbb{R})$$

Just watch the minus signs!!! Thus:

$$(1-x)^n = 1 + n(-x) + \frac{n(n-1)}{2!}(-x)^2 + \frac{n(n-1)(n-2)}{3!}(-x)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}(-x)^4$$

$$(a+bx)^n = \left[a\left(1+\frac{bx}{a}\right)\right]^n = a^n\left(1+\frac{bx}{a}\right)^n$$

$$= a^n\left[1 + n\frac{b}{a}x + \frac{n(n-1)}{2!}\left(\frac{b}{a}x\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}x\right)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{b}{a}x\right)^4\right]$$

Valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \frac{a}{b}$

- For the general Binomial Theorem any rational value of n can be used (i.e. fractional or negative values, and not just positive integers).
- For these expansions, the binomial must start with a 1 in the brackets. For binomials of the form $(a+bx)^n$, the a term must be factored out.

  Therefore, the binomial $(a+bx)^n$ must be changed to $a^n\left(1+\frac{b}{a}x\right)^n$.
- When n is a positive integer the series is finite and gives an exact value of $(1+x)^n$ and is valid for all values of x. The expansion terminates after $n+1$ terms, because coefficients after this term are zero.
- When n is either a fractional and/or a negative value, the series will have an infinite number of terms. and the coefficients are never zero.
  - In these cases the series will either diverge and the value will become infinite or they will converge, with the value converging towards the value of binomial $(1+x)^n$.
  - The general Binomial Theorem will converge when $|x| < 1$ (*i.e.* $-1 < x < 1$). This is the condition required for convergence and we say that the series is valid for this condition.
  - For binomials of the form $a^n\left(1+\frac{b}{a}x\right)^n$, the series is only valid when $\left|\frac{b}{a}x\right| < 1$, or $|x| < \frac{a}{b}$
  - The range must always be stated.
  - When the series is convergent it will make a good approximation of $(1+x)^n$ depending on the number of terms used, and the size of x. Small is better.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \ldots$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \ldots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$$

All valid for $|x| < 1$

Note that when the sign inside the bracket is different from the index, the signs in the expansion alternate, and when they are the same the signs in the expansion are all positive.

69.3 Area Under a Curve

69.3.1 Trapezium Rule

For a function $f(x)$ the approximate area is given by:

$$\int_a^b f(x)\ dx = \int_{x_0}^{x_n} f(x)\ dx \approx \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right]$$

where $h = \dfrac{b-a}{n}$ and n = number of strips

The value of the function for each ordinate is given by:

$$y_i = f(x_i) = f(a + ih)$$

and where i is the ordinate number.
In simpler terms:

$$A \approx \frac{\text{width}}{2}\left[\left(\text{First} + \text{last}\right) + 2 \times \text{the sum of the middle } y \text{ values}\right]$$

69.3.2 Mid-ordinate Rule

For a function $f(x)$ the approximate area is given by:

$$\int_a^b f(x)\ dx = \int_{x_0}^{x_n} f(x)\ dx \approx h\left[y_{1/2} + y_{3/2} + \ldots + y_{n-3/2} + y_{n-1/2}\right]$$

where $h = \dfrac{b-a}{n}$ and n = number of strips

69.3.3 Simpson's Rule

For a function $f(x)$ the approximate area is given by:

$$\int_a^b f(x)\ dx = \int_{x_0}^{x_n} f(x)\ dx \approx \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{n-2}\right)\right]$$

where $h = \dfrac{b-a}{n}$ and n = an EVEN number of strips

In simpler terms:

$$\int_a^b f(x)\ dx \approx \frac{h}{3}\left[\left(\text{first} + \text{last ordinate}\right) + 4\left(\text{sum of odd ordinates}\right) + 2\left(\text{sum of even ordinates}\right)\right]$$

69.4 Parametric Equations

Circle centre (0, 0) radius r:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Circle centre (a, b) radius r:

$$x = a + r\cos\theta \qquad y = b + r\sin\theta$$

69.5 Vectors

Vector Equation:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{p} \qquad \text{Through } A \text{ with direction } \boldsymbol{p}$$

$$\boldsymbol{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB} \qquad \text{Through points } A \text{ \& } B$$

$$\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$$

$$\boldsymbol{r} = (1 - \lambda)\boldsymbol{a} + \lambda \boldsymbol{b}$$

Dot Product:

$$\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}| \cos\theta$$

$$\therefore \qquad \cos\theta = \frac{\boldsymbol{p} \bullet \boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$$

$$\left|\overrightarrow{OQ}\right| = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{a^2 + b^2 + c^2}$$

$$\boldsymbol{a} \bullet \boldsymbol{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \bullet \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_x b_x \\ a_y b_y \\ a_z b_z \end{pmatrix} = (a_x \times b_x) + (a_y \times b_y) + (a_z \times b_z) = a_x b_x + a_y b_y + a_z b_z$$

If $\theta = 90°$ $\quad \Rightarrow \quad$ $\cos\theta = 0$

$\therefore \quad \boldsymbol{p} \bullet \boldsymbol{q} = 0$ $\quad$ if 2 vectors are perpendicular.

The inclusion of $\cos\theta$ in the equation brings some useful results:

- If $\boldsymbol{p}$ and $\boldsymbol{q}$ are parallel then $\theta = 0$, $\therefore \cos\theta = 1$ and $\boldsymbol{p} \bullet \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|$
- If $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular then $\theta = 90$, $\therefore \cos\theta = 0$ and $\boldsymbol{p} \bullet \boldsymbol{q} = 0$
- If the angle θ is acute then $\cos\theta > 0$ and $\boldsymbol{p} \bullet \boldsymbol{q} > 0$
- If the angle θ is between 90° & 180° then $\cos\theta < 0$ and $\boldsymbol{p} \bullet \boldsymbol{q} < 0$
- If $\boldsymbol{p} \bullet \boldsymbol{q} = 0$, then either $|\boldsymbol{p}| = 0$, $|\boldsymbol{q}| = 0$ or $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular
- Recall that $\cos\theta = -\cos(180 - \theta)$ (2nd quadrant)

Note also that:

$\boldsymbol{i} \bullet \boldsymbol{j} = 0 \quad \boldsymbol{i} \bullet \boldsymbol{k} = 0 \quad \boldsymbol{j} \bullet \boldsymbol{k} = 0$ $\qquad$ (unit vectors perpendicular)

$\boldsymbol{i} \bullet \boldsymbol{i} = 1 \quad \boldsymbol{j} \bullet \boldsymbol{j} = 1 \quad \boldsymbol{k} \bullet \boldsymbol{k} = 1$ $\qquad$ (unit vectors parallel)

$\boldsymbol{p} \bullet \boldsymbol{q} = \boldsymbol{q} \bullet \boldsymbol{p}$ $\qquad$ (commutative law)

$\boldsymbol{s} \bullet (\boldsymbol{p} + \boldsymbol{q}) = \boldsymbol{s} \bullet \boldsymbol{p} + \boldsymbol{s} \bullet \boldsymbol{q}$ $\qquad$ (distributive over vector addition)

$\boldsymbol{p} \bullet (k\boldsymbol{q}) = (k\boldsymbol{p}) \bullet \boldsymbol{q} = k(\boldsymbol{p} \bullet \boldsymbol{q})$ $\qquad$ (k is a scalar)

70 • Apdx • Trig Rules & Identities

70.1 Basic Trig Rules

Degrees	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*AT*	0	*AT*	0
sin^2	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1			
cos^2	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0			
tan^2	0	$\frac{1}{3}$	1	3	*ND*			

Where *AT* means function approaches an asymtote and *ND* means 'not defined'.

$$360° = 2\pi \text{ radians} \qquad 1 \text{ radian} = \frac{180}{\pi} \approx 57{\cdot}3°$$

Examples: $$\cos x = \sin x = \frac{1}{\sqrt{2}} \qquad \tan 30 = \frac{1}{\sqrt{3}}$$

Recall: *SOHCAHTOA*

$$\sin x = \frac{opposite}{hypotenuse}$$

$$\cos x = \frac{adjacent}{hypotenuse}$$

$$\tan x = \frac{opposite}{adjacent}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$cosec\, x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin\alpha = \cos\beta \qquad \tan\alpha = \cot\beta \qquad \sec\alpha = cosec\,\beta$$

$$\cos\alpha = \sin\beta \qquad \cot\alpha = \tan\beta \qquad cosec\,\alpha = \sec\beta$$

where $\alpha + \beta = 90° \qquad \beta = 90° - \alpha$

$$\sin(\theta) = \cos(90° - \theta) \qquad \cos(\theta) = \sin(90° - \theta)$$

$$\sin(-\theta) = -\sin\theta \qquad cosec\,(-\theta) = -cosec\,\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Second quadrant

$$\sin\theta = \sin(180° - \theta) \qquad \operatorname{cosec}\theta = \operatorname{cosec}(180° - \theta)$$
$$\cos\theta = -\cos(180° - \theta) \qquad \sec\theta = -\sec(180° - \theta)$$
$$\tan\theta = -\tan(180° - \theta) \qquad \cot\theta = -\cot(180° - \theta)$$

Third quadrant

$$\sin\theta = -\sin(\theta - 180°) \qquad \operatorname{cosec}\theta = -\operatorname{cosec}(\theta - 180°)$$
$$\cos\theta = -\cos(\theta - 180°) \qquad \sec\theta = -\sec(\theta - 180°)$$
$$\tan\theta = \tan(\theta - 180°) \qquad \cot\theta = \cot(\theta - 180°)$$

Fourth quadrant

$$\sin\theta = -\sin(360° - \theta) \qquad \operatorname{cosec}\theta = -\operatorname{cosec}(360° - \theta)$$
$$\cos\theta = \cos(360° - \theta) \qquad \sec\theta = \sec(360° - \theta)$$
$$\tan\theta = -\tan(360° - \theta) \qquad \cot\theta = -\cot(360° - \theta)$$

70.2 General Trig Solutions

- Cosine
  - The principal value (PV) of $\cos\theta = k$ is as per your calculator where $\theta = \cos^{-1}k$
  - A second solution (SV) is found at $\theta = 360 - \cos^{-1}k \qquad (\theta = 2\pi - \cos^{-1}k)$
  - Thereafter, add or subtract multiples of 360° (or 2π)
  - k valid only for $-1 \leqslant k \leqslant 1$
- Sine
  - The principal value (PV) of $\sin\theta = k$ is as per your calculator where $\theta = \sin^{-1}k$
  - A second solution (SV) is found at $\theta = 180 - \sin^{-1}k \qquad (\theta = \pi - \sin^{-1}k)$
  - Thereafter, add or subtract multiples of 360° (or 2π)
  - k valid only for $-1 \leqslant k \leqslant 1$
- Tan
  - The principal value (PV) of $\tan\theta = k$ is as per your calculator where $\theta = \tan^{-1}k$
  - A second solution (SV) is found at $\theta = 180 + \tan^{-1}k \qquad (\theta = \pi + \tan^{-1}k)$
  - Thereafter, add or subtract multiples of 360° (or 2π)
  - k valid for $k \in \mathbb{R}$

70.3 Sine & Cosine Rules

Sine rule: $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin B}$$

Cosine rule: $$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Area of a triangle: $$A = \frac{1}{2}ab\sin C = \frac{1}{2}bh \text{ (Half base} \times \text{vert height)}$$

70.4 Trig Identities

70.4.1 Trig Identities

$$sin\,\theta \equiv cos\left(\frac{1}{2}\pi - \theta\right) \qquad sin\,x = cos\left(90° - x\right)$$

$$cos\,\theta \equiv sin\left(\frac{1}{2}\pi - \theta\right) \qquad cos\,x = sin\left(90° - x\right)$$

$$tan\,\theta \equiv \frac{sin\,\theta}{cos\,\theta}$$

70.4.2 Pythagorean Identities

$$cos^2\,\theta + sin^2\,\theta \equiv 1 \qquad (1)$$

$$1 + cot^2\,\theta \equiv cosec^2\,\theta \qquad \text{(Division of (1) by } sin^2\,\theta\text{)}$$

$$1 + tan^2\,\theta \equiv sec^2\,\theta \qquad \text{(Division of (1) by } cos^2\,\theta\text{)}$$

70.4.3 Compound Angle (Addition) Identities

$$sin\,(A \pm B) \equiv sin\,A\,cos\,B \pm cos\,A\,sin\,B$$

$$cos\,(A \pm B) \equiv cos\,A\,cos\,B \mp sin\,A\,sin\,B$$

$$tan\,(A \pm B) \equiv \frac{tan\,A \pm tan\,B}{1 \mp tan\,A\,tan\,B}$$

70.4.4 Double Angle Identities

$$sin\,2A \equiv 2\,sin\,A\,cos\,A$$

$$cos\,2A \equiv cos^2 A - sin^2 A$$

$$\equiv 2\,cos^2 A - 1 \qquad (sin^2\,\theta = 1 - cos^2\,\theta)$$

$$\equiv 1 - 2 sin^2 A \qquad (cos^2\,\theta = 1 - sin^2\,\theta)$$

$$tan\,2A \equiv \frac{2 tan\,A}{1 - tan^2 A}$$

70.4.5 Triple Angle Identities

$$sin\,3A \equiv 3 sin\,A - 4 sin^3 A$$

$$cos\,3A \equiv 4 cos^3 A - 3 cos\,A$$

$$tan\,3A \equiv \frac{3 tan\,A - tan^3 A}{1 - 3 tan^2 A}$$

70.4.6 Half Angle Identities

$$cos^2\,\frac{A}{2} \equiv \frac{1}{2}\left(1 + cos\,A\right)$$

$$sin^2\,\frac{A}{2} \equiv \frac{1}{2}\left(1 + cos\,A\right)$$

70.4.7 Factor formulæ:

Sum to Product rules:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Or $$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$ *Note the gotcha in the signs*

Alternative format:

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$$

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$$

Product to Sum rules:

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$$

70.4.8 Small t Identities

If $t = \tan \tfrac{1}{2}\theta$

$$\sin\theta \equiv \frac{2t}{1+t^2}$$

$$\cos\theta \equiv \frac{1-t^2}{1+t^2}$$

$$\tan\theta \equiv \frac{2t}{1-t^2}$$

70.4.9 Small Angle Approximations

$$\sin\theta \approx \theta$$

$$\tan\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

θ in radians!!!!!!!

70.5 Harmonic (Wave) Form: $a \cos x + b \sin x$

$$a \sin x \pm b \cos x \equiv R \sin (x \pm \alpha)$$

$$a \cos x \pm b \sin x \equiv R \cos (x \mp \alpha) \qquad \text{(watch the signs)}$$

$$R = \sqrt{a^2 + b^2} \qquad R \cos \alpha = a \qquad R \sin \alpha = b$$

$$\tan \alpha = \frac{b}{a} \qquad 0 < a < \frac{\pi}{2}$$

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}} \qquad \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

Recall

$$\sin (A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

70.6 Formulæ for integrating cos A cos B, sin A cos B, & sin A sin B

$$2 \sin A \cos B \equiv \sin (A - B) + \sin (A + B)$$

$$2 \cos A \cos B \equiv \cos (A - B) + \cos (A + B)$$

$$2 \sin A \sin B \equiv \cos (A - B) - \cos (A + B)$$

$$2 \sin A \cos A \equiv \sin 2A$$

$$2 \cos^2 A \equiv 1 + \cos 2A$$

$$2 \sin^2 A \equiv 1 - \cos 2A$$

70.7 For the Avoidance of Doubt

The expressions $\sin^{-1} \theta$ and $(\sin \theta)^{-1}$ are not the same.
$\sin^{-1} \theta \equiv \arcsin \theta$ and is the inverse of $\sin \theta$ in the same way that $f^{-1}(x)$ is the inverse of $f(x)$.

$(\sin \theta)^{-1}$ is the reciprocal of $\sin \theta$ i.e. $\quad (\sin \theta)^{-1} = \frac{1}{\sin \theta}$

The confusion is made worse by the fact that we use: $\sin^2 \theta = (\sin \theta)^2$.

70.8 Geometry

70.8.1 Straight Lines

Equation of straight line

$$y = mx + c$$

$$y - y_1 = m(x - x_1) \qquad \text{Line thro' } (x_1, y_1)$$

$$m_1 m_2 = -1$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \text{Line thro' } (x_1, y_1), (x_2, y_2)$$

Dist between 2 points $\quad D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Mid point co-ordinates of a line $\quad M = \left(\frac{x_2 - x_1}{2}, \frac{y_2 - y_1}{2}\right)$

70.8.2 Equation of a Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Circle centre $= (-g, -f) \qquad$ radius $= \sqrt{(g^2 + f^2 - c)}$

$$(x - x_1)^2 + (y_2 - y_1)^2 = r^2$$

70.8.3

$180° = \pi$ radians

Arc length $= r\theta \qquad L = \dfrac{\pi r \theta}{180} \quad$ (θ in degrees)

Length of chord $= 2r\, sin\, \dfrac{\theta}{2} \qquad$ (θ in degreeas or radians)

Area of sector $= ½\, r^2 \theta \qquad$ (θ in radians) $\qquad A = \dfrac{\pi r^2 \theta}{360} \quad$ (θ in degrees)

Area of segment $= ½\, r^2 (\theta - sin\, \theta) \qquad$ (θ in degreeas or radians)

70.8.4 Areas

Area of a triangle: $\quad A = \frac{1}{2} ab \sin C = \frac{1}{2} bh$ (*Half base × vertical height*)

Area of a sector: $\quad A = \frac{1}{2} r^2 \theta \qquad$ (θ in radians)

Arc length: $\quad l = r\theta \qquad$ (θ in radians)

71 • Apdx • Logs & Exponentials

71.1 Log & Exponent Rules Summarised

Exponents	Logarithms	
$N = b^x$	$log_b N = x$	$b > 0$
$b^0 = 1$	$log_b 1 = 0$	
$b^1 = b$	$log_b b = 1$	
$a^m a^n = a^{(m+n)}$	$log_a(MN) = log_a M + log_a N$	
$\frac{a^m}{a^n} = a^{(m-n)}$	$log_a\left(\frac{M}{N}\right) = log_a M - log_a N$	
$\frac{1}{a^n} = a^{(-n)}$	$log_a\left(\frac{1}{N}\right) = -log_a N$	
$\sqrt[n]{m} = m^{\frac{1}{n}}$	$log_a \sqrt[n]{M} = \frac{1}{n} log_a M$	
$(a^m)^n = a^{(mn)}$	$log_a M^n = n\, log_a M$	
$(a^m)^{\frac{1}{n}} = a^{\left(\frac{m}{n}\right)}$	$log_a M^{\frac{1}{n}} = \frac{1}{n} log_a M$	
Change of base $\Rightarrow$	$log_a N = \frac{log_b N}{log_b a}$	
	$log_a b = \frac{1}{log_b a}$	
$\frac{a}{b} = \left(\frac{b}{a}\right)^{-1}$	$ln\frac{a}{b} = -ln\frac{b}{a}$	
	$a^{log_a m} = m$	
$a^{log_a x} = x$	$log_a(a^x) = x$	
$e^{ln\,x} = x$	$ln\, e^x = x$	
$e^{a\,ln\,x} = x^a$	$a\, ln\, e^x = ax$	*

Tips:

To solve problems like $a^x = b$ take logs on both sides first.

Note:

$$log\,x \Leftrightarrow log_{10}\,x \qquad \& \qquad ln\,x \Leftrightarrow ln_e\,x$$

71.2 Handling Exponentials

$$e^{ax+c} = e^{ax}.e^c$$

If $e^c = A$ then: $e^{ax+c} = A\,e^{ax}$

71.3 Heinous Howlers

Don't make up your own rules!

- $log\,(x + y)$ is ***not the same*** as $log\,x + log\,y$. Study the above table and you'll find that there's nothing you can do to split up $log\,(x + y)$ or $log\,(x - y)$.
- $\dfrac{log\,(x)}{log\,(y)}$ is ***not the same*** as $log\left(\dfrac{x}{y}\right)$. When you divide two logs to the same base, you are in fact using the change-of-base formula backwards. Note that $\dfrac{log\,(x)}{log\,(y)} = log_y\,(x)$, ***NOT*** $log\left(\dfrac{x}{y}\right)$!
- $(log\,x)\,(log\,y)$ is ***not the same*** as $log\,(xy)$. There's really not much you can do with the product of two logs when they have the same base.

Handling logs causes many problems, here are a few to avoid.

1 $ln\,(y + 2) = ln\,(4x - 5) + ln\,3$

You cannot just remove all the ln's so: $(y + 2) \neq (4x - 5) + 3$

To solve, put the RHS into the form of a single log first: $ln\,(y + 2) = ln\,[3\,(4x - 5)]$

$\therefore \quad (y + 2) = 3\,(4x - 5)$

2 $ln\,(y + 2) = 2\,ln\,x$

You cannot just remove all the ln's so: $(y + 2) \neq 2x$

To solve, put the RHS into the form of a single log first: $ln\,(y + 2) = ln\,x^2$

$\therefore \quad (y + 2) = x^2$

3 $ln\,(y + 2) = x^2 + 3x$

You cannot convert to exponential form term by terrm like this: $(y + 2) \neq e^{x^2} + e^{3x}$

To solve, raise e to the whole of the RHS : $(y + 2) = e^{x^2 + 3x}$

72.1 Differentiation

General differential of a function:

$$y = [f(x)]^n \Rightarrow \frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$$

Inverse Rule:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Product Rule:

$$y = uv \qquad\qquad y = f(x)g(x)$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad\qquad \frac{dy}{dx} = f(x)g'(x) - f'(x)g(x)$$

Quotient Rule:

$$y = \frac{u}{v} \qquad\qquad y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad\qquad \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2} \qquad *$$

Trig Rules:

$$y = sin^n x \Rightarrow \frac{dy}{dx} = n\, sin^{n-1}x\, cos\, x$$

$$y = cos^n x \Rightarrow \frac{dy}{dx} = -n\, cos^{n-1}x\, sin\, x$$

$$y = tan^n x \Rightarrow \frac{dy}{dx} = n\, tan^{n-1}x\, sec^2 x$$

$$y = a^x \Rightarrow \frac{dy}{dx} = a^x\, ln\, a$$

72.2 Integration

Standard integrals (useful for substitution or by inspection):

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$$

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\tan f(x)\,dx = \ln|\sec f(x)| + c \qquad etc$$

By Parts:

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx \qquad *$$

Vol of revolution:

Basic Vol of revolution: $V = \pi\int_a^b (radius)^2\,dx$

x-axis vol of revolution: $V = \pi\int_a^b y^2\,dx$ $\quad x$-axis $\quad a$ & b are x *limits*

y-axis vol of revolution: $V = \pi\int_a^b x^2\,dy$ $\quad y$-axis $\quad a$ & b are y *limits*

72.3 Differential Equations

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}\frac{dy}{dx} = x$$

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + c$$

$$e^{\ln y} = e^{\frac{1}{2}x^2 + c}$$

$$y = e^{\frac{1}{2}x^2}e^c$$

$$y = Ae^{\frac{1}{2}x^2}$$

Function $f(x)$	*Differential* $\frac{dy}{dx} = f'(x)$	
a	0	
x^n	nx^{n-1}	
e^x	e^x	
e^{ax}	ae^{ax}	
$e^{f(x)}$	$f'(x)\,e^{f(x)}$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\sin kx$	$k\cos kx$	
$\cos kx$	$-k\sin kx$	
$\tan kx$	$k\sec^2 kx$	*
$\sin f(x)$	$f'(x)\cos f(x)$	
$\cos f(x)$	$-f'(x)\sin f(x)$	
$\tan f(x)$	$f'(x)\sec^2 f(x)$	
$\cot x$	$-\mathrm{cosec}^2 x$	*
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x\cot x$	*
$\sec x$	$\sec x\tan x$	*

For all trig: x **in radians**

$\ln x$	$\frac{1}{x}$ $(x > 0)$	
$\ln ax$	$\frac{1}{x}$ $(x > 0)$	
$\ln f(x)$	$\frac{f'(x)}{f(x)}$	
$u\,v$	$uv' + vu'$	
$\frac{u}{v}$	$\frac{vu' - uv'}{v^2}$	*

$y = f(x)$	*Integral* $\int f(x)\,dx = F(x) + c$	
a	$ax + c$	
x^n	$\frac{x^{n+1}}{n+1} + c$ $(n \neq -1)$	
e^x	$e^x + c$	
e^{ax}	$\frac{1}{a}e^{ax} + c$ $(a \neq 0)$	
$\sin x$	$-\cos x + c$	
$\cos x$	$\sin x + c$	
$\tan x$	$\ln\lvert\sec x\rvert + c$	*
$\tan x$	$-\ln\lvert\cos x\rvert + c$	
$\sin kx$	$-\frac{1}{k}\cos kx + c$	
$\cos kx$	$\frac{1}{k}\sin kx + c$	
$\cos\,(kx + n)$	$\frac{1}{k}\sin(kx + n) + c$	
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert + c$	
$\cot x$	$\ln\lvert\sin x\rvert + c$	*
$\mathrm{cosec}\, x\cot x$	$-\mathrm{cosec}\, x + c$	
$\sec x\tan x$	$\sec x + c$	
$\mathrm{cosec}\, x$	$\ln\lvert\tan\frac{1}{2}x\rvert + c$	*
$\mathrm{cosec}\, x$	$-\ln\lvert\mathrm{cosec}\, x + \cot x\rvert + c$	*
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + c$	*
$\sec x$	$\ln\left\lvert\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right\rvert + c$	*
$\sec^2 kx$	$\frac{1}{k}\tan kx + c$	*
$\mathrm{cosec}^2 x$	$-\cot x + c$	
$\frac{1}{x}$	$\ln\lvert x\rvert + C$ $(x \neq 0)$	
$\ln x$	$x\ln(x) - x + C$	
$u\,v'$	$uv - \int u'v\,dx + C$	
$(ax + b)$	$\frac{(ax+b)^{n+1}}{a(n+1)} + C$ $(n \neq -1)$	
a^x	$\frac{a^x}{\ln a} + C$	

Function $y = f(x)$	Differential $\frac{dy}{dx} = f'(x)$	
$\sin^{-1} x$	$\frac{1}{\sqrt{1 - x^2}}$	*
$\cos^{-1} x$	$-\frac{1}{\sqrt{1 - x^2}}$	*
$\tan^{-1} x$	$\frac{1}{x^2 + 1}$	*
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$	
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2 - x^2}}$	
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{x^2 + a^2}$	

General differential of a function:

$$y = ax^n \Rightarrow \frac{dy}{dx} = ax^{n-1}$$

$$y = [f(x)]^n \Rightarrow \frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$y = uv$

Product Rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$y = \frac{u}{v}$

Quotient Rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$y = \sin^n x \Rightarrow \frac{dy}{dx} = n \sin^{n-1} x \cos x$$

$$y = \cos^n x \Rightarrow \frac{dy}{dx} = -n \cos^{n-1} x \sin x$$

$$y = \tan^n x \Rightarrow \frac{dy}{dx} = n \tan^{n-1} x \sec^2 x$$

$$y = a^x \Rightarrow \frac{dy}{dx} = a^x \ln a$$

$y = f(x)$	Integral $\int f(x)\,dx$	
$\frac{f'(x)}{f(x)}$	$\ln \lvert f(x) \rvert + C$	
$\frac{1}{ax + b}$	$\frac{1}{a} \ln \lvert ax + b \rvert + C$	
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln \left\lvert \frac{x - a}{x + a} \right\rvert + C$	*
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left\lvert \frac{a + x}{a - x} \right\rvert + C$	*
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + C$	*
$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$	*
$\frac{1}{x^2 + 1}$	$\tan^{-1} x + C$	

Standard integrals (useful for substitution or by inspection):

$$\int ax^n dx = \frac{a}{n + 1} x^{n+1} + C$$

$$\int_a^b ax^n dx = -\int_b^a ax^n dx$$

$$\int f'(x)[f(x)]^n dx = \frac{1}{n + 1}[f(x)]^{n+1} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln \lvert f(x) \rvert + C$$

$$\int f'(x) e^{f(x)} dx = e^{f(x)} + C$$

$$\int f'(x) \cos f(x)\,dx = \sin f(x) + C$$

$$\int f'(x) \sin f(x)\,dx = -\cos f(x) + C$$

$$\int f'(x) \tan f(x)\,dx = \ln \lvert \sec f(x) \rvert + C \qquad \text{etc}$$

By Parts: $\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b v \frac{du}{dx} dx$ *

Basic Vol of revolution: $V = \pi \int_a^b (radius)^2 dx$

x-axis: vol of revolution: $V = \pi \int_a^b y^2 dx$ x limits

y-axis: vol of revolution: $V = \pi \int_a^b x^2 dy$ y limits

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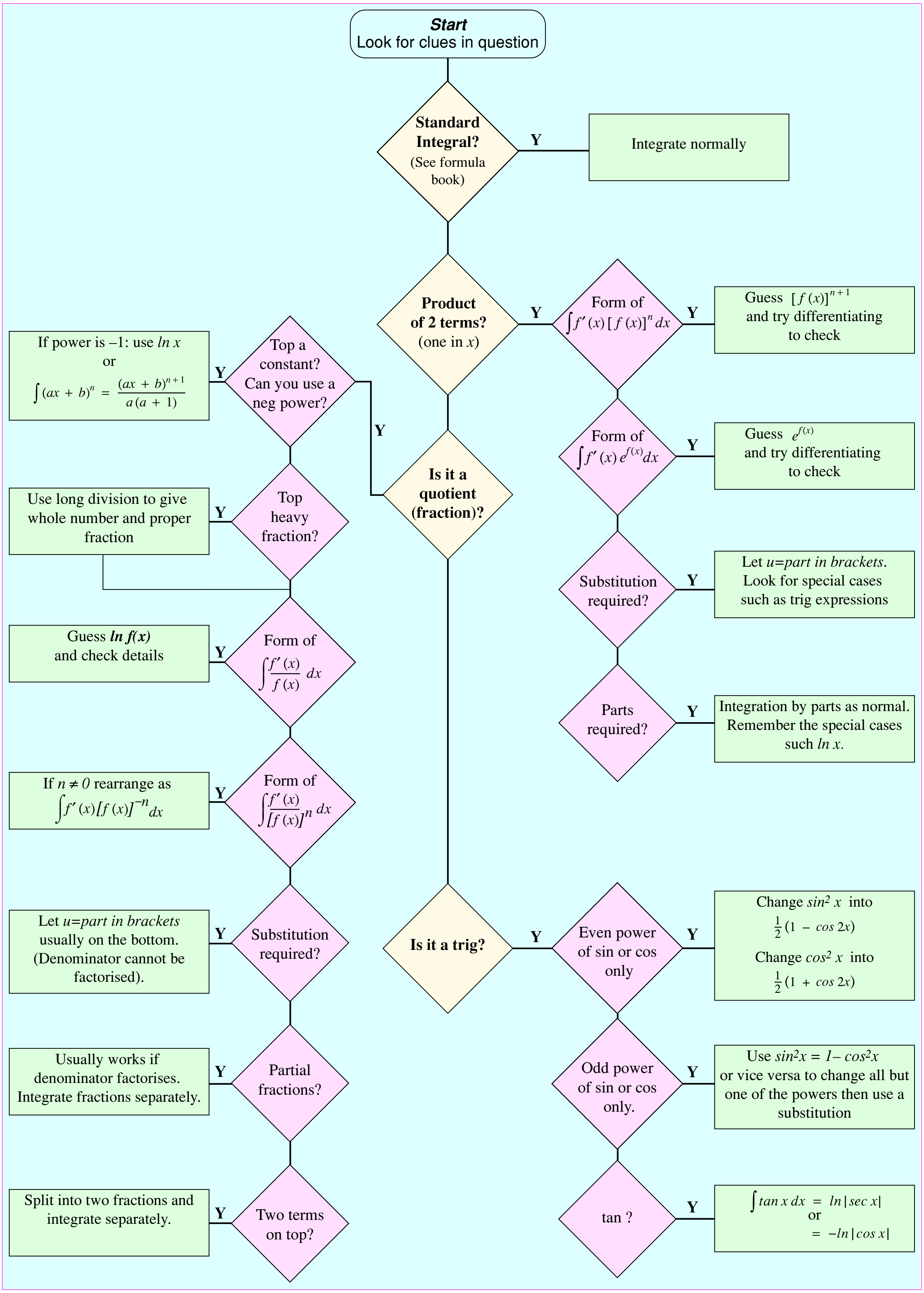

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